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STABILITY AND OSCILLATION OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

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Abstract. We established necessary and sufficient conditions for the asymptotic stability of the trivial solution and sufficient conditions for the oscillation of all solutions of the first order neutral delay differential equation with piecewise constant argument

$$\frac{d}{dt}\left(y(t)+py(t-1)\right)+qy([t-1])=0, \quad t\geq 0,$$

where p and q are real numbers and $[\cdot]$ designates the greatest-integer function.

We also obtained sufficient conditions for the oscillation of all solutions of the second order neutral delay differential equation with piecewise constant argument

$$\frac{d^2}{dt^2}\left(y(t)+py(t-1)\right)+qy([t-1])=0, \quad t\geq 0$$

and proved that the trivial solution is not asymptotically stable.

1. Introduction. In this paper we study the oscillatory behavior of solutions and the stability of the trivial solution of the first and second order neutral delay differential equations with piecewise constant argument (NEPCA)

$$\frac{d}{dt}\left(y(t)+py(t-1)\right)+qy([t-1])=0, \quad t\geq 0 \tag{1}$$

and

$$\frac{d^2}{dt^2}\left(y(t)+py(t-1)\right)+qy([t-1])=0, \quad t\geq 0, \tag{2}$$

where p and q are real numbers and $[\cdot]$ designates the greatest-integer function.

First order equations with piecewise constant arguments, of non-neutral type have been the subject of many recent investigations originated by Cooke and Wiener [2] and Shah and Wiener [4]. See also [1] and the references cited therein. These equations may have applications in vertically transmitted diseases. They are also connected to difference equations as

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continuity of the solution at the endpoints of consecutive integral intervals implies recursion relations for the values of the solution at such points.

By a solution of Eq. (1), we mean a function y which is defined on $[-1,\infty)$ and satisfies the following three conditions:

(i) $y \in C[[-1,\infty), \mathbf{R}]$;

(ii) $\frac{d}{dt}(y(t)+py(t-1))$ exists on $[0,\infty)$ with the possible exception at the points $[t] \in [0,\infty)$ where one-sided derivatives exist;

(iii) Eq. (1) is satisfied on each interval $[n, n+1)$ with $n = 0, 1, 2, \dots$.

It follows by the method of steps that for every initial function

$$y_0 \in C[[-1,0], \mathbf{R}],$$

Eq. (1) has a unique solution $y(t)$ satisfying

$$y(t) = y_0(t), \quad -1 \le t \le 0.$$

By a solution of Eq. (2), we mean a function y which is defined on $[-1,\infty)$ and satisfies the following three conditions:

(i) $y \in C^1[[-1,\infty), \mathbf{R}]$;

(ii) $\frac{d^2}{dt^2}(y(t)+py(t-1))$ exists on $[0,\infty)$ with the possible exception at the points $[t] \in [0,\infty)$ where one-sided second derivatives exist;

(iii) Eq. (2) is satisfied on each interval $[n, n+1)$ with $n = 0, 1, 2, \dots$.

It follows by the method of steps that for every initial function $y_0 \in C^1[[-1,0], \mathbf{R}]$ and every real number B_0, Eq. (2) has a unique solution $y(t)$ satisfying

$$y(t) = y_0(t), \quad -1 \le t \le 0,$$

and

$$\frac{d}{dt}(y(t)+py(t-1)) = B_0 \quad \text{at} \quad t = 0.$$

As is customary, a solution of Eq. (1) or Eq. (2) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

2. First Order Equation.

Here, we will investigate the stability nature and the oscillatory behavior of the solutions of the NEPCA (1).

Let $y(t)$ be the unique solution of Eq. (1) which satisfies the initial condition

$$y(t) = y_0(t), \quad -1 \le t \le 0, \tag{3}$$

where $y_0 \in C[[-1,0], \mathbf{R}]$. Set

$$A_n = y(n), \quad n = -1, 0, 1, 2, \dots.$$

For each $n = 0, 1, 2, \dots$, Eq. (1) reduces to

$$\frac{d}{dt}(y(t)+py(t-1)) = -qA_{n-1}, \quad t \in [n, n+1). \tag{4}$$

By integrating Eq. (4) from n to t, we obtain

$$y(t) + py(t-1) = A_n + pA_{n-1} - qA_{n-1}(t-n)\,, \tag{5}$$

which, because of the continuity of the solution $y(t)$ at $t = n+1$ yields the second order difference equation

$$A_{n+1} + pA_n = A_n + pA_{n-1} - qA_{n-1}\,,$$

or

$$A_{n+2} + (p-1)A_{n+1} + (q-p)A_n = 0, \quad n = -1, 0, 1, \dots, \tag{6}$$

with initial values

$$A_{-1} = y_0(-1) \quad \text{and} \quad A_0 = y_0(0). \tag{7}$$

First, we will investigate the oscillation and the stability of the solutions of the general second order linear difference equation with constant coefficients of the form

$$A_{n+2} + a_1 A_{n+1} + a_2 A_n = 0, \quad n = -1, 0, 1, \dots, \tag{8}$$

where

$$a_1,\ a_2,\ A_{-1},\ A_0 \in \mathbf{R}.$$

The characteristic equation of Eq. (8) is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0. \tag{9}$$

The next lemma gives necessary and sufficient conditions for the oscillation of all solutions of Eq. (8). We say that a solution $\{A_n\}$ of a difference equation oscillates if and only if A_n is not eventually of fixed sign.

