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THE FUNCTIONAL SPACE $C^{-1,\alpha}$ AND ANALYTIC SEMIGROUPS

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Abstract. The functional space $C^{-1,\alpha}$ of derivatives of Hölder continuous functions is introduced. Using integral estimates, it is also proved that a variational elliptic operator generates an analytic semigroup in $C^{-1,\alpha}$. After characterizing the interpolation and extrapolation spaces, some applications to linear and quasilinear parabolic equations are given.

1. Introduction. The main result of this paper is the proof of generation of analytic semigroups by variational elliptic operators with Dirichlet boundary conditions in the space $(C^{-1,\alpha}(\Omega))^N$, $(0<\alpha<1)$, where Ω is any bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$, and $(C^{-1,\alpha}(\Omega))^N$ (in the sequel we will omit the index N when there is no danger of confusion) is the set of all N-uples of functions belonging to $C^{-1,\alpha}(\Omega)$. The space $C^{-1,\alpha}(\Omega)$ is the set of all functions $u\in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$) having a α-Hölder continuous representation: i.e., there are $f_0,\dots,f_n\in C^\alpha(\bar\Omega)$, such that $u=f_0+\sum_{i=1}^n D_if_i$ in the distributional sense. $C^{-1,\alpha}(\Omega)$ is a Banach space with the norm

$$\|u\|_{-1,\alpha}=\inf\Big\{\sum_{i=0}^n\|f_i\|_{0,\alpha}\,;\ f_i\in C^\alpha(\bar\Omega),\ u=f_0+\sum_{i=1}^n D_if_i\Big\}. \tag{1.1}$$

If $A_{i,j},\ A_i:\bar\Omega\to C^{N^2}$ $(i,j=1,\dots,n)$ are α-Hölder continuous, satisfy the ellipticity condition

$$\sum_{i,j=1}^n \eta_i\eta_j\ \mathrm{Re}\,\langle A_{ij}^{hk}\pi_h,\pi_k\rangle\geq\nu|\eta|^2|\pi|^2\quad\text{ for each }\eta\in\mathbf{R}^n\text{ and }\pi\in\mathbf{C}^N,$$

with $\nu>0$ and B_i and $C:\Omega\to\mathbf{C}^{N^2}$ are bounded and measurable, we define the operator

$$\begin{cases} Eu=-\sum_{i,j=1}^n D_i(A_{ij}D_ju)-\sum_{i=1}^n D_i(A_iu)+\sum_{i=1}^n B_iD_iu+Cu\\ E:D(E)=\{u\in C^{1,\alpha}(\bar\Omega)\cap C_0^0(\bar\Omega)\}\to C^{-1,\alpha}(\Omega),\end{cases}\tag{1.2}$$

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and we show that $E : D(E) \to C^{-1,\alpha}(\Omega)$ generates an analytic semigroup. More precisely, we prove that there exists $\omega \in \mathbf{R}$, $M > 0$, such that for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda > \omega$ and for every $f \in C^{-1,\alpha}(\Omega)$, the system

$$\lambda u + Eu = f$$

has a unique solution in $H_0^1(\Omega)$, which belongs also to $D(E)$ and satisfies the inequality

$$(|\lambda| - \omega)\|u\|_{-1,\alpha} + (|\lambda| - \omega)^{1/2}\|u\|_{0,\alpha} + \|u\|_{1,\alpha} \le M\|f\|_{-1,\alpha} .$$

The result is proved in three steps. First we obtain some integral estimates in the case that A_{ij} are constant and A_i, B_i, C vanish. Then, using such inequalities, we prove the generation theorem in the functional space $H^{-1}_{(\mu)}(\Omega)$, with $0 < \mu < n$. Finally, this result is used to prove the generation in $C^{-1,\alpha}(\Omega)$. Such a result may be employed to study a class of (linear and nonlinear) variational parabolic systems. A detailed treatment of nonlinear systems will appear in a forthcoming paper. Since interpolation and extrapolation spaces have shown to be useful tools in the study of nonlinear systems, we also characterize here the interpolation space $D_E(\theta,\infty)$ $(0 < \theta < 1)$. As a consequence, using also some general interpolation theory results, we show that the operator

$$\begin{cases} \tilde{E} : D(\tilde{E}) = \{u \in C^{1,\alpha}(\bar{\Omega}) \cap C_0^0;\ \tilde{E}u \in C_0^\alpha(\bar{\Omega})\} \to C_0^\alpha(\bar{\Omega}) \\ \tilde{E}u = Eu \quad \text{for each } u \in D(\tilde{E}) \end{cases}$$

generates an analytic semigroup in $C_0^\alpha(\bar{\Omega})$ for $0 < \alpha < 1$, and we also characterize the interpolation and the extrapolation spaces. Generation of analytic semigroup in Hölder spaces by non variational operators and characterization of interpolation spaces has been widely studied (see [2, 13, 14, 8, 9, 18]). Variational operators have been considered only in [6] (in the case of a single equation), but nor the domain of the elliptic operator neither interpolation spaces have been characterized.

In §2, of this paper we give notations and preliminary results and we recall well known properties of variational systems. Section 3 is devoted to proving some integral estimates. In sections 4 and 5, the generation in the spaces $H^{-1}_{(\mu)}(\Omega)$ and $C^{-1,\alpha}(\Omega)$ is proved. In §6, we characterize the interpolation and the extrapolation spaces. In the last section, we use these results to give examples of applications of abstract semigroup theory to linear and quasilinear variational parabolic equations.

2. Notations and preliminary results. In this section, we recall some basic definitions and well known properties of elliptic systems which will be frequently used in the sequel. If $N \ge 1$ is an integer, we denote by $(\,|\,)_N$ and $|\ |_N$ respectively, the scalar product and the norm in $\mathbf{C}^N$. We shall also omit the subscript when no confusion is possible. If $x_0 \in \mathbf{R}^n$ and $\sigma > 0$, set

$$B(x_0,\sigma) = \{x \in \mathbf{R}^n : |x_0 - x| < \sigma\}, \qquad B(0,\sigma) = B(\sigma).$$

Write for $x \in \mathbf{R}^n$ $x = (x, x_n)$ and set

$$B^+(\sigma) = \{x \in B(\sigma) : x_n > 0\}, \qquad \Gamma(\sigma) = \{x \in B(\sigma) : x_n = 0\}.$$

Let Ω be a bounded domain in $\mathbf{R}^n$ and denote the diameter of Ω by d_Ω. Now we recall the definitions and a few properties of some function spaces (a systematic exposition can be found in [5]). Let A be a measurable subset of $\mathbf{R}^n$ with positive measure. If $u : A \to \mathbf{C}^N$ is integrable over A, set

$$u_A = (\operatorname{meas} A)^{-1} \int_A u(x)\, dx .$$

For $0 \le \mu < n+2$, $\mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N)$ is the Banach space of the functions $u : \Omega \to \mathbf{C}^N$ such that $u \in L^2(\Omega, \mathbf{C}^N)$ and

$$[u]^2_{\mathcal{L}^{2,\mu}(\Omega)} = \sup_{x\in\Omega, 0<r\le d_\Omega} r^{-\mu} \int_{\Omega\cap B(x,r)} |u(y) - u_{\Omega\cap B(x,r)}|^2\, dy < +\infty .$$

The norm of $\mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N)$ is

$$\|u\|_{\mathcal{L}^{2,\mu}(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{\mathcal{L}^{2,\mu}(\Omega)} .$$

If $0 \le \mu < n$, we denote by $L^{2,\mu}(\Omega, \mathbf{C}^N)$ the Morrey space of the functions $u \in L^2(\Omega, \mathbf{C}^N)$, such that

$$\|u\|^2_{L^{2,\mu}(\Omega)} = \sup_{x\in\Omega, 0<r\le d_\Omega} r^{-\mu} \int_{\Omega\cap B(x,r)} |u(y)|^2\, dy < +\infty .$$

If $0 < \alpha < 1$, denote by $C^{0,\alpha}(\bar\Omega, \mathbf{C}^N)$ the functional space of the functions $u : \Omega \to \mathbf{C}^N$, such that

$$[u]_{0,\alpha,\Omega} = \sup_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|u(x) - u(y)|_N}{|x-y|_n^\alpha} < +\infty .$$

$C^{0,\alpha}(\bar\Omega, \mathbf{C}^N)$ is a Banach space with the norm

$$|u|_{0,\alpha,\bar\Omega} = |u|_{L^\infty(\Omega)} + [u]_{0,\alpha,\bar\Omega} .$$

Denote $D_i = (\partial/\partial x_i)$, $D_{ij} = D_i D_j$ for $i,j = 1,2,\dots,n$; and if α is the multi-index $(\alpha_1, \alpha_2, \dots, \alpha_n)$, let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$. Set $D = (D_1, D_2, \dots, D_n)$. Define $C^{k,\alpha}(\bar\Omega, \mathbf{C}^N)$ as the space of functions $u : \Omega \to \mathbf{C}^N$, such that $D^\beta u \in C^{0,\alpha}$ for each β with $|\beta| \le k$. Denote the norm and the seminorm in $C^{k,\alpha}(\bar\Omega, \mathbf{C}^N)$ by $|u|_{k,\alpha,\bar\Omega}$ and $[u]_{k,\alpha,\bar\Omega}$, respectively. We say that $u \in C_0^{k,\alpha}(\bar\Omega, \mathbf{C}^N)$ if $u \in C^{k,\alpha}(\bar\Omega, \mathbf{C}^N)$ and $u = 0$ on $\partial\Omega$. Let $1 \le p \le \infty$ and $m \ge 0$ be an integer. We denote by $H^{m,p}(\Omega, \mathbf{C}^N)$ and $H_0^{m,p}(\Omega, \mathbf{C}^N)$ the usual Sobolev spaces, which are the completion, respectively, of $C^\infty(\bar\Omega, \mathbf{C}^N)$ and of $C_0^\infty(\Omega, \mathbf{C}^N)$ with respect to the norms

$$\|u\|_{m,p,\Omega} = \Big(\sum_{j=0}^m |u|^2_{j,p,\Omega} \Big)^{1/2} ,$$

where

$$|u|_{j,p,\Omega} = \Big[\int_\Omega \Big(\sum_{|\alpha|=j} |Du|^2 \Big)^{p/2} dx \Big]^{1/p} .$$

If $p = 2$, we just drop the p's from our notations (e.g. $H^m(\Omega, \mathbf{C}^N)$ will stand for $H^{m,2}(\Omega, \mathbf{C}^N)$). If $0 \le \mu < n+2$, denote by $H^m_{(\mu)}(\Omega, \mathbf{C}^N)$ the space of vectors $u \in H^m(\Omega, \mathbf{C}^N)$, such that $D^\alpha u \in \mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N)$ for each α with $|\alpha| \le m$. Define $B(\Omega, \mathbf{C}^N)$ as the space of the functions that are bounded in Ω. If $0 < \mu < n$, we denote by $H^{-1}_{(\mu)}(\Omega, \mathbf{C}^N)$ the space of the functions $u \in H^{-1}(\Omega)$ that admit a representation in $\mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N)$.

The following result is due to Campanato [3].

Theorem 2.1. *Assume that Ω has the cone property. If $0 \le \mu < n$, then $\mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N) = L^{2,\mu}(\Omega, \mathbf{C}^N)$, and there is $c(\Omega, \mu) \ge 1$, such that*

$$[u]_{\mathcal{L}^{2,\mu}(\Omega)} \le \|u\|_{L^{2,\mu}(\Omega)} \le c(\Omega, \mu)\|u\|_{\mathcal{L}^{2,\mu}(\Omega)}\,.$$

If $n < \mu < n+2$, then $\mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N) = C^{0,\alpha}(\bar{\Omega}, \mathbf{C}^N)$ with $\alpha = (\mu - n)/2$ and there are $c(n)$, $c(\Omega, n)$, such that

$$c(n)[u]_{\mathcal{L}^{2,\mu}(\Omega)} \le [u]_{0,\alpha,\bar{\Omega}} \le c(\Omega, n)[u]_{\mathcal{L}^{2,\mu}(\Omega)}\,.$$

Moreover, if $u \in H^1_{(\mu)}(\Omega, \mathbf{C}^N)$, then $u \in \mathcal{L}^{2,\mu+2}(\Omega, \mathbf{C}^N)$, $0 \le \mu < n$.

We say that a bounded domain Ω is of class $C^{1,\alpha}$ if there exists a finite open covering $\{U_j\}$ of $\partial\Omega$ and if there exists a one-to-one transformation $\phi_j : U_j \to B(1)$ with the following properties:

$$\phi_j \text{ and } \phi_j^{-1} \text{ have } \alpha\text{-Hölder continuous first derivatives, on } \bar{U}_j \text{ and } \overline{B(1)}, \text{ resp.} \tag{2.1}$$

$$\phi_j(U_j \cap \Omega) = B^+(1), \quad \phi_j(U_j \cap \partial\Omega) = \Gamma(1)\,. \tag{2.2}$$

Let A_{ij}, A_i, B_i $(i, j = 1, 2, \ldots, n)$ and C be $N \times N$ complex matrix-valued functions on Ω, such that

$$A_{ij},\ A_i \ \text{ are continuous in } \bar{\Omega} \tag{2.3}$$

$$B_i \ \text{ and } \ C \ \text{ are bounded and measurable in } \Omega\,. \tag{2.4}$$

Throughout the paper, we assume the Legendre-Hadamard ellipticity conditions: there exists $\nu > 0$ such that for each $x \in \Omega$, for each $\eta \in \mathbf{R}^n$ and each $\pi \in \mathbf{C}^N$ we have

$$\sum_{i,j=1}^{n} \eta_i\eta_j \operatorname{Re}\left(A_{ij}(x)\pi|\pi\right)_N \ge \nu|\eta|_n^2\,|\pi|_N^2\,. \tag{2.5}$$

Consider the elliptic operator in divergence form:

$$Eu = -\sum_{i,j=1}^{n} D_i(A_{ij}D_ju) - \sum_{i=1}^{n} D_i(A_iu) + \sum_{i=1}^{n} B_iD_iu + Cu\,. \tag{2.6}$$

The following result was obtained using variational techniques ([5, 8]).

Lemma 2.2. *Suppose that the operator E satisfies the hypotheses (2.3)-(2.5) in a domain Ω that satisfies conditions (2.1)-(2.2). Then there exists $\omega \in \mathbf{R}^+$ such that, for each λ with $Re\lambda \ge \omega$, the Dirichlet problem*

$$\begin{cases} u \in H_0^1(\Omega, \mathbf{C}^N) \\ (\lambda + E)u = -\sum_{i=1}^n D_if_i + f_0, \quad f_i \in \mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N), \ \ 0 \le \mu < n+2 \end{cases}$$

has a unique solution u and

$$(|\lambda| - \omega)^{1/2}|u|_{0,\Omega} + |u|_{1,\Omega} \le c\sum_{i=0}^{n} |f_i|_{0,\Omega} \le c d_\Omega^{\mu/2}\|f_i\|_{\mathcal{L}^{2,\mu}(\Omega,\mathbf{C}^N)}\,, \tag{2.7}$$

where the constant c is independent of λ and u.

To study some properties of solutions of elliptic systems, it is necessary to control the norm of higher order derivatives with the norm of the lower order ones. These sort of estimates are called Caccioppoli type inequalities and we will use the following (see [8]).

Lemma 2.3. *Let $u \in H^2(B(R), \mathbf{C}^N)$ be a solution of the system*

$$\lambda u - \sum_{i,j=1}^{n} A_{ij}^0 D_{ij} u = 0 \quad \text{in } B(R),$$

where $\lambda \in \mathbf{C}$, $Re\,\lambda \geq 0$ and A_{ij}^0 are constant elliptic matrices. Then for each $r \in]0, R[$, we have

$$|u|_{2,B(r)} \leq c(R-r)^{-1}|u|_{1,B(R)} \tag{2.8}$$

and

$$|u|_{1,B(r)} \leq c(R-r)^{-1}(R/r)^{n/2}|u - u_{B(R)}|_{0,B(R)}, \tag{2.9}$$

where the constant c does not depend on λ, R, r and u.

Lemma 2.4. *Let $u \in H^2(B^+(R), \mathbf{C}^N)$ be such that $u = 0$ on $\Gamma(R)$ and*

$$\lambda u - \sum_{i,j=1}^{n} A_{ij}^0 D_{ij} u = 0 \quad \text{in } B^+(R),$$

where A_{ij}^0 are constant elliptic matrices. Then, for each $r \in]0, R[$,

$$|u|_{1,B^+(r)} \leq c(R-r)^{-1}|u|_{0,B^+(R)}, \tag{2.10}$$

and

$$\sum_{s=1}^{n-1} |D_s u|_{1,B^+(r)} \leq c(R-r)^{-1}|u|_{1,B^+(R)}, \tag{2.11}$$

where the constant c does not depend on λ, R, r and u.

We now need some technical lemmas. The first one allows us to characterize the space $C^{-1,\alpha}$, while the second one (see [7]) is an algebraical lemma useful to get integral estimates.

Lemma 2.5. *Let Δ be the Laplace operator; then for each $u \in C^{1,\alpha}$, $\Delta u \in C^{-1,\alpha}$. Moreover, if $f \in C^{-1,\alpha}$, there exists $u \in C_0^{1,\alpha}$, such that $\Delta u = f$ (or equivalently, there exist $f_1, f_2, \ldots, f_n \in C^{0,\alpha}$, such that $f = \sum_{i=1}^n D_i f_i$). Set $g = g_0 - \sum_{i=1}^n D_i g_i$ where $g_i \in \mathcal{L}^{2,\mu}(\Omega, \mathbf{C}^N)$ and $g_0 \in \mathcal{L}^{2,(\mu-2)\vee 0}(\Omega, \mathbf{C}^N)$. Then, if $0 < \mu < n$, $g \in H^{-1}_{(\mu)}(\Omega, \mathbf{C}^N)$, while if $\mu = n + 2\alpha$, with $0 < \alpha < 1$, $g \in C^{-1,\alpha}(\Omega, \mathbf{C}^N)$.*

Proof: The first part of Lemma 2.5 is trivial. In order to prove the second part, consider the Dirichlet problem

$$\begin{cases} \Delta u = f \in C^{-1,\alpha}(\Omega, \mathbf{C}^N) \\ u \in H_0^1(\Omega, \mathbf{C}^N). \end{cases}$$

By classical regularity results, we have that $u \in C_0^{1,\alpha}(\Omega, \mathbf{C}^N)$. Therefore,

$$f = \Delta u = \sum_{i=1}^{n} D_i(D_i u) = \sum_{i=1}^{n} D_i f_i,$$

where $f_i = D_i u$. The last part of the lemma follows by well known regularity results too.

Lemma 2.6. [7]. *Let $d > 0$, assume that ϕ, ψ and θ are non-negative functions on $]0, d]$, and let ψ be non-decreasing. Assume that $\lim_{r\to 0}\theta(r) = 0$ and there exists A, α, β, with $\beta < \alpha$, such that for every $t \in]0,1]$ and each $r \in]0,d]$, we have*

$$\phi(tr) \le (At^\alpha + \theta(r))\phi(r) + r^\beta \psi(r).$$

Then for each $\epsilon > 0$, there exists $d_\epsilon \in]0, d]$, such that

$$\phi(tr) \le k_1 t^{\alpha-\epsilon}\phi(r) + k_\epsilon (tr)^\beta \psi(r).$$

3. Local estimates. An essential tool in the proof of the generation of analytic semigroups in the space of Hölder continuous functions is a suitable class of integral inequalities (see [4, 8]). This remains true for the functional space $C^{-1,\alpha}$ also. Let A_{ij}^{hk}, $i, j = 1, 2, \dots, n$; $h, k = 1, 2, \dots, N$, be constant matrices satisfying the ellipticity condition; i.e., there exists a positive constant ν such that

$$\sum_{i,j=1}^{n}\sum_{h,k=1}^{N} A_{ij}^{hk}\pi_i\pi_j\eta_h\eta_k \ge \nu|\pi|_n^2\,|\eta|_N^2 \quad \text{for each } \pi \in \mathbf{R}^n,\ \eta \in \mathbf{C}^N.$$

Then the following results hold (for the proof see [8]).

Theorem 3.1. *If $u \in H^1(B(R), \mathbf{C}^N)$ is a solution of the elliptic system*

$$\lambda u - \sum_{i,j=1}^{n} A_{ij}D_{ij}u = 0 \quad \text{in } B(R) \tag{3.1}$$

with $\lambda \in \mathbf{C}$ and $Re\,\lambda > 0$, then for each $t \in]0,1]$ and $r \in]0,R]$, we have

$$|u|^2_{1,B(tr)} \le ct^n |u|^2_{1,B(r)}, \tag{3.2}$$

where the constant c does not depend on λ, R and u.

Theorem 3.2. *Let $u \in H^2(B^+(R), \mathbf{C}^N)$ vanish on $\Gamma(R)$ and solve the elliptic system*

$$\lambda u - \sum_{i,j=1}^{n} A_{ij}D_{ij}u = 0 \quad \text{in } B^+(R), \tag{3.3}$$

with $\lambda \in \mathbf{C}$ and $Re\,\lambda > 0$. Then for each $\epsilon \in]0,n]$, there exists a constant c_ϵ independent of λ, R and u, such that for each $t \in]0,1]$ and $r \in]0,R]$, we have

$$|u|^2_{1,B^+(tr)} \le c_\epsilon t^{n-\epsilon}|u|^2_{1,B^+(r)}. \tag{3.4}$$

An easy consequence of Theorem 3.1, of Poincare's inequality and of the Caccioppoli type inequality (2.9) is the following:

Theorem 3.3. *If $u \in H^2(B(R), \mathbf{C}^N)$ is a solution of the elliptic system (3.1), then for each $t \in]0,1]$ and $r \in]0,R]$*

$$\sum_{i=1}^{n} |D_i u - (D_i u)_{B(tr)}|^2_{0,B(tr)} \le ct^{n+2}\sum_{i=1}^{n} |D_i u - (D_i u)_{B(r)}|^2_{0,B(r)}. \tag{3.5}$$

Theorem 3.3 has a counterpart for half balls, but the proof is not simple and may be deduced from the following lemmas. The first one will be only stated here; a proof can be found in [8].

Lemma 3.4. *Under the assumptions of Theorem 3.2, set*

$$U(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x', -x_n) & \text{if } x_n < 0 . \end{cases} \tag{3.6}$$

Then u belongs to $H^2(B(R), \mathbf{C}^N)$ and solves in $B(R)$ the system

$$\lambda U - \sum_{i=1}^{n} A_{ii} D_{ii} U = F ,$$

where

$$F = \sum_{i \neq j}^{1,n-1} A_{ij} D_{ij} U + (\operatorname{sgn} x_n) \sum_{i=1}^{n-1} (A_{in} + A_{ni}) D_{in} U . \tag{3.7}$$

Lemma 3.5. *Under the hypotheses of Theorem 3.3, for each $t \in]0,1]$ and $r \in]0,R]$, we have*

$$\int_{B^+(tr/2)} \Big(\sum_{i=1}^{n-1} |D_i u| + |D_n u - (D_n u)_{B^+(tr/2)}| \Big)^2 dx$$
$$\leq ct^{n+2} \int_{B^+(r/2)} \Big(\sum_{i=1}^{n-1} |D_i u| + |D_n u - (D_n u)_{B^+(r/2)}| \Big)^2 dx + cr^2 \int_{B^+(r/2)} \sum_{i=j} |D_{ij} u|^2 \, dx , \tag{3.8}$$

where c is a constant independent of λ, r and u.

Proof: Consider the function U defined in Lemma 3.4. $U \in H^2(B(R), \mathbf{C}^N)$ and for each $\phi \in H_0^1(B(R), \mathbf{C}^N)$ we have

$$\lambda \int_{B(R)} (U \mid \phi) \, dx - \int_{B(R)} \sum_{i=1}^{n} (A_{ii} D_i U \mid D_i \phi) \, dx = \int_{B(R)} (F \mid \phi) \, dx ,$$

where F is defined in (3.7). Split the function U on $B(r/2)$, into the sum $v + w$, where $w \in H_0^1(B(r/2, \mathbf{C}^N))$ and

$$\lambda \int_{B(r/2)} (w \mid \phi) \, dx + \int_{B(r/2)} \sum_{i=1}^{n} (A_{ii} D_i w \mid D_i \phi) \, dx = \int_{B(r/2)} (F \mid \phi) \, dx \tag{3.9}$$

for each $\phi \in H_0^1(B(r/2), \mathbf{C}^N)$; whereas, $v \in H^1(B(r/2), \mathbf{C}^N)$ and

$$\lambda \int_{B(r/2)} (v \mid \phi) \, dx + \int_{B(r/2)} \sum_{i=1}^{n} (A_{ii} D_i v \mid D_i \phi) \, dx = 0$$

for each $\phi \in H_0^1(B(r/2), \mathbf{C}^N)$. Choosing in (3.9), $\phi = w$, from Poincaré's inequality we obtain

$$\int_{B(r/2)} \sum_{i=1}^{n} |D_i w|^2 \, dx \leq c(n, \nu) r^2 \int_{B(r/2)} |F|^2 \, dx .$$

By Theorem 3.3, we get, for each $t \in]0,1]$

$$\int_{B(tr/2)} \sum_{i=1}^{n} \left| D_i v - (D_i v)_{B(tr/2)} \right|^2 dx \leq ct^{n+2} \sum_{i=1}^{n} \int_{B(r/2)} \left| D_i v - (D_i v)_{B(r/2)} \right|^2 dx .$$

From the previous two estimates, the following one derives:

$$\begin{aligned}&\int_{B(tr/2)}\sum_{i=1}^{n}\left|D_iU-(D_iU)_{B(tr/2)}\right|^2dx\\&\le 2\int_{B(tr/2)}\Big(\sum_{i=1}^{n}\left|D_iv-(D_iv)_{B(tr/2)}\right|^2+\sum_{i=1}^{n}\left|D_iw-(D_iw)_{B(tr/2)}\right|^2\Big)dx\\&\le ct^{n+2}\int_{B(r/2)}\sum_{i=1}^{n}\left|D_iU-(D_iU)_{B(r/2)}\right|^2dx+cr^2\int_{B(r/2)}|F|^2dx\,.\end{aligned}$$

Now we obtain the statement of Lemma 3.5 as a consequence of the definitions (3.6) and (3.7).

Theorem 3.6. *Let $u\in H^2(B^+(R),\mathbf{C}^N)$ vanish on $\Gamma(R)$ and solve the system (3.3) with $\lambda\in\mathbf{C}$ and $Re\,\lambda>0$. Then for each $\epsilon\in]0,n+2]$, there exists a constant c_ϵ independent of λ, R and u, such that for each $t\in]0,1]$ and $r\in]0,R]$,*

$$\begin{aligned}&\int_{B^+(tr)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(tr)}|^2\Big)dx\\&\le c_\epsilon t^{n+2-\epsilon}\int_{B^+(r)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(r)}|^2\Big)dx\,.\end{aligned}\tag{3.10}$$

Proof: First, we observe that for each $k=1,2,\ldots,n-1$, D_ku is a solution of system (3.3) that vanishes on $\Gamma(R)$ and can therefore be estimated by (3.4) for each $z\in]0,1]$ and $r\in]0,R]$:

$$\int_{B^+(zr/2)}\sum_{i=1}^{n}\sum_{k=1}^{n-1}|D_{ik}u|^2dx\le cz^n\int_{B^+(r/2)}\sum_{i=1}^{n}\sum_{k=1}^{n-1}|D_{ik}u|^2dx\,.\tag{3.11}$$

Now, by Lemma 3.5, we obtain for each $s\in]0,1]$,

$$\begin{aligned}&\int_{B^+(zrs/2)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(zrs/2)}|^2\Big)dx\\&\le c_\epsilon s^{n+2-\epsilon/2}\int_{B^+(zr/2)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(zr/2)}|^2\Big)dx\\&\quad+c_\epsilon(zr)^2\int_{B^+(zr/2)}\sum_{i=j}|D_{ij}u|^2dx\,,\end{aligned}$$

and so by (3.11) and (2.11), we get

$$\begin{aligned}&\int_{B^+(szr/2)}\Big(\sum_{i=1}^{n+1}|D_iu|^2+|D_nu-(D_nu)_{B^+(szr/2)}|^2\Big)dx\\&\le c_\epsilon s^{n+2-\epsilon/2}\int_{B^+(zr/2)}\sum_{i=1}^{n-1}\Big(|D_iu|^2+|D_nu-(D_nu)_{B^+(zr/2)}|^2\Big)dx\\&\quad+c_\epsilon z^{n+2-\epsilon}\int_{B^+(r)}\sum_{i=1}^{n-1}|D_iu|^2dx\end{aligned}$$

for each $s,\ z\in]0,1]$ and each $r\in]0,R]$. From Lemma 2.6, we conclude that for each $\epsilon\in]0,n+2]$, each $z,\ s\in]0,1]$ and each $r\in]0,R]$

$$\begin{aligned}&\int_{B^+(zrs/2)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(zrs/2)}|^2\Big)\,dx\\ &\le c_\epsilon s^{n+2-\epsilon}\int_{B^+(zr/2)}\Big(\sum_{i=1}^{n-1}|D_iu|^2+|D_nu-(D_nu)_{B^+(zr/2)}|^2\Big)\,dx\\ &\quad+c_\epsilon z^{\epsilon/2}(sz)^{n+2-\epsilon}\int_{B^+(r)}\sum_{i=1}^{n-1}|D_iu|^2\,dx\,,\end{aligned}$$

that gives (3.10) for $z=1$.

The following estimates on U may be deducted from Caccioppoli's estimate and Poincaré's inequality and from Theorem 3.3 and 3.6.

Theorem 3.7. *If $u\in H^1(B(R),\mathbf{C}^N)$ is a solution of the system (3.1), with $\lambda\in\mathbf{C}$ and $Re\,\lambda>0$, then for each $t\in]0,1]$ and $r\in]0,R]$,*

$$|u-u_{B(tr)}|^2_{0,B(tr)}\le ct^{n+2}|u-u_{B(r)}|^2_{0,B(r)}\,,$$

with c independent of λ, r and u.

Theorem 3.8. *Let $u\in H^1(B^+(R),\mathbf{C}^N)$ be a solution of the elliptic system (3.3) with $\lambda\in\mathbf{C}$ and $Re\,\lambda>0$. Then for each $\epsilon\in]0,n+2]$, there exists a constant c_ϵ independent of λ, R and u, such that for every $t\in]0,1]$ and $r\in]0,R]$,*

$$|u|^2_{0,B^+(tr)}\le c_\epsilon t^{n+2-\epsilon}|u|^2_{0,B^+(r)}\,.$$

4. The estimates in the space of derivatives of Morrey functions. In this section, we prove the generation of analytic semigroups in the functional space $H^{-1}_{(\mu)}(\Omega,\mathbf{C}^N)$, $0\le\mu<n$. This result is an essential step in the proof of the generation in the functional space $C^{-1,\alpha}$, $0<\alpha<1$ (see next section). Before starting with our analysis, let us recall the assumption we have to make in order to study the problem. Let Ω be a bounded C^1 domain in $\mathbf{R}^n$. Consider the variational second order differential operator E defined in (2.6), satisfying the hypotheses (2.3)-(2.5). From a variational inequality, it is well known that there exists a positive number ω, such that for every $\lambda\in\mathbf{C}$ with $\mathrm{Re}\,\lambda>\omega$ the Dirichlet problem

$$\begin{cases}u\in H^1_0(\Omega,\mathbf{C}^N)\\ (\lambda+E)u=f\in H^{-1}(\Omega,\mathbf{C}^N)\end{cases}\tag{4.1}$$

has a unique solution. In this section we will set forth the following result:

Theorem 4.1. *There exists $\omega>0$, such that for every λ with $Re\,\lambda>\omega$ and for every $f\in H^{-1}_{(\mu)}(\Omega,\mathbf{C}^N)$ $(0\le\mu<n)$, the solution u of problem (4.1) belongs to $H^1_{(\mu)}(\Omega,\mathbf{C}^N)$, and*

$$(|\lambda|-\omega)\|u\|_{H^{-1}_{(\mu)}(\Omega)}+(|\lambda|-\omega)^{1/2}\|u\|_{L^{2,\mu}(\Omega)}+\|u\|_{H^1_{(\mu)}(\Omega)}\le c\|f\|_{H^{-1}_{(\mu)}(\Omega)}\,,\tag{4.2}$$

where c is independent of λ, f and u.

Theorem 4.1 is a consequence of the local estimates of the previous section. Let us start with a lemma concerning the solutions of particular elliptic systems.

Lemma 4.2. *For every ball $B(x_0,r)\subset\Omega$, $s\in]0,r]$ and solution $U\in H^1(B(x_0,r),\mathbf{C}^N)$ of the system*

$$\lambda U-\sum_{i,j=1}^{n}D_i(A_{ij}D_jU)=g\in H^{-1}_{(\mu)}(B(x_0,r),\mathbf{C}^N),$$

the following estimate holds:

$$|U|^2_{1,2,B(x_0,s)}\le cs^{\mu}\left(\|g\|^2_{H^{-1}_{(\mu)}(B(x_0,r))}+|U|^2_{1,2,B(x_0,r)}\right), \tag{4.3}$$

where c is independent of λ, x_0 and s.

Proof: By the standard variational theory we know that for every $s\in]0,r]$ there exists a unique $w\in H^1_0(B(x_0,s),\mathbf{C}^N)$ that solves the Dirichlet problem

$$\lambda w-\sum_{i,j=1}^{n}D_i(A_{ij}(x_0)D_jw)=g-\sum_{i,j=1}^{n}D_i\big((A_{ij}(x_0)-A_{ij}(x)\big)D_jU \quad\text{in } B(x_0,s),$$

and

$$\begin{aligned}|w|^2_{1,B(x_0,s)}&\le c\|g\|^2_{H^{-1}(B(x_0,s)}+\omega(s)|u|^2_{1,B(x_0,s)}\\&\le cs^{\mu}\|g\|^2_{H^{-1}_{(\mu)}(B(x_0,s))}+\omega(s)|u|^2_{1,B(x_0,s)},\end{aligned} \tag{4.4}$$

where $\omega(s)$ is a nondecreasing function, such that $\omega(s)\to 0$ for $s\to 0$. Hence, setting $v=U-w$ on $B(x_0,s)$, we have

$$\begin{cases} v\in H^1(B(x_0,s),\mathbf{C}^N)\\ \lambda v-\sum_{i,j=1}^{n}D_i(A_{ij}(x_0)D_jv)=0 \quad\text{in } B(x_0,s).\end{cases}$$

Therefore, by Theorem 3.1, we have

$$|v|^2_{1,B(x_0,ts)}\le ct^n|v|^2_{1,B(x_0,ts)}. \tag{4.5}$$

Combining together inequalities (4.3) and (4.4), we obtain

$$\begin{aligned}|U|^2_{1,B(x_0,ts)}&\le 2|v|^2_{1,B(x_0,ts)}+2|w|^2_{1,B(x_0,ts)}\\&\le ct^n|v|^2_{1,B(x_0,s)}+2|w|^2_{1,B(x_0,s)}\\&\le c(t^n+\omega(s))|U|^2_{1,B(x_0,s)}+cs^{\mu}\|g\|^2_{H^{-1}_{(\mu)}(B(x_0,s))}.\end{aligned} \tag{4.6}$$

Now, we apply Lemma 2.6 to (4.6), choosing $\epsilon=n-\mu$. We thus conclude that there exists a $r_\mu\in[0,r[$, such that for each $t\in]0,1]$, each $s\le r_\mu$

$$|U|^2_{1,B(x_0,ts)}\le ct^{\mu}|U|^2_{1,B(x_0,s)}+c(ts)^{\mu}\|g\|^2_{H^{-1}_{(\mu)}(B(x_0,s)}.$$

If $s\le r_\mu$, estimate (4.3) is proved. On the other hand, if $s\ge r_\mu$, (4.3) is trivial.

Remark 4.3. Let $\Omega_1\subset\Omega$ and set $d_1=\operatorname{dist}(\Omega_1,\Omega)$. For each $x_0\in\Omega_1$, the solution u of the problem (4.1) is such that

$$\lambda u-\sum_{i,j=1}^{n}D_i(A_{ij}D_ju)=g=f+\sum_{i=1}^{n}D_i(A_iu)-\sum_{i=1}^{n}B_iD_iu-Cu \quad\text{in } B(x_0,d_1).$$

By well known estimates in Morrey spaces (see [5]), we have that $u \in L^{2,2}(\Omega, \mathbf{C}^N)$ and $g \in H^{-1}_{(\mu_*)}$ with $\mu_* = \mu \wedge 2$ and

$$\|g\|_{H^{-1}_{(\mu_*)}} \le c\|f\|_{H^{-1}_{(\mu)}} + c|u|_{1,2}.$$

Therefore, by Lemma 4.2, $u \in H^1_{(\mu_*)}(\Omega_1, \mathbf{C}^N)$ and

$$\|u\|_{H^1_{(\mu_*)}(\Omega_1)} \le c\{\|f\|_{H^{-1}_{(\mu)}(\Omega)} + |u|_{1,2,\Omega}\},$$

where c is independent of λ. By iterating this procedure a finite number of times, we get the following conclusion: for each subdomain $\Omega' \subset \Omega$, $u \in H^1_{(\mu)}(\Omega, \mathbf{C}^N)$ and

$$\|u\|_{H^1_{(\mu)}(\Omega')} \le c\{\|f\|_{H^{-1}_{(\mu)}(\Omega)} + |u|_{1,2,\Omega}\}. \tag{4.7}$$

In order to estimate the first derivatives of the solution u near the boundary, we prove two lemmas concerning the unit half ball.

Lemma 4.4. *Let A_{ij} be continuous elliptic matrix-valued functions defined in $B^+(1)$ and $\lambda \in \mathbf{C}$, such that $Re\,\lambda > 0$. If $U \in H^1(B^+(1), \mathbf{C}^N)$ is a solution of the system*

$$\lambda U - \sum_{i,j=1}^{n} D_i(A_{ij} D_j U) = g \in H^{-1}_{(\mu)}(B^+(1), \mathbf{C}^N), \tag{4.8}$$

and vanishes on $\Gamma(1)$, then for each $s \in]0,1]$, the following integral estimate holds

$$|U|^2_{1,B^+(s)} \le c s^{\mu}\left(\|g\|^2_{H^{-1}_{(\mu)}(B^+(1))} + |U|^2_{1,B^+(1)}\right), \tag{4.9}$$

where c is independent of λ, s and U.

Proof: Let $s \in]0,1]$, let $w \in H^1_0(B^+(s))$ be the solution of the equation

$$\lambda w - \sum_{i,j=1}^{n} D_i(A_{ij}(0) D_j w) = g + \sum_{i,j=1}^{n} D_i((A_{ij}(x) - A_{ij}(0)) D_j U) \quad \text{in } B^+(s).$$

By standard variational results, the following inequality holds:

$$|w|^2_{1,B^+(s)} \le c s^{\mu} |g|^2_{H^{-1}_{(\mu)}} + \omega(s)|U|^2_{1,B^+(s)}, \tag{4.10}$$

where $\omega(s)$ tends to 0 as $s \to 0$. Let $v = U - w$; then v belongs to $H^1(B^+(s), \mathbf{C}^N)$, vanishes on $\Gamma(s)$ and satisfies

$$\lambda v - \sum_{i,j=1}^{n} A_{ij}(0) D_{ij} v = 0 \quad \text{in } B^+(s).$$

Therefore, by Theorem 3.2, for each $t \in]0,1]$, we have

$$|v|^2_{1,B^+(ts)} \le c t^{n+\mu/2} |v|^2_{1,B^+(s)}, \tag{4.11}$$

where c is independent of λ, s and v. Arguing exactly as in Lemma 4.2 and using estimates (4.10) and (4.11) and the algebraic Lemma 2.6, (4.9) easily follows.

Remark 4.5. Also in this situation, a system which does not consist only in its leading part can be treated with a simple modification of the argument used in the previous lemma. Assume that $u \in H^1(B^+(1), \mathbf{C}^N)$, $u = 0$ on $\Gamma(1)$, is a solution of the system

$$\lambda u - \sum_{i,j=1}^{n} D_i(A_{ij} D_j u) - \sum_{i=1}^{n} D_i(A_i u) + \sum_{i=1}^{n} B_i D_i u + Cu = g \in H^{-1}_{(\mu)},$$

where A_{ij} are elliptic matrices, A_{ij} and A_i are continuous matrix-valued functions and B_i are bounded and measurable. Set $G = g + \sum_{i=1}^n D_i(A_i u) - \sum_{i=1}^n B_i D_i u - Cu$. By known properties of the Morrey spaces, we get $G \in H^{-1}_{(\mu_*)}(B^+(1))$, where $\mu_* = \mu \wedge 2$. Moreover,

$$\|G\|_{H^{-1}_{(\mu_*)}(B^+(1))} \le c\|g\|_{H^{-1}_{(\mu)}(B^+(1))} + c\|u\|_{1,B^+(1)}.$$

Applying Lemma 4.4, we have that for every $s \in]0,1]$,

$$|u|_{1,B^+(s)} \le c s^{\mu_*/2} \left(\|g\|_{H^{-1}_{(\mu)}(B^+(1))} + \|u\|_{1,B^+(1)}\right). \tag{4.12}$$

From (4.12) and Remark 4.3, we can deduce (see [4]) that $u \in H^1_{(\mu_*)}(B^+(r), \mathbf{C}^N)$ for each $r \in]0,1[$ and

$$\|u\|_{H^1_{(\mu_*)}(B^+(r))} \le c\{\|g\|_{H^{-1}_{(\mu)}(B^+(1))} + \|u\|_{1,B^+(1)}\};$$

this new information implies that $G \in H^{-1}_{(\mu_{**})}(B^+(r), \mathbf{C}^N)$, where $\mu_{**} = \mu \wedge 4$ and

$$\|G\|_{H^{-1}_{(\mu_{**})}(B^+(r))} \le c\{\|g\|_{H^{-1}_{(\mu)}(B^+(1))} + \|u\|_{1,B^+(1)}\}.$$

Iterating the previous argument a finite numbers of times, it follows that for each $r \in]0,1[$, $u \in H^1_{(\mu)}(B^+(r), \mathbf{C}^N)$ and

$$\|u\|_{H^1_{(\mu)}(B^+(r))} \le c\|g\|_{H^{-1}_{(\mu)}(B^+(1))} + c\|u\|_{1,B^+(1)}, \tag{4.13}$$

where c does not depend on λ and u. Combining the information given by Remarks 4.3 and 4.5, we are able to estimate the first derivatives of u.

Proposition 4.6. *Under the same assumptions of Theorem 4.1,*

$$\|u\|_{H^1_{(\mu)}(\Omega)} \le c\|g\|_{H^{-1}_{(\mu)}(\Omega)}, \tag{4.14}$$

where c is independent of λ and u.

Proof: As $\partial\Omega$ is of class C^1, there exists a finite open covering $\{U_1, U_2, \dots, U_m\}$ of $\partial\Omega$ and a corresponding set $\{\phi_1, \phi_2, \dots, \phi_m\}$ of one-to-one transformations taking U_k onto $B(1)$, such that conditions (2.1) and (2.2) are satisfied. Set $U^k(x) = u(\phi_k^{-1}(x))$, $x \in B^+(1)$ and $k = 1, 2, \dots, m$. Then, $U^k = 0$ on $\Gamma(1)$ and solves in $B^+(1)$ the system

$$\lambda U^k - \sum_{i,j=1}^{n} D_i(A^k_{ij} D_j U^k) - \sum_{i=1}^{n} D_i(A^k_i U^k) + \sum_{i=1}^{n} B^k_i D^k_i U^k + C^k U^k = f^k \in H^{-1}_{(\mu)}(B^+(1)).$$

From Remark 4.5, it follows that

$$\|U^k\|_{H^1_{(\mu)}(B^+(r))} \le c\{\|f^k\|_{H^{-1}_{(\mu)}(B^+(1))} + \|U^k\|_{1,2,B^+(1)}\}, \tag{4.15}$$

with c independent of λ, k and u. Choose r, such that $\{\phi_1^{-1}(B(r)), \dots, \phi_m^{-1}(B(r))\}$ is still a covering of $\partial\Omega$. By inequalities (4.15) and (4.7), we have that $u \in H^1_{(\mu)}(\Omega, \mathbf{C}^N)$ and

$$\|u\|_{H^1_{(\mu)}(\Omega)} \le c\{\|f\|_{H^{-1}_{(\mu)}(\Omega)} + \|u\|_{1,2,\Omega}\},$$

where c is independent of λ. Estimate (4.14) comes from the last inequality and from Lemma 2.2.

Using the last result, now we are ready to get estimates on u.

Lemma 4.7. *Under the same hypotheses of Theorem 4.1, for each $x_0 \in \Omega$, set $r = 1/2\, dist(x_0, \partial\Omega)$. Then, if u is the solution of the equation (4.1) for every $s \in]0, r[$, the following estimate holds:*

$$(|\lambda| - \omega)^{1/2} |u - u_{B(s)}|_{0,B(s)} \leq c s^{\mu/2} \|g\|_{H^{-1}_{(\mu)}(\Omega)}, \tag{4.16}$$

where c is independent of s, r, u and g.

Proof: In $B(x_0, s)$, u solves the elliptic system

$$\lambda u - \sum_{i,j=1}^{n} D_i(A_{ij}(x_0) D_j u) = G = g + \sum_{i=1}^{n} D_i(A_{ij}(x) - A_{ij}(x_0) D_j u) + \sum_{i=1}^{n} D_i(A_i u) - \sum_{i=1}^{n} B_i D_i U + CU .$$

By Proposition 4.6,

$$G \in H^{-1}_{(\mu)}(B(r)) \quad \text{and} \quad \|G\|_{H^{-1}_{(\mu)}(B(r))} \leq c\|g\|_{H^{-1}_{(\mu)}(\Omega)} .$$

Let $s \in [0, r[$ and consider the solution $w \in H^1_0(B(s))$ of the system

$$\lambda w - \sum_{i,j=1}^{n} D_i(A_{ij}(x_0) D_j w) = G .$$

By standard variational estimates, we get

$$(|\lambda| - \omega)^{1/2} |w|_{0,B(s)} \leq c s^{\mu/2} \|g\|_{H^{-1}_{(\mu)}(\Omega)} . \tag{4.17}$$

Define $v = u - w \in H^1(B(s))$, where v is a solution of the system

$$\lambda v - \sum_{i,j=1}^{n} D_i(A_{ij}(x_0) D_j v) = 0 .$$

By Theorem 3.7, for every $t \in [0, n+2[$, we have

$$|v - v_{B(ts)}|^2_{0,B(ts)} \leq c t^{n+2} |v - v_{B(s)}|^2_{0,B(s)} . \tag{4.18}$$

Estimate (4.16) follows from inequalities (4.17) and (4.18) using the same arguments of the final part of the proof of Lemma 4.2.

Lemma 4.8. *Under the same assumptions of Remark 4.5, for every $s \in [0, 1[$, we have*

$$(|\lambda| - \omega)|u|^2_{0,B^+(s)} \leq c s^{\mu} \big\{ \|g\|_{H^{-1}_{(\mu)}(B^+(1))} + \|u\|^2_{1,B^+(1)} \big\} , \tag{4.19}$$

where c does not depend on λ, s and u.

Sketch of the proof: Also in this situation, in $B^+(s)$, $0 < s < 1$, we write $u = v + w$, where $w \in H^1_0(B^+(s), \mathbf{C}^N)$ solves

$$\lambda w - \sum_{i,j=1}^{n} D_i(A_{ij}(x_0) D_{ij} w) = G = g - \sum_{i,j=1}^{n} D_i\big((A_{ij}(x_0) - A_{ij}(x)) D_j u\big) + \sum_{i=1}^{n} D_i(A_i u) - \sum_{i=1}^{n} B_i D_i u - Cu .$$

Therefore, v solves the system

$$\lambda v - \sum_{i,j=1}^{n} D_i(A_{ij}(0)D_j v) = 0.$$

The statement of this lemma follows, combining together the estimates on v and w given respectively by Theorem 3.8 and by standard variational techniques and the algebraic Lemma 2.6.

Proposition 4.9. *Under the same conditions of Theorem 4.1, we have*

$$(|\lambda| - \omega)^{1/2}\|u\|_{L^{2,\mu}(\Omega)} \leq c\|g\|_{H^{-1}_{(\mu)}(\Omega)}, \tag{4.20}$$

with c independent of λ and u.

Sketch of the proof: Arguing exactly as in Proposition 4.6, and using estimates (4.16) and (4.19) instead of (4.3) and (4.9), one obtains

$$(|\lambda| - \omega)^{1/2}\|u\|_{\mathcal{L}^{2\mu}(\Omega)} \leq c|g|_{H^{-1}_{(\mu)}(\Omega)}.$$

Inequality (4.19) derives from the last one, recalling that $\mathcal{L}^{2,\mu}$ is isomorphic to $L^{2,\mu}$.

We are now ready to prove the main result of this section.

Proof of Theorem 4.1: We obtain the complete estimate directly from system (4.1). In fact,

$$|\lambda|\,|u|_{H^{-1}_{(\mu)}(\Omega)} = \left\| \sum_{i,j=1}^{n} D_i(A_{ij}D_j u) + \sum_{i=1}^{n} D_i(A_i u) - \sum_{i=1}^{n} B_i D_i u - Cu + f \right\|_{H^{-1}_{(\mu)}(\Omega)}.$$

Therefore, the estimate on u in $H^{-1}_{(\mu)}(\Omega)$ derives from estimates (4.16), (4.20) and from the previous equality.

5. The estimate in the space of derivatives of Hölder continuous functions.

Before stating the main result of this section, we make stronger assumptions on the regularity of the coefficients and the boundary of Ω. Namely, fix $\alpha \in]0,1[$ and let Ω be a $C^{1,\alpha}$ domain in $\mathbf{R}^n$. Assume further the following conditions on the coefficients of system (4.1):

$$A_{ij} \text{ and } A_i \text{ are Hölder continuous in } \bar{\Omega} \text{ with exponent } \alpha \in]0,1[, \tag{5.1}$$

$$B_i \text{ and } C \text{ are bounded and measurable.} \tag{5.2}$$

Remark 5.1. The result of this section holds with less stronger regularity assumptions:

$$B_i \in L^{n/(1-\alpha)}(\Omega) \quad \text{and} \quad C \in L^{n/(2-\alpha)}(\Omega). \text{ (see [9])}$$

Recalling that the Banach space $\mathcal{L}^{2,n+2}(\Omega, \mathbf{C}^N)$ is isomorphic to the space of α-Hölder continuous functions on $\bar{\Omega}$, we start proving the result of generation in $C^{-1,\alpha}(\Omega, \mathbf{C}^N)$.

Theorem 5.2. *Under conditions (5.1) and (5.2), there exists a positive number ω, such that if $Re\,\lambda > \omega$, then for each $f \in C^{-1,\alpha}(\Omega, \mathbf{C}^N)$, the solution u of the problem*

$$\begin{cases} (\lambda - E)u = f \\ u \in H_0^1(\Omega, \mathbf{C}^N) \end{cases} \tag{5.3}$$

belongs to $C_0^{1,\alpha}(\Omega, \mathbf{C}^N)$. Moreover,

$$(|\lambda| - \omega)\|u\|_{-1,\alpha,\Omega} + (|\lambda| - \omega)^{1/2}\|u\|_{0,\alpha,\Omega} + \|u\|_{1,\alpha,\Omega} \le c\|f\|_{-1,\alpha,\Omega}\,, \tag{5.4}$$

where c is independent of λ.

Also in this case, we first prove some integral inequalities in the interior of Ω.

Lemma 5.3. *Under the same hypotheses of Theorem 5.2, assume that for each $\epsilon > 0$, the solution $u \in H^1_{(n-\epsilon)}(\Omega, \mathbf{C}^N)$. For each subdomain $\Omega' \subset \Omega$, for each $x_0 \in \Omega'$, for each $r \le d/2$, where $d = \mathrm{dist}\,(\Omega', \partial\Omega)$, the following estimate holds for every $t \in]0,1]$:*

$$\sum_{i=1}^n |D_i u - (D_i u)_{B(x_0,tr)}|^2_{0,B(x_0,tr)} \le ct^{n+2}\{\|f\|^2_{-1,\alpha,\Omega} + \|u\|^2_{H^1_{(n-\alpha)}(\Omega)}\}\,, \tag{5.5}$$

where c is independent of λ and r.

Proof: Let $\Omega'' \subset \Omega'$ be a subdomain of Ω. Set $d_1 = \mathrm{dist}(\Omega'', \partial\Omega)$ and fix $x_0 \in \Omega''$. In $B(x_0, r)$, $r \le d_1/2$, split the solution u into the sum $u = v + w$, where $w \in H_0^1(B(r), \mathbf{C}^N)$ solves the equation

$$\lambda w - \sum_{i,j=1}^n D_i(A_{ij}(x_0)D_j w) = -\sum_{i,j=1}^n D_i(A_{ij}(x_0) - A_{ij}(x)D_j u) + G(x)\,, \tag{5.6}$$

where

$$G(x) = f(x) + \sum_{i=1}^n D_i(A_i u) - \sum_{i=1}^n B_i D_i u - Cu \in C^{-1,\alpha}(\Omega, \mathbf{C}^N)\,.$$

Hence, by Lemma 2.2, we obtain (choosing $\epsilon = \alpha$)

$$\begin{aligned} |w|^2_{1,2,B(r)} &\le cr^{n+2\alpha}\|G\|^2_{-1,\alpha,\Omega} + c\sum_{i,j=1}^n \|A_{ij}(x_0) - A_{ij}(x)\|^2_{L^\infty(B(r))} r^{n-\epsilon}\|u\|^2_{H^1_{(n-\epsilon)}} \\ &\le cr^{n+\alpha}\{\|G\|^2_{-1,\alpha,\Omega} + \|u\|^2_{H^1_{(n-\alpha)}(\Omega)}\}\,. \end{aligned} \tag{5.7}$$

On the other hand, v is a solution of the system

$$\lambda v - \sum_{i,j=1}^n D_i(A_{ij}(x_0)D_j v) = 0\,.$$

By Theorem 3.3, for each $t \in]0, n+2]$, we have

$$\sum_{i=1}^n |D_i v - (D_i v)_{B(tr)}|^2_{0,B(tr)} \le ct^{n+2}\sum_{i=1}^n |D_i v - (D_i v)_{B(r)}|^2_{0,B(r)}\,. \tag{5.8}$$

From (5.7) and (5.8), we obtain that for each $r \leq d_1/2$, for each $x_0 \in \Omega''$,

$$\sum_{i=1}^{n} \left| D_i u - (D_i u)_{B(x_0,r)} \right|^2_{0,B(x_0,r)} \leq ct^{n+\alpha} \{ \|f\|^2_{-1,\alpha,\Omega} + \|u\|^2_{H^1_{(n-\epsilon)}(\Omega)} \} . \tag{5.9}$$

The last inequality (see [4]) means that $u \in C^{1,\alpha/2}(\Omega'', \mathbf{C}^N)$ and

$$\sum_{i=1}^{n} |D_i u|^2_{L^\infty(\Omega'')} \leq c \{ \|f\|^2_{-1,\alpha,\Omega} + \|u\|^2_{H^1_{(n-\epsilon)}(\Omega)} \} . \tag{5.10}$$

This new information allows us to refine (5.7) (obviously for each $x_0 \in \Omega''' \subset \Omega''$ and for each $r \leq d_2/2$ where $d_2 = \text{dist}(\Omega''', \partial\Omega'')$); i.e.,

$$\begin{aligned} |w|^2_{1,2,B(r)} &\leq cr^{n+2} \{ \|G\|^2_{-1,\alpha,\Omega''} + \sum_{i=1}^{n} \|D_i u\|^2_{L^\infty(\Omega'')} \} \\ &\leq cr^{n+2} \{ \|f\|^2_{-1,\alpha,\Omega} + \|u\|^2_{H^1_{(n-\alpha)}(\Omega)} \} . \end{aligned} \tag{5.11}$$

Using (5.8) and (5.11), it is now possible to improve (5.9) and obtain (5.5). Also in this case, it is useful to estimate the derivatives of u on a half ball.

Lemma 5.4. *For each $\epsilon > 0$, let $u \in H^1_{(n-\epsilon)}(B^+(1), \mathbf{C}^N)$, $u = 0$ on $\Gamma(1)$, be a solution of the system*

$$u - \sum_{i,j=1}^{n} D_i(A_{ij} D_j u) - \sum_{i=1}^{n} D_i(A_i u) + \sum_{i=1}^{n} B_i D_i u + Cu = f \in C^{-1,\alpha}(B^+(1), \mathbf{C}^N) ,$$

where A_{ij} are elliptic, A_{ij} and $A_i \in C^{0,\alpha}(B^+(1), \mathbf{C}^N)$; B_i and $C \in L^\infty(B^+(1), \mathbf{C}^N)$. Then for each $t \in]0,1[$ and $r \in]0,1[$,

$$\begin{aligned} &\sum_{j=1}^{n-1} |D_j u|^2_{0,B^+(tr)} + |D_n u - (D_n u)_{B^+(tr)}|^2_{0,B^+(tr)} \\ &\qquad \leq ct^{n+2} \{ \|f\|^2_{-1,\alpha,B^+(1)} + \|u\|^2_{H^1_{(n-\alpha)}(B^+(1))} \} , \end{aligned} \tag{5.12}$$

where c is independent of λ.

Proof: For each $r \in]0,1[$, split u in $v + w$, where $w \in H^1_0(B^+(r), \mathbf{C}^N)$ and solves (5.6) in $B^+(1)$. As in the previous lemma, one obtains

$$|w|^2_{1,B^+(r)} \leq cr^{n+2} \{ \|G\|^2_{-1,\alpha,B^+(1)} + \|u\|^2_{H^1_{(n-\alpha)}(B^+(1))} \} . \tag{5.13}$$

If Theorem 3.6 is applied to v, one obtains

$$\begin{aligned} &\sum_{j=1}^{n-1} |D_j v|^2_{0,B^+(tr)} + |D_n v - (D_n v)_{B^+(tr)}|^2_{0,B^+(tr)} \\ &\qquad \leq ct^{n+2} \Big\{ \sum_{j=1}^{n-1} |D_j v|^2_{0,B^+(r)} + |D_n v - (D_n v)_{B^+(r)}|^2_{0,B^+(r)} \Big\} . \end{aligned} \tag{5.14}$$

From (5.12) and (5.13), choosing $\epsilon = 1 - \alpha/4$ and arguing in a standard way, a weaker version of (5.11) is obtained:

$$\begin{aligned}&\sum_{j=1}^{n-1} |D_j u|^2_{0,B^+(tr)} + |D_n u - (D_n u)_{B^+(tr)}|^2_{0,B^+(tr)} \\ &\quad \le c t^{n+\alpha}\{\|f\|_{-1,\alpha,B^+(1)} + \|u\|^2_{H^1_{(n-\alpha)}(B^+(1))}\}.\end{aligned} \tag{5.15}$$

Now the estimates (5.15) and (5.9) imply that for each $r \in]0,1[$, we have $u \in C^{1,\alpha/2}(B^+(r), \mathbf{C}^N)$; therefore, $Du \in L^\infty(B^+(r), \mathbf{C}^N)$ and

$$\sum_{j=1}^{n} |D_j u|_{L^\infty(B^+(r))} \le c(r)\{\|f\|_{-1,\alpha,B^+(1)} + \|u\|_{H^1_{(n-\alpha)}(B^+(1))}\}. \tag{5.16}$$

In view of this, we can argue as in the previous lemma, first we improve (5.13) and successively (5.15), thus obtaining (5.12).

Proof of Theorem 5.2. By the standard covering argument already used in §4, we reduce the problem into several problems in a half-ball and a problem in the interior of Ω. By Lemmas 5.3 and 5.4 and Theorem 4.1, we can deduce (see also Theorem 4.1) that

$$\sum_{i=1}^{n} \|D_i u\|_{0,\alpha,\bar\Omega} \le c\|f\|_{-1,\alpha,\Omega},$$

with c independent of λ. The estimate on u, i.e.,

$$(|\lambda| - \omega)\|u\|^2_{0,\alpha,\bar\Omega} \le c\|f\|^2_{-1,\alpha,\Omega}$$

follows by repeating the argument of Lemmas 4.7 and 4.8 and Proposition 4.9. At this point, the estimate on u in the topology of $C^{-1,\alpha}(\Omega, \mathbf{C}^N)$ follows directly from the equation and from the previous inequalities.

6. Interpolation and extrapolation spaces. The goal of this section is the characterization of some interpolation spaces for variational operators and extrapolation spaces for non variational ones. We begin with interpolation spaces between $C^{-1,\alpha}(\Omega)$ and the domain $C_0^{1,\alpha}(\bar\Omega)$ of the operator E defined in (1.2). We recall that if a linear operator E generates an analytic semigroup in a Banach space X, the interpolation space $D_E(\theta,\infty)$ $(0<\theta<1)$ is defined by

$$\begin{cases} D_E(\theta,\infty) = \{x \in X;\ [x]_{\theta,\infty} = \sup_{0<t\le 1} t^{-\theta}\|e^{tE}x - x\|_X < +\infty\} \\ \|x\|_{\theta,\infty} = \|x\|_X + [x]_{\theta,\infty} \end{cases}$$

and it coincides (algebraically and topologically) with interpolation space $(D(E),X)_{1-\theta,\infty}$ (see [12, 17]).

Theorem 6.1. *Let Ω be a bounded open set in $\mathbf{R}^N$ with $C^{3,\alpha}$ boundary. Let A_{ij}, A_i, B_i, and C satisfy the assumptions of Theorem 5.2, and let $E : C_0^{1,\alpha}(\bar{\Omega}) \to C^{-1,\alpha}(\Omega)$ be defined by (1.2). Then ,*

$$D_E(\theta,\infty) = \begin{cases} C^{-1,2\theta+\alpha}(\Omega), & 0<\theta<(1-\alpha)/2 \\ C_0^{0,2\theta+\alpha-1}(\bar{\Omega}), & (1-\alpha)/2<\theta<1-\alpha/2 \\ C_0^{1,2\theta+\alpha-2}(\bar{\Omega}), & 1-\alpha/2<\theta<1. \end{cases}$$

Proof: The Laplace operator $\Delta : C_0^{1,\alpha}(\bar{\Omega}) \to C^{-1,\alpha}(\Omega)$, $\Delta(u_1,\dots,u_N) = (\Delta u_1,\dots,\Delta u_N)$ satisfies, obviously, the assumptions of Theorem 5.2. Since

$$D_E(\theta,\infty) = (C_0^{1,\alpha}(\bar{\Omega}), C^{-1,\alpha}((\Omega))_{1-\theta,\infty} = D_\Delta(\theta,\infty),$$

we need only to characterize $D_\Delta(\theta,\infty)$. Let ψ belong to $D_\Delta(\theta,\infty)$. In particular, since ψ is in $C^{-1,\alpha}(\Omega)$ and $\Delta : C_0^{1,\alpha}(\bar{\Omega}) \to C^{-1,\alpha}(\Omega)$ is invertible, there is a unique $w \in C_0^{1,\alpha}(\bar{\Omega})$, such that $\psi = \Delta w$. Then,

$$\sup_{0<t\leq 1} t^{-\theta}\|e^{t\Delta}(\Delta w) - \Delta w\|_{-1,\alpha} = \sup_{0<t\leq 1} t^{-\theta}\|\Delta(e^{t\Delta}w - w)\|_{-1,\alpha} < +\infty.$$

This implies (see Lemma 2.5)

$$\sup_{0<t\leq 1} t^{-\theta}\|e^{t\tilde{\Delta}}w - w\|_{1,\alpha} < +\infty;$$

i.e., w belongs to $D_{\tilde{\Delta}}(\theta,\infty)$, where

$$\tilde{\Delta} : D(\tilde{\Delta}) = \{u \in C_0^{3,\alpha}(\bar{\Omega});\ \Delta u|_{\partial\Omega} = 0\} \to C_0^{1,\alpha}(\bar{\Omega}),\quad \tilde{\Delta}u = \Delta u.$$

By the characterization given in [13], it follows that $D_{\tilde{\Delta}}(\theta,\infty)$ coincides (algebraically and topologically) with $C_0^{1+\alpha+2\theta}(\bar{\Omega})$ (such that $\Delta u|_{\partial\Omega} = 0$ for $\theta > (1-\alpha)/2$) if $\alpha + 2\theta$ is not an integer (we have used the notation $C_0^{\beta}(\bar{\Omega}) = C_0^{[\beta],\beta-[\beta]}(\bar{\Omega})$, $[\beta]$ is the integral part of β). Hence, $\psi \in C^{\alpha+2\theta-1}(\bar{\Omega})$ $(\wedge C_0^0(\bar{\Omega})$ if $\theta > (1-\alpha)/2)$. In order to prove the other inclusion assume that $f \in C^{-1+\alpha+2\theta}(\bar{\Omega})$, then there exists $w \in C_0^{1+\alpha+2\beta}(\bar{\Omega})$ such that $\Delta w = f$. This implies

$$\sup_{0<t\leq 1} t^{-\theta}\|e^{t\tilde{\Delta}}w - w\|_{1+\alpha} < +\infty,$$

so that

$$\sup_{0<t\leq 1} t^{-\theta}\|\Delta(e^{t\Delta}w - w)\|_{-1+\alpha} = \sup_{0<t\leq 1} t^{-\theta}\|e^{t\Delta}f - f\|_{-1+\alpha} < +\infty.$$

Now, we characterize some interpolation spaces $D_{\tilde{E}}(\theta,\infty)$ for small $\theta > 0$, where the operator $\tilde{E}$ is so defined.

$$\tilde{E} : D(\tilde{E}) = \{u \in C_0^1(\bar{\Omega});\ Eu \in C_0^{0,\beta}(\bar{\Omega})\} \to C_0^{0,\beta}(\bar{\Omega})$$

$\tilde{E}u = Ew$ for each $w \in D(\tilde{E})$.

Theorem 6.2. *Assume that $\tilde{E}$ satisfies the regularity conditions (5.1)-(5.2) and the ellipticity condition (2.5); then for every $0 < \theta < (1 + \alpha - \beta)/2$,*

$$D_{\tilde{E}}(\theta, \infty) = \begin{cases} C_0^{0,\beta+2\theta}(\bar{\Omega}), & 0 < \theta < (1-\beta)/2 \\ C_0^{1,2\theta+\beta-1}(\bar{\Omega}), & \theta > (1-\beta)/2. \end{cases}$$

Proof: Recalling the notations of Theorem 6.1, note that

$$D(\tilde{E}) = D_E\big((3+\beta-\alpha)/2, \infty\big) \quad \text{and} \quad C_0^{0,\beta}(\bar{\Omega}) = D_E\big((1+\beta-\alpha)/2, \infty\big).$$

Therefore, this theorem is a direct consequence of Theorem 6.1 and Reiteration Theorem (see [17]).

Consider the operator $A : C_0^{2,\alpha}(\bar{\Omega}) \to C_0^{0,\alpha}(\bar{\Omega})$, where

$$A = -\sum_{i,j=1}^{n} A_{ij} D_{ij} + \sum_{i=1}^{n} B_i D_i + C$$

and A_{ij} are continuously differentiable in $\bar{\Omega}$ and B_i and C belong to $C^{0,\alpha}(\bar{\Omega})$. Moreover, assume that A_{ij} satisfy ellipticity condition (2.5). In this section, we characterize the extrapolation spaces $D_A(\theta - 1, \infty)$ for $(1-\alpha)/2 < \theta < 1$. Let us recall some properties of the extrapolation spaces (for more details see [11], where these spaces were introduced with a different approach; see also [19]). Let B be a Banach space with norm $|\cdot|$, and let Z be a closed linear operator with domain $D(Z) \subset B$, $D(Z)$ dense in B. $D(Z)$ is a Banach space with the graph norm $\|\cdot\|$. Let us assume that there exists $\lambda \in \mathbf{R}$, such that $Z + \lambda$ is invertible. On B we consider the following norm:

$$|||x||| = \|(Z+\lambda)^{-1}\|.$$

Let F_Z be the completion of the Banach space B with respect to the norm $|||\cdot|||$. For every $0 < \theta < 1$, define the extrapolation spaces $D_Z(\theta - 1, \infty) = (B, F_Z)_{1-\infty}$. If D_Z is not dense in B, assume that there exists a Banach space $C \supset B$, such that there exists a realization of Z in C with dense domain, i.e., a linear operator $Z_1 : D(Z_1) \to C$, $Z_1(x) = Z(x)$ for each $x \in D(Z)$. If $D(Z)$ is dense in C, consider the extrapolation space F_{Z_1} and the extrapolated operator $\bar{Z}_1 : C \to F_{Z_1}$ ($\bar{Z}_1 = Z_1$ in $D(Z)$, $\bar{Z}_1(x)$ defined by completion in C). Define $F_Z = (Z_1 + \lambda)(B)$, F_Z is a Banach space with the norm $|||x||| = |(Z_1+\lambda)^{-1}x|$. Define analogously, for each $0 < \theta < 1$, $D_Z(\theta - 1, \infty) = (B, F_Z)_{1-\theta,\infty}$.

Theorem 6.3.

$$D_A(\theta - 1, \infty) = \begin{cases} C_0^{0,2\theta+\alpha-2}(\bar{\Omega}), & 1 - \alpha/2 < \theta < 1 \\ C_0^{-1,2\theta+\alpha-1}(\Omega), & (1-\alpha)/2 < \theta < 1 - \alpha/2\,. \end{cases}$$

Proof: Let us consider $\tilde{A} : \{u \in C_0^{3,\beta} : Au = 0 \text{ on } \partial\Omega\} \to C_0^{1,\beta}(\bar{\Omega})$, $\tilde{A}u = Au$, $\beta \in]0, \alpha[$. It is obvious that $F_{\tilde{A}} = C^{-1,\beta}(\Omega)$ and therefore, the extrapolation spaces related to $\tilde{A}$ were characterized in Theorem 6.1. On the other hand, note that $C_0^{1,\beta}(\bar{\Omega}) = D_{\tilde{A}}((1+\beta-\alpha)/2, \infty)$ and therefore $C^{-1,\beta}(\Omega) = D_{\tilde{A}}((\beta-\alpha-1)/2, \infty)$. Hence, the statement follows by Reiteration Theorem.

7. Applications to parabolic equations. The results obtained in the last section can be applied to study the regularity of the solutions of the linear autonomous Cauchy problem:

$$\begin{cases} u_t(t,x) = Eu(t,x) + f(t,x) \\ u(0,x) = 0 \quad \text{for each } x \in \Omega, \quad u(t,x) = 0 \quad \text{for } t > 0, \ x \in \partial\Omega, \end{cases} \tag{7.1}$$

where

$$Ev = \sum_{i,j=1}^{n} D_i(A_{ij}(x)D_jv) + \sum_{i=1}^{n} D_i(A_i(x)v) + \sum_{i=1}^{n} B_i(x)D_iv + C(x)v\,.$$

Assume that A_{ij} satisfy the ellipticity condition (2.5), A_{ij}, $A_i \in C^{0,\alpha}(\bar{\Omega}, \mathbf{C}^N)$, B_i and C are bounded and measurable. Recalling Theorem 6.2, it is easy to prove

Theorem 7.1. ([16]). *Let $f \in C([0,T];\, C_0^{0,\alpha}(\bar{\Omega})) \cap B([0,T];\, C_0^{0,\beta}(\bar{\Omega}))$ for some $\beta \in]\alpha, 1[$. Then, problem (7.1) has a strict solution such that*

$$Eu,\ u_t \in C\big([0,T];\, C_0^{0,\alpha}(\bar{\Omega})\big) \cap B\big([0,T];\, C_0^{0,\beta}(\bar{\Omega})\big).$$

Let $f \in C^{0,\gamma}\big([0,T];\, C_0^{0,\alpha}(\bar{\Omega})\big)$ with $\gamma \neq 1/2$ and $f(0) \in C_0^{2\gamma}(\bar{\Omega}, \mathbf{C}^N)$, then

$$u_t,\ Eu \in C^{0,\gamma}\big([0,T];\, C_0^{0,\alpha}(\bar{\Omega})\big) \cap B\big([0,T];\, C_0^{\alpha+2\gamma}(\bar{\Omega})\big).$$

Remark 7.2. Results similar to the ones of Theorem 7.1 may be shown also for non autonomous systems using the abstract theory of [1] instead of [16].

So far, we have focused our attention to the linear problem. It is interesting to study also quasilinear and fully nonlinear variational parabolic equations. We give now an example of a quasilinear equation, but in a forthcoming paper the applications of the fundamental spaces $C^{-1,\alpha}$ to nonlinear equations will be analyzed in detail.

Theorem 7.3. ([14]). *Consider the problem*

$$\begin{cases} u_t(t,x) = \displaystyle\sum_{i,j=1}^{n} D_i\big(A_{ij}(x,u(t,x))D_ju(t,x)\big) + f(x,u(t,x),Du(t,x)) \\ \qquad\quad\ = A(x,u)u + f(x,u) \\ u(0,x) = U_0(x), \quad x \in \Omega, \ u(t,x) = 0 \quad \text{for } t > 0 \text{ and } x \in \partial\Omega\,, \end{cases} \tag{7.2}$$

where f is locally Lipschitz continuous. Assume that for each $U_0 \in C_0^{1,\alpha}(\bar{\Omega})$, $A(t,U_0)$ satisfies the ellipticity condition (2.5). Moreover, suppose also that A_{ij} is locally Lipschitz continuous with its first derivatives; then, for each $U_0 \in C_0^{1,\alpha}(\bar{\Omega})$, there exists $t_1 > 0$ such that problem (7.2) has a unique classical solution in $[0, t_1[\times\bar{\Omega}$.

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