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ON THE SET OF SOLUTIONS TO LIPSCHITZIAN DIFFERENTIAL INCLUSIONS

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Introduction. In this paper we consider a Lipschitzian differential inclusion

$$x'(t) \in F(t,x), \quad x(t_0) = a\,, \tag{F}$$

where the values of F are compact but not convex. We prove that the map that associates to the initial point a the set of solutions to (F), $S_F(a)$, admits a selection, continuous from R^n to the space of absolutely continuous functions. The images of this map are sets that are not decomposable. In particular, the map from a to the attainable set at T, $A_T(a)$, admits a continuous selection. It is known that this map, in general, has no closed values.

Construction of the selection. We will use a further refinement of the selection technique of [1], [3], [6] and [8]. The main tools are a careful use of Liapunov's Theorem on the range of vector measures (see [6]) and of Filippov's extension of Gronwall's inequality ([5]; see also [2] p. 120). In what follows, $|a|$ is the Euclidean norm of a, $D(A,B)$ the Hausdorff distance of the sets A and B; $C(I)$ is the space of continuous mappings from I into R^n, with $\|f\|_C$ the sup norm. By $AC(I)$ we mean the space of absolutely continuous maps with the norm

$$\|f\|_{AC} = |f(t_0)| + \int |f'(s)|\,ds\,.$$

The map F will satisfy the following assumption.

Assumption (H). F is defined on an open Ω in R^{n+1}, bounded by M on it and such that

α) $t \to F(t,x)$ is measurable for fixed x;
β) $x \to F(t,x)$ is Lipschitzian with constant $K(t)$, $K \in L^1_{\text{loc}}$
γ) the value of F are compact;
δ) there exists a compact $A \subset R^n$ such that

$$\{(t,\, a + v(t-t_0)) : a \text{ in } A,\ v \text{ such that } |v| \leq M,\ t \text{ in } [t_0,T]\} \subset \Omega\,.$$

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We also set

$$\gamma(t) = \int_{t_0}^{t} k(s)\,ds\,.$$

We shall restrict ourselves to a subinterval $I = [t_0, t_1]$ of $[t_0, T]$ such that

$$\lambda = 3\left(e^{\gamma(t_1)} - 1\right) < 1.$$

On $[t_0, T]$, the result can be obtained by applying the theorem below on each subinterval satisfying the above condition and by taking composition of maps.

We will denote by Γ the integral

$$\int_{t_0}^{t_1} e^{\gamma(t_1)-\gamma(s)}\,ds\,.$$

Theorem. *Let F satisfy assumption (H). Then there exists a map $a \to \tilde{g}(a)$, continuous from $A \subset R^n$ into $AC(I)$ such that for every a, $\tilde{g}(a)$ is a solution to*

$$x'(t) \in F(t, x(t)), \quad x(t_0) = a. \tag{F}$$

Proof: (a) We claim that there exist two sequences of maps $(g^n(a))$ and $(s^n(a))$ such that

i) $s^n(a) \in S_F(a)$; $g^n(a)$ is absolutely continuous and $g^n(a)(t_0) = a$.

ii) $\|g^n(a) - s^n(a)\|_C \le M\Gamma\lambda^{n-1}$

iii) for every $\epsilon > 0$ there exist $\delta(\epsilon)$, depending on n, and a function $\phi^n(a,\epsilon)$ from I to R^+,

$$\int_I \phi^n(a,\epsilon)\,ds \le 2M\epsilon$$

such that

$$|g^n(a)'(t) - g^n(a')'(t)| \le \phi^n(a,\epsilon)(t)$$

whenever $|a - a'| \le \delta(\epsilon)$

iv) $|g^n(a)'(t) - s^n(a)'(t)| \le 3M\Gamma\lambda^{n-2}k(t)e^{\gamma(t)}, \;\; n \ge 2$

v) $\left\|g^n(a) - g^{n-1}(a)\right\|_{AC} \le 3M\Gamma\lambda^{n-1}, \;\; n \ge 3.$

(b) Set $g^1(a) = a$. Then

$$d\left(g^1(a)'(t),\, F(t,\, g^1(a)(t)\right) = d\left(0,\, F(t,a)\right) \le M$$

and by Filippov's construction there exists $s^1(a)$ in $S_F(a)$ such that

$$\left|g^1(a)(t) - s^1(a)(t)\right| \le \int_{t_0}^{t} e^{\gamma(t)-\gamma(s)} M\,ds \le M\Gamma\,.$$

The above shows that g^1, s^1 satisfy i), ii) with $n = 1$. Point iii) holds setting ϕ^1 to be 0.

Assume that we have defined g^ν and s^ν satisfying i)–iii) up to $n-1$. We claim that we can define g^n and s^n satisfying i)–iv) for $n \ge 2$.

(c) To simplify our notation we will denote g^{n-1} by g and s^{n-1} by s. The map $a \to g(a)$ is uniformly continuous on A. Let $r > 0$ satisfy the following conditions: $r \le \delta(\Gamma\lambda^{n-1})$, where δ is defined in iii); $r \le M\Gamma\lambda^{n-2}/3$; a' a'' in $B[a,r]$ imply

$$\left\|g(a') - g(a'')\right\|_C \le M\Gamma\lambda^{n-2}/3\,.$$

Let $(B[a_i,r])_i$ be a finite open cover of A and π_i a partition of unity subordinate to it, $i=1,\ldots,m$. Set $\sigma(j,a)$ to be $\Sigma_{1\le i\le j}\pi_i(a)$. Also set $s_i(t)$ to be $s(a_i)(t)$.

Let $\delta>0$ be such that $(t_1-t_0)/\delta$ is an integer, m', and $\delta\le\Gamma\lambda^{n-2}/12$. The intervals

$$J(j)=[t_0+(j-1)\delta,\, t_0+j\delta)$$

partition $I=[t_0,t_1)$. Consider the family of maps from $[t_0,t_1)$ into R^n given by

$$d_{ij}(t)=s_i'(t)\,1_{J(j)}(t):i=1,\ldots,m;\;\; j=1,\ldots,m'.$$

Let $(A(\alpha))_\alpha$ be a nested family of measurable subsets of $[t_0,t_1)$, $A(0)=\emptyset$, $A(1)=[t_0,t_1)$, such that

$$\int_{A(\alpha)} d_{ij}=\alpha\int_{t_0}^{t_1} d_{ij} \quad\text{and}\quad \mu(A(\alpha))=\alpha(t_1-t_0). \tag{1}$$

Such a family exists by a Corollary to Liapunov's Theorem (see [6]). Define $g^n(a)$ to be

$$g^n(a)(t)=a+\sum_i\int_{t_0}^{t} s_i'(u)\,1_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))}(u)\,du. \tag{2}$$

The function $g^n(a)$ is absolutely continuous. To show that it satisfies iii), note that $g^n(a)'$ and $g^n(a')'$ differ only on

$$\bigcup_i\{(A(\sigma(i,a))\setminus A(\sigma(i-1,a)))\Delta(A(\sigma(i,a'))\setminus A(\sigma(i-1,a')))\}$$

and that this set is contained in

$$\bigcup_i\{A(\sigma(i,a))\Delta A(\sigma(i,a'))\}. \tag{3}$$

Hence, fix any ϵ and let $\xi(\epsilon)$ be the (common) modulus of continuity of the functions $a\to\sigma(i,a)$. Then, whenever

$$|a-a'|<\xi(\epsilon/2m),$$

the set (3) is contained in

$$E(a,\epsilon)=\bigcup_i\{A(\sigma(i,a)+\epsilon/2m)\setminus A(\sigma(i,a)-\epsilon/2m)\} \tag{4}$$

and the total measure of this set is bounded by ϵ or,

$$\int_I 1_{E(a,\epsilon)}<\epsilon.$$

Then,

$$|g^n(a)'(t)-g^n(a')'(t)|\le 2M\,1_{E(a,\epsilon)}(t)$$

and claim iii) follows with

$$\delta(\epsilon)=\xi(\epsilon/2m)\quad\text{and}\quad \phi^n(a,\epsilon)=2M\,1_{E(a,\epsilon)}.$$

(d) We wish to estimate the distance of $g^n(a)$ from $S_F(a)$. Let t be in $[t_0+r\delta,\, t_0+(r+1)\delta)$. At $t_0+r\delta$, the integral in (2) can be written as

$$\begin{aligned}
&\sum_i \int_{t_0}^{t_0+r\delta} s_i'\, 1_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))} \\
&= \sum_i \sum_{l\le r} \int s_i'\, 1_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))}\, 1_{J(l)} \\
&= \sum_i \sum_{l\le r} \int d_{i,1}\, 1_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))} \\
&= \sum_i \sum_{l\le r} \int_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))} d_{i,l}
\end{aligned}$$

and by (1) and the definition of σ,

$$\begin{aligned}
&= \sum_{l\le r} \sum_i \pi_i(a) \int_I d_{i,l} \\
&= \sum_i \sum_{l\le r} \pi_i(a)\{s_i(t_0+l\,\delta) - s_i(t_0+(l-1)\delta)\} \\
&= \sum_i \pi_i(a)\{s_i(t_0+r\delta) - s_i(t_0)\};
\end{aligned}$$

i.e.,

$$g^n(a)(t_0+r\delta) - a = \sum_i \pi_i(a)(s_i(t_0+r\delta) - a_i)\,.$$

For any j we can write

$$\begin{aligned}
|g^n(a)(t) - s_j(t)| \le{}& |g^n(a)(t_0+r\delta) - s_j(t_0+r\delta)| \\
&+ |g^n(a)(t_0+r\delta) - g^n(a)(t)| + |s_j(t) - s_j(t_0+r\delta)|.
\end{aligned}$$

Since the derivatives of g^n and s_j are bounded by M, by our choice of δ, the sum of the two last terms is bounded by $M\Gamma\lambda^{n-2}/3$. Hence,

$$\begin{aligned}
|g^n(a)(t) - s_j(t)| \le{}& \Big|a - \sum \pi_i(a)a_i\Big| + \\
&\Big|\sum \pi_i(a)(s_i(t_0+r\delta) - s_j(t_0+r\delta))\Big| + M\Gamma\lambda^{n-2}/3.
\end{aligned} \tag{5}$$

By our choice of r in (c), whenever $\pi_i(a) > 0$,

$$|a - a_i| \le M\Gamma\lambda^{n-2}/3,$$

hence the same is true for the first term at the r.h.s. of (5). Moreover,

$$\begin{aligned}
|s_i(t_0+r\delta) - s_j(t_0+r\delta)| \le{}& |s_i(t_0+r\delta) - g(a_i)(t_0+r\delta)| + \\
&|g(a_i)(t_0+r\delta) - g(a_j)(t_0+r\delta)| + |g(a_j)(t_0+r\delta) - s_j(t_0+r\delta)|\,.
\end{aligned} \tag{6}$$

When both $\pi_i(a) > 0$ and $\pi_j(a) > 0$, again by the choice of r, the second term on the right of (6) is bounded by $M\Gamma\lambda^{n-2}/3$. By ii) and the recursive assumption, we finally have

$$|g^n(a)(t) - s_j(t)| \le 3M\Gamma\lambda^{n-2}\,.$$

The above holds for every j such that $\pi_j(a) > 0$. At any given t, except on a set of measure zero, $g^n(a)'(t) = s'_j(t)$ for some $j : \pi_j(a) > 0$. Since $s'_j(t) \in F(t, s_j(t))$,

$$\begin{aligned} d\,(g^n(a)'(t),\, F(t, g^n(a)(t))) &\le D\,(F(t, s_j(t)),\, F(t, g^n(a)(t))) \\ &\le 3M\Gamma\lambda^{n-2}k(t)\,. \end{aligned}$$

This estimate is independent of j, hence it holds on I. Again by the generalization of Gronwall's inequality, there exists a function $s^n(a)$ in $S_F(a)$ such that

$$|s^n(a)(t) - g^n(a)(t)| \le 3M\Gamma\lambda^{n-2}(e^{\gamma(t)} - 1) \le M\Gamma\lambda^{n-1}$$

and

$$|s^n(a)'(t) - g^n(a)'(t)| \le 3M\Gamma\lambda^{n-2}k(t)e^{\gamma(t)}\,.$$

The above inequality proves, ii) and iv) for all $n \ge 2$.

(e) It is left to show that if i)–iv) hold up to $n-1$, v) holds for n. With the same notations, fix any t and let j be such that $g^n(a)(t) = s'_j(t)$ (hence, $\pi_j(a) > 0$). Then

$$\begin{aligned} |g^n(a)'(t) - g^{n-1}(a)'(t)| &= |s'_j(t) - g(a)'(t)| \\ &\le |s'_j(t) - g(a_j)'(t)| + |g(a_j)'(t) - g(a)'(t)|\,. \end{aligned}$$

By iv) the first term is bounded by $3M\Gamma\lambda^{n-2}K(t)e^{\gamma(t)}$ while by the choice of r the second is bounded by $\phi^{n-1}(a,\, \Gamma\lambda^{n-1})$. Neither of these bounds depends on j, so they hold on I. Finally,

$$\left\|g^n(a) - g^{n-1}(a)\right\|_{AC} \le M\Gamma\lambda^{n-1} + 2M\Gamma\lambda^{n-1}$$

proving v).

(f) By iii) we have that each map g^n is (uniformly) continuous from A to $AC(I)$. Point v), recalling that $\lambda < 1$, shows that the sequence of maps is Cauchy, hence that it converges to a $\tilde{g}$, continuous from A to $AC(I)$. By construction, $g^n(a)'$ converges in $L^1(I)$ to $\tilde{g}'$, hence a subsequence converges to $\tilde{g}'$ pointwise a.e. By iv) the distance of $g'^n(t)$ to $F(t, g^n(t))$ converges to zero. Since the images of F are closed and F is continuous, $\tilde{g}(a)'(t) \in F(t,\, \tilde{g}(a)(t))$.

Corollary. *Let $A_T(a)$ be $\{s(a)(T) : s(a) \in S_F(a)\}$. Then the map $a \to A_T(a)$ admits a continuous selection. In particular, assume that for some convex compact set K, a in K implies $A_T(a) \subset K$. Then, $a \to A_T(a)$ has a fixed point.*

It is known [7] that under the above conditions the values of A_T need not be closed. For convex valued upper-semicontinuous maps the fixed point result appeared in [4].

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