Lemma 1. *Every solution of Eq. (8) oscillates if and only if Eq. (9) has no positive real root.*

Proof: Assume that Eq. (9) has no positive real root and let λ_1 and λ_2 denote the two roots of Eq. (9).

When the roots λ_1 and λ_2 are real and equal the general solution of Eq. (8) is of the form

$$A_n = (c_1 + c_2 n)\lambda_1^n\,,$$

where c_1 and c_2 are arbitrary constants. Hence, eventually the sign of every nontrivial solution alternates.

When the roots λ_1 and λ_2 are real and distinct with, say, $\lambda_1 < \lambda_2$ the general solution of Eq. (8) is of the form

$$A_n = c_1\lambda_1^n + c_2\lambda_2^n = \lambda_1^n\left(c_1 + c_2(\frac{\lambda_2}{\lambda_1})^n\right)\,, \tag{10}$$

where c_1 and c_2 are arbitrary constants. When $c_1 = 0$, then clearly A_n oscillates. When $c_1 \neq 0$, then for every nontrivial solution, the bracket in (10) is of fixed sign and so A_n oscillates. Finally, when the roots λ_1 and λ_2 are not real, the general solution of Eq. (8) is of the form

$$A_n = r^n(c_1 \cos n\theta + c_2 \sin n\theta),$$

where c_1, c_2 are arbitrary constants, $r = |\lambda_1| = |\lambda_2| > 0$ and $\theta = \operatorname{Arg}\lambda_1 = -\operatorname{Arg}\lambda_2$ with

$$\theta \neq 0 \quad \text{and} \quad |\theta| < \pi.$$

For every nontrivial solution of Eq. (8), let $\theta_0 \in [0, 2\pi)$ be such that

$$\cos\theta_0 = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\theta_0 = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

Then

$$A_n = r^n \sqrt{c_1^2 + c_2^2}\,(\cos\theta_0 \cos n\theta + \sin\theta_0 \sin n\theta)$$

or

$$A_n = r^n \sqrt{c_1^2 + c_2^2}\,\cos(n\theta - \theta_0).$$

We will show that the sequence

$$\cos(n\theta - \theta_0), \quad n = 1, 2, \ldots,$$

oscillates. Assume, for the sake of contradiction, that

$$\cos(n\theta - \theta_0) > 0 \quad \text{for} \quad n \geq n_0 \tag{11}$$

for some $n_0 \in N$. The case where $\cos(n\theta - \theta_0) < 0$ is similar and it will be omitted. Set

$$\theta_n = n\theta - \theta_0, \quad n = 1, 2, \ldots,$$

and observe that

$$|\theta_{n+1} - \theta_n| = |\theta| < \pi, \quad n = 1, 2, \ldots. \tag{12}$$

As $\cos t > 0$ if and only if the arc t terminates in the first or fourth quadrant, it follows from (11) and (12) that for $n \geq n_0$ the arcs θ_n all lie in the first or in the fourth quadrant. This is clearly impossible and the proof that A_n oscillates is complete.

Conversely, if Eq. (9) has a positive real root λ_0, then Eq. (8) has the nonoscillatory solution λ_0^n. The proof is complete.

The following corollary gives conditions in terms of the coefficients a_1 and a_2 of Eq. (8) so that Eq. (9) has no positive real root.

Corollary 1. *Every solution of Eq. (8) oscillates if and only if one of the following three conditions is satisfied:*

$$a_2 = \left(\frac{a_1}{2}\right)^2 \quad \textit{and} \quad a_1 \geq 0;$$

$$0 \leq a_2 < \left(\frac{a_1}{2}\right)^2 \quad \textit{and} \quad a_1 > 0;$$

$$a_2 > \left(\frac{a_1}{2}\right)^2.$$

The following lemma gives necessary and sufficient conditions for the asymptotic stability of the trivial solution of Eq. (8). See [3, p. 371].

Lemma 2. *The trivial solution of Eq. (8) is asymptotically stable if and only if*

$$|a_1| - 1 < a_2 < 1. \tag{13}$$

Proof: The trivial solution of Eq. (8) is asymptotically stable if and only if each root λ of Eq. (9) lies in the unit disk $|\lambda| < 1$. Under the Möbius transformation

$$\lambda = \frac{z+1}{z-1},$$

the unit disk $|\lambda| < 1$ is mapped onto the negative half-plane $\operatorname{Re} z < 0$ and so each root λ of $P(\lambda) = 0$ lies in the unit disk $|\lambda| < 1$ if and only if each root z of $P((z+1)/(z-1)) = 0$, or equivalently, of

$$(1 + a_1 + a_2)z^2 + 2(1 - a_2)z + (1 - a_1 + a_2) = 0,$$

lies in the negative half-plane $\operatorname{Re} z < 0$. By using the Routh-Hurwitz criteria, this is equivalent to either

$$1 + a_1 + a_2 > 0, \;\; 1 - a_2 > 0, \;\; 1 - a_1 + a_2 > 0 \tag{14}$$

or

$$1 + a_1 + a_2 < 0, \;\; 1 - a_2 < 0, \;\; 1 - a_1 + a_2 < 0. \tag{15}$$

Conditions (14) are equivalent to (13), while conditions (15) are inconsistent. The proof is complete.

The following theorem is an application of Corollary 1 to Eq. (6).

Theorem 1. *Assume that one of the following three conditions is satisfied:*

(C_1) $4q = (p+1)^2$ *and* $p \geq 1$;
(C_2) $4p \leq 4q < (p+1)^2$ *and* $p > 1$;
(C_3) $4q > (p+1)^2$.

Then every solution of Eq. (1) oscillates.

The next lemma enables us to present explicitly the unique solution $y(t)$ of (1) and (3). The idea of the proof is based on the variation of constants formula for difference equations. See [5, pp. 173-174].

Lemma 3. *The solution to the first order linear difference equation*

$$y(t) + py(t-1) = z(t), \quad t \geq 0 \tag{16}$$

with initial function

$$y(t) = y_0(t), \quad -1 \leq t \leq 0,$$

where $y_0 \in C[[-1,0], \mathbf{R}]$ *and* $z \in C[[0,\infty), \mathbf{R}]$, *is a continuous function given by*

$$y(t) = (-p)^{n+1} y_0(\theta - 1) + \sum_{k=0}^{n} (-p)^{n-k} z(k+\theta), \quad t \geq 0, \tag{17}$$

where

$$t = n + \theta \quad \text{with} \quad 0 \leq \theta \leq 1 \quad \text{and} \quad n = 0, 1, 2, \ldots.$$

Proof: When $p = 0$, the proof is obvious. Also, in the case where $y_0(t_0) = 0$ for some $t_0 \in [-1, 0]$, by setting $\theta = t_0 + 1$ and using (17), we see that Eq. (16) is satisfied for all points t of the form $t = n + \theta$, $n = 0, 1, 2, \ldots$. In the remainder of the proof, assume that $p \neq 0$ and that $y_0(t) \neq 0$ for $-1 \leq t \leq 0$.

For any $t \geq -1$ set $t = n + \theta$ with $0 \leq \theta \leq 1$ and $n = -1, 0, 1, \ldots$. The solution to the corresponding homogenous problem; that is, when $z(t) \equiv 0$, is given by

$$y_h(t) = -py(t-1) = (-p)^{n+1} y(t-n-1) = (-p)^{n+1} y_0(\theta - 1), \quad t \geq -1. \tag{18}$$

For the general solution of the nonhomogeneous problem (16), we look for a solution of the form

$$y(t) = y_h(t)x(t), \quad t \geq -1. \tag{19}$$

From (19), for $t \in [-1, 0]$, we obtain

$$y_0(t) = y_0(t)x(t)$$

and so

$$x(t) = 1, \quad -1 \leq t \leq 0. \tag{20}$$

Also, for $t \geq 0$, (19) implies that

$$y_h(t)x(t) + py_h(t-1)x(t-1) = z(t), \tag{21}$$

while from (18), we have that

$$py_h(t-1) = -y_h(t). \tag{22}$$

From (21) and (22), we obtain

$$y_h(t)x(t) - y_h(t)x(t-1) = z(t)$$

or

$$x(t) - x(t-1) = \frac{z(t)}{y_h(t)}, \quad t \geq 0.$$

Summing up, we find

$$x(t) = x(\theta - 1) + \sum_{k=0}^{n} \frac{z(k+\theta)}{(-p)^{k+1} y_0(\theta - 1)}, \quad t \geq 0. \tag{23}$$

By using (18), (20) and (23) in (19), we obtain Eq. (17). The proof is complete.

By applying Lemma 3 to the difference equation (5), we obtain the following explicit formula for the solution $y(t)$ of Eq. (1) with initial function given by (3):

$$y(t) = (-p)^{n+1} y_0(\theta - 1) + \sum_{k=0}^{n} (-p)^{n-k} \left(A_k + (p - q\theta)A_{k-1}\right), \tag{24}$$

where

$$t = n + \theta \quad \text{with} \quad 0 \leq \theta \leq 1 \quad \text{and} \quad n = 0, 1, 2, \ldots.$$

and $A_n = y(k)$, $k = -1, 0, 1, \cdots$, is the solution of (6) and (7).

The following theorem describes the stability nature of the trivial solution of Eq. (1).

Theorem 2. *(i) Assume that the initial function $y_0(t)$, $-1 \le t \le 0$, is not linear. Then the solution of Eq. (1) tends to zero as $t \to \infty$, if and only if*

$(\mathbf{C}_4)$ $$|p| < 1 \text{ and } 0 < q < 1 + p.$$

(ii) Assume that the initial function $y_0(t)$, $-1 \le t \le 0$, is a linear function. Then the solution of Eq. (1) tends to zero as $t \to \infty$, if and only if

$(\mathbf{C}_5)$ $$|p-1| - 1 < q - p < 1.$$

Proof: From (24), we obtain

$$y(n) = (-p)^{n+1}\left(y_0(-1) + \sum_{k=0}^{n}(-p)^{-k-1}(A_k + pA_{k-1})\right),$$

$$y(n+1) = (-p)^{n+1}\left(y_0(0) + \sum_{k=0}^{n}(-p)^{-k-1}(A_k + pA_{k-1}) - \sum_{k=0}^{n}(-p)^{-k-1}qA_{k-1}\right),$$

$$y(n+\theta) = (-p)^{n+1}\left(y_0(\theta-1) + \sum_{k=0}^{n}(-p)^{-k-1}(A_k + pA_{k-1}) - \theta\sum_{k=0}^{n}(-p)^{-k-1}qA_{k-1}\right),$$

where $n = 0, 1, 2, \dots$, $0 \le \theta \le 1$ and $A_k = y(k)$, $k = -1, 0, 1, \dots$. By eliminating the summations, we find

$$y(n+\theta) = (-p)^{n+1}\left(y_0(\theta-1) - \theta y_0(0) + (\theta-1)y_0(-1)\right) + (1-\theta)y(n) + \theta y(n+1)$$

or

$$y(t) = (-p)^{n+1}\left(y_0(\theta-1) - \theta A_0 + (\theta-1)A_{-1}\right) + (1-\theta)A_n + \theta A_{n+1}, \tag{25}$$

where $t = n + \theta$ with $0 \le \theta \le 1$ and $n = 0, 1, 2, \dots$, $A_k = y(k)$, $k = -1, 0, 1, \dots$, is the solution of (6) and (7), and $y_0(t)$, $-1 \le t \le 0$ is the given initial function.

(i) Assume $(\mathbf{C}_4)$ holds. Then applying Lemma 2 to Eq. (6), we see that

$$y(n) = A_n \to 0 \quad \text{as} \quad n \to \infty \tag{26}$$

and so from (25),

$$y(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{27}$$

Conversely, if (27) holds, then (26) holds and so from Eq. (25), by using the hypothesis, we have that $|p| < 1$. The above condition, together with Lemma 2 applied to Eq. (6), implies that $(\mathbf{C}_4)$ holds.

(ii) Observe that the hypothesis reduces Eq. (25) to

$$y(t) = (1-\theta)A_n + \theta A_{n+1},$$

from which a necessary and sufficient condition for

$$y(t) \to 0 \quad \text{as} \quad t \to \infty,$$

is the asymptotic stability of the trivial solution of Eq. (6). By using Lemma 2, we obtain the desired result. The proof is complete.

In the special case where $p = 0$; that is, in the non-neutral case, our results agree with those in [1] and can be summarized as follows:

Corollary 2. *Consider the first order EPCA*

$$\dot{y}(t) + qy([t-1]) = 0, \quad t \geq 0. \tag{1'}$$

Then the following statements hold:

(i) Every solution of Eq. (1′) tends to zero, as $t \to \infty$, if and only if $0 < q < 1$.

(ii) Every solution of Eq. (1′) oscillates if and only if $q > 1/4$.

3. Second order equation. In this section, we will investigate the stability nature and the oscillatory behavior of the solutions of the NEPCA (2).

Let $y(t)$ be the unique solution of Eq. (2) which satisfies the initial conditions

$$y(t) = y_0(t), \quad -1 \leq t \leq 0, \tag{28}$$

and

$$\frac{d}{dt}\left(y(t) + py(t-1)\right) = B_0 \;\text{ at }\; t = 0, \tag{29}$$

where $y_0 \in C^1[[-1,0], \mathbf{R}]$ and $B_0 \in \mathbf{R}$. Set $A_n = y(n)$, $n = -1, 0, 1, \ldots$, and

$$B_n = \frac{d}{dt}\left(y(t) + py(t-1)\right) \;\text{ at }\; t = n, \;\; n = 0, 1, 2, \ldots,.$$

For each $n = 0, 1, 2, \ldots$, Eq. (2) reduces to

$$\frac{d^2}{dt^2}\left(y(t) + py(t-1)\right) = -qA_{n-1} \;\text{ for }\; t \in [n, n+1).$$

By integrating this equation from n to t, we obtain

$$\frac{d}{dt}\left(y(t) + py(t-1)\right) = B_n - qA_{n-1}(t-n). \tag{30}$$

A second integration from n to t yields

$$y(t) + py(t-1) = A_n + pA_{n-1} + B_n(t-n) - \frac{1}{2}qA_{n-1}(t-n)^2. \tag{31}$$

By using continuity at $t = n+1$, (30) and (31) yield

$$B_{n+1} = B_n - qA_{n-1}, \tag{32}$$

and

$$A_{n+1} + pA_n = A_n + pA_{n-1} + B_n - \frac{1}{2}qA_{n-1} \tag{33}$$

respectively.

By eliminating B_n and B_{n+1} in the system (32)-(33), we obtain the third order difference equation

$$A_{n+3} + (p-2)A_{n+2} + (1 - 2p + \frac{q}{2})A_{n+1} + (p + \frac{q}{2})A_n = 0, \tag{34}$$

where $n = -1, 0, 1, \ldots$, $A_{-1} = y_0(-1)$, $A_0 = y_0(0)$ and

$$A_1 = (1-p)y_0(0) + (p - \frac{q}{2})y_0(-1) + B_0\,. \tag{35}$$

We will first investigate the stability nature and the oscillatory behavior of solutions of the general third order linear difference equation with constants coefficients of the form

$$A_{n+3} + a_1A_{n+2} + a_2A_{n+1} + a_3A_n = 0, \quad n = -1, 0, 1, \ldots, \tag{36}$$

where a_1, a_2, a_3, A_{-1}, A_0, $A_1 \in \mathbf{R}$. The characteristic equation of Eq. (36) is

$$F(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{37}$$

The following lemma gives necessary and sufficient conditions for the asymptotic stability of the trivial solution of Eq. (36).

Lemma 4. *The trivial solution of Eq. (36) is asymptotically stable if and only if the following three conditions hold:*

$$\begin{aligned} &|a_1 + a_3| < a_2 + 1, \\ &|a_1 - 3a_3| < 3 - a_2, \\ &a_3^2 + a_2 - a_1 a_3 - 1 < 0. \end{aligned} \tag{38}$$

Proof: The trivial solution of Eq. (36) is asymptotically stable if and only if each root λ of Eq. (37) satisfies

$$|\lambda| < 1. \tag{39}$$

Under the Möbius transformation $\lambda = (z+1)/(z-1)$, condition (39) is equivalent to

$$\operatorname{Re} z < 0 \tag{40}$$

for each root z of the equation $F((z+1)/(z-1)) = 0$, or equivalently of

$$(a_1 + a_2 + a_3 + 1)z^3 + (3 + a_1 - a_2 - 3a_3)z^2 + (3 - a_1 - a_2 + 3a_3)z + (1 - a_1 + a_2 - a_3) = 0.$$

Using the Routh-Hurwitz criteria, condition (40) is equivalent to either

$$\begin{gathered} a_1 + a_2 + a_3 + 1 > 0, \quad -a_1 + a_2 - a_3 + 1 > 0, \\ 3 + a_1 - a_2 - 3a_3 > 0, \quad 3 - a_1 - a_2 + 3a_3 > 0, \quad a_3^2 + a_2 - a_1 a_3 - 1 < 0, \end{gathered} \tag{41}$$

or

$$\begin{gathered} a_1 + a_2 + a_3 + 1 < 0, \quad -a_1 + a_2 - a_3 + 1 < 0, \\ 3 + a_1 - a_2 - 3a_3 < 0, \quad 3 - a_1 - a_2 + 3a_3 < 0, \quad a_3^2 + a_2 - a_1 a_3 - 1 < 0. \end{gathered} \tag{42}$$

Now, it is easily seen that conditions (41) are equivalent to conditions (38), while conditions (42) are inconsistent. The proof is complete.

The next lemma gives necessary and sufficient conditions for the oscillation of all solutions of Eq. (36).

Lemma 5. *Every solution of Eq. (36) oscillates if and only if Eq. (37) has no positive real root.*

Proof: When Eq. (37) has a positive real root, then clearly, Eq. (36) has a nonoscillatory solution.

Now, assume Eq. (37) has no positive real root. Let λ_1, λ_2 and λ_3 denote the three roots (no necessarily distinct) of Eq. (37). When $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we obtain the trivial solution of Eq. (36) which oscillates. When all three roots are real, with say, $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$ and $\lambda_1 < 0$, then the general solution of Eq. (36) is of the form

$$A_n = \sum_{i=1}^{3} \mathbf{C}_i(n)\lambda_i^n = \lambda_1^n \sum_{i=1}^{3} \mathbf{C}_i(n)(\frac{\lambda_i}{\lambda_1})^n, \tag{43}$$

where $\mathbf{C}_i$, $i = 1, 2, 3$ are polynomials in n of degree at most two.

It is easily seen that the summation in the right-hand side of (43) is eventually of one sign and so A_n oscillates because λ_1^n does.

When λ_1 is real and λ_2, λ_3 are nonzero complex conjugates, the solution of Eq. (36) can be written in the form

$$A_n = c_1\lambda_1^n + r^n(c_2 \cos n\theta + c_3 \sin n\theta), \tag{44}$$

where c_1, c_2, c_3 are arbitrary real numbers, $\lambda_1 \leq 0$, $r = |\lambda_2| = |\lambda_3| > 0$ and $\theta = \operatorname{Arg}\lambda_2 = -\operatorname{Arg}\lambda_3$ with $|\theta| < \pi$ and $\theta \neq 0$. If $c_2 = c_3 = 0$, then clearly, Eq. (44) oscillates. When $|c_2| + |c_3| > 0$, we can rewrite Eq. (44), as in the proof of Lemma 1, in the form

$$A_n = c_1\lambda_1^n + r^n\sqrt{c_2^2 + c_3^2}\cos(n\theta - \theta_0), \tag{45}$$

where $\theta_0 \in [0, 2\pi)$ is such that

$$\cos\theta_0 = \frac{c_2}{\sqrt{c_2^2 + c_3^2}} \quad \text{and} \quad \sin\theta_0 = \frac{c_3}{\sqrt{c_2^2 + c_3^2}}.$$

When $c_1\lambda_1 = 0$ the proof that Eq. (45) oscillates follows as in the proof of Lemma 1. So assume $c_1\lambda_1 \neq 0$. We also assume, for the sake of contradiction, that A_n is eventually of one sign, say

$$A_n = c_1\lambda_1^n + r^n\sqrt{c_2^2 + c_3^2}\cos(n\theta - \theta_0) > 0, \quad n \geq n_0. \tag{46}$$

The case where $A_n < 0$ for $n \geq n_0$ is similar.

Case 1: Assume that $|\lambda_1| > r$. Then, (46) implies that

$$A_n = \lambda_1^n\left(c_1 + (\frac{r}{\lambda_1})^n\sqrt{c_2^2 + c_3^2}\cos(n\theta - \theta_0)\right) > 0, \quad n \geq n_0$$

which leads to a contradiction, since $\lambda_1 < 0$ and $(r/\lambda_1)^n \to 0$, as $n \to \infty$.

Case 2: Assume that $|\lambda_1| = r$. Then, (46) implies that

$$A_n = r^n\sqrt{c_2^2 + c_3^2}\left(\cos(n\theta - \theta_0) + (-1)^n\frac{c_1}{\sqrt{c_2^2 + c_3^2}}\right) > 0, \quad n \geq n_0,$$

which shows that

$$\cos(n\theta - \theta_0) > (-1)^n\frac{-c_1}{\sqrt{c_2^2 + c_3^2}}, \quad n \geq n_0. \tag{47}$$

When $|c_1| \geq \sqrt{c_2^2 + c_3^2}$, (47) leads to a contradiction, while when $|c_1| < \sqrt{c_2^2 + c_3^2}$ we have that there exists a $k \geq n_0$ such that $\cos(k\theta - \theta_0) > |c_1|/\sqrt{c_2^2 + c_3^2}$. Then,

$$2k_0\pi - \operatorname{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2 + c_3^2}} < k\theta - \theta_0 < 2k_0\pi + \operatorname{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2 + c_3^2}}, \tag{48}$$

for some $k_0 \in \mathbf{Z}$, which, since $|\theta| < \pi$, implies that

$$2k_0\pi - \operatorname{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2 + c_3^2}} - \pi < (k+1)\theta - \theta_0 < 2k_0\pi + \operatorname{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2 + c_3^2}} + \pi.$$

From this and the fact that $\cos((k+1)\theta - \theta_0) > -|c_1|/\sqrt{c_2^2 + c_3^2}$, we find

$$2k_0\pi - \operatorname{Cos}^{-1}\frac{-|c_1|}{\sqrt{c_2^2 + c_3^2}} < (k+1)\theta - \theta_0 < 2k_0\pi + \operatorname{Cos}^{-1}\frac{-|c_1|}{\sqrt{c_2^2 + c_3^2}}, \tag{49}$$

from which, since $|\theta| < \pi$, we obtain

$$2k_0\pi - \operatorname{Cos}^{-1}\frac{-|c_1|}{\sqrt{c_2^2 + c_3^2}} - \pi < (k+2)\theta - \theta_0 < 2k_0\pi + \operatorname{Cos}^{-1}\frac{-|c_1|}{\sqrt{c_2^2 + c_3^2}} + \pi. \tag{50}$$

But (50), together with $\cos((k+2)\theta - \theta_0) > |c_1|/\sqrt{c_2^2+c_3^2}$, implies that

$$2k_0\pi - \mathrm{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2+c_3^2}} < (k+2)\theta - \theta_0 < 2k_0\pi + \mathrm{Cos}^{-1}\frac{|c_1|}{\sqrt{c_2^2+c_3^2}}. \tag{51}$$

From (48), (49) and (51), it follows by induction that the sequence

$$\theta_n = n\theta - \theta_0, \quad n = k, k+1, k+2, \ldots,$$

is bounded and this contradiction completes the proof in this case.

Case 3: Assume that $|\lambda_1| < r$. Then (46) implies that

$$A_n = r^n\sqrt{c_2^2+c_3^2}\left(\cos(n\theta-\theta_0) + (\frac{\lambda_1}{r})^n\frac{c_1}{\sqrt{c_2^2+c_3^2}}\right) > 0, \quad n \geq n_0. \tag{52}$$

Set

$$\theta_n = n\theta - \theta_0, \quad b_n = -(\frac{\lambda_1}{r})^n\frac{c_1}{\sqrt{c_2^2+c_3^2}}, \quad n = 1, 2, \ldots,$$

and let, as before, Cos^{-1} denote the principal part of the inverse cosine function. As $\lim_{n\to\infty} b_n = 0$ and $|\theta| < \pi$, there exists an $n_1 \geq n_0$ such that

$$\mathrm{Cos}^{-1}|b_{n+1}| + \mathrm{Cos}^{-1}(-|b_{n+2}|) > \pi - (\pi - |\theta|) = |\theta|, \quad n \geq n_1. \tag{53}$$

Also, from (52), there exists a $k \geq n_1$ such that $\cos\theta_k > b_k > 0$ and $b_k < 1$. Then

$$2k_0\pi - \mathrm{Cos}^{-1}b_k < \theta_k < 2k_0\pi + \mathrm{Cos}^{-1}b_k, \tag{54}$$

for some $k_0 \in \mathbf{Z}$, which, since $|\theta| < \pi$, implies that

$$2k_0\pi - \mathrm{Cos}^{-1}b_k - \pi < \theta_{k+1} < 2k_0\pi + \mathrm{Cos}^{-1}b_k + \pi.$$

From this and the fact that $\cos\theta_{k+1} > b_{k+1} > -b_k$, $b_{k+1} < 0$ we find

$$2k_0\pi - \mathrm{Cos}^{-1}b_{k+1} < \theta_{k+1} < 2k_0\pi + \mathrm{Cos}^{-1}b_{k+1} \tag{55}$$

which, since $|\theta| < \pi$, implies that

$$2k_0\pi - \mathrm{Cos}^{-1}b_{k+1} - \pi < \theta_{k+2} < 2k_0\pi + \mathrm{Cos}^{-1}b_{k+1} + \pi. \tag{56}$$

But $\cos\theta_{k+2} > b_{k+2} > 0$ and $b_{k+2} < -b_{k+1}$ and so either

$$2k_0\pi - \mathrm{Cos}^{-1}b_{k+2} < \theta_{k+2} < 2k_0\pi + \mathrm{Cos}^{-1}b_{k+2} \tag{57}$$

or

$$|\theta| = |\theta_{k+2} - \theta_{k+1}| > \mathrm{Cos}^{-1}|b_{k+1}| + \mathrm{Cos}^{-1}(-|b_{k+2}|)$$

which contradicts (53). Thus, (57) holds and in fact, since θ_n is monotone, (55) can be replaced by

$$2k_0\pi - \mathrm{Cos}^{-1}b_{k+2} < \theta_{k+1} < 2k_0\pi + \mathrm{Cos}^{-1}b_{k+2}. \tag{58}$$

Now, from (54), (58) and (57) and the fact that $1 > b_k > b_{k+2} > 0$, we see that

$$|\theta_n - 2k_0\pi| < \frac{\pi}{2}, \quad n = k, k+1, k+2. \tag{59}$$

By induction, (59) can be shown to hold for all $n \geq k$, which contradicts the fact that $|\theta_n| = |n\theta - \theta_0| \to \infty$, as $n \to \infty$. The proof is complete.

The following result is an application of Lemma 4 to Eq. (34).

Theorem 3. *The trivial solution of Eq. (2) is not asymptotically stable.*

Proof: Applying Lemma 4 to Eq. (34), we obtain that the trivial solution of Eq. (34) is asymptotically stable if and only if

$$|2p-2+\frac{q}{2}|<2-2p+\frac{q}{2},$$
$$|-2p-2-\frac{3q}{2}|<2+2p-\frac{q}{2} \quad \text{and}$$
$$(p+\frac{q}{2})(3+\frac{q}{2})<3p.$$

But these three conditions are inconsistent and the proof is complete.

By using Lemma 5, we obtain the following sufficient conditions for all solutions of Eq. (2) to oscillate.

Theorem 4. *Assume that the equation*

$$f(\lambda)=\lambda^3+(p-2)\lambda^2+(1-2p+\frac{q}{2})\lambda+p+\frac{q}{2}=0 \tag{60}$$

has no positive real root. Then every solution of Eq. (2) oscillates.

Proof: Observe that Eq. (60) is the characteristic equation of Eq. (34). When Eq. (60) has no positive real root, then by Lemma 5, every solution of Eq. (34) and hence of Eq. (2) oscillates. The proof is complete.

The next corollary gives conditions in terms of the coefficients p and q of Eq. (60) so that Eq. (60) has no positive real root.

Corollary 3. *Assume that the following condition holds:*

$$2\le p\le \frac{q+2}{4}. \tag{$\mathbf{C}_6$}$$

Then every solution of Eq. (2) oscillates.

Proof: When ($\mathbf{C}_6$) holds, then all four coefficients of the third degree polynomial given in (60) are non-negative and so Eq. (60) has no positive real root. Then by Theorem 4 every solution of Eq. (2) oscillates. The proof is complete.

By applying Lemma 3 to the difference equation (31), we can obtain a formula for the unique solution of Eq. (2) with initial data given by (28) and (29), as follows:

Corollary 4. *The unique solution of Eq. (2) with initial data given by (28) and (29) can be written in the form*

$$y(t)=(-p)^{n+1}\Big(y_0(\theta-1)-\theta y_0(0)-(1-\theta)y_0(-1)\Big)+(1-\theta)A_n+\theta A_{n+1}$$
$$+\theta(1-\theta)\frac{q}{2}\sum_{k=0}^{n}(-p)^{n-k}A_{k-1} \tag{61}$$

where $t=n+\theta$ with $0\le\theta\le1$ and $n=0,1,2,\ldots$, and $A_n=y(n)$, $n=0,1,2,\ldots$, is the solution to the initial value problem given by (34) and (35).

Proof: By using (31) and (33), we obtain

$$y(t)+py(t-1)=A_n+pA_{n-1}+\Big(A_{n+1}+pA_n-A_n-pA_{n-1}+\frac{q}{2}A_{n-1}\Big)\theta-\frac{q}{2}A_{n-1}\theta^2. \tag{62}$$

By Lemma 3 the solution to the difference equation (62) is given by

$$y(t) = (-p)^{n+1}\{y_0(\theta-1) + \sum_{k=0}^{n}(-p)^{-k-1}\Big(A_k + pA_{k-1} + (A_{k+1} + pA_k - A_k - pA_{k-1} + \frac{q}{2}A_{k-1})\theta - \frac{q}{2}A_{k-1}\theta^2\Big)\}. \tag{63}$$

From (63) and for each $n = 0, 1, 2, \ldots$, we obtain

$$y(n) = (-p)^{n+1}y_0(-1) + \sum_{k=0}^{n}(-p)^{n-k}(A_k + pA_{k-1}), \tag{64}$$

and

$$y(n+1) = (-p)^{n+1}y_0(0) + \sum_{k=0}^{n}(-p)^{n-k}(A_{k+1} + pA_k). \tag{65}$$

Solving for the summations in (64) and (65), and substituting back to (63), we obtain Eq. (61). The proof is complete.

In the special case where $p = 0$; that is, in the non-neutral case, the solution of Eq. (2) given by (61) reduces to

$$y(t) = \frac{\theta(1-\theta)}{2}qA_{n-1} + (1-\theta)A_n + \theta A_{n+1} \tag{66}$$

and the following necessary and sufficient condition for oscillation holds:

Theorem 5. *Consider the second order EPCA*

$$y''(t) + qy([t-1]) = 0, \quad t \geq 0. \tag{2'}$$

Then every solution of Eq. (2′) oscillates if and only if $q > 0$.

Proof: Assume that $q > 0$. Then for $p = 0$, Eq. (60) takes the form

$$f(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + \frac{q}{2}(\lambda+1) = 0$$

or

$$f(\lambda) = \lambda(\lambda-1)^2 + \frac{q}{2}(\lambda+1) = 0 \tag{67}$$

which clearly has no positive real root. Thus, by Theorem 4, every solution of Eq. (2′) oscillates.

Conversely, assume that every solution of Eq. (2′) oscillates and for the sake of contradiction also assume that $q \leq 0$. Then

$$f(1) = q \leq 0 \quad \text{and} \quad f(\infty) > 0 \tag{68}$$

and so Eq. (67) has a positive real root λ_0. But then, by using Lemma 5, Eq. (34), with $p = 0$ has the nonoscillatory solution $A_n = \lambda_0^n$ for $n = -1, 0, 1, \ldots$. When $q = 0$, it follows from Eq. (66) (as well as directly from Eq. (2′)) that the solution $y(t)$ is nonoscillatory, which is a contradiction. When $q < 0$, observe that (68) implies that $\lambda_0 > 1$ and so Eq. (66) yields

$$\begin{aligned} y(t) &= \lambda_0^n \left(\theta(1-\theta)\frac{q}{2\lambda_0} + 1 - \theta + \lambda_0\theta \right) \\ &= \lambda_0^n \left(\theta(\theta-1)\frac{(\lambda_0-1)^2}{\lambda_0+1} + \theta(\lambda_0-1) + 1 \right) \\ &= \frac{\lambda_0^n}{\lambda_0+1}\left(\theta(\theta-1)(\lambda_0-1)^2 + \theta(\lambda_0^2-1) + \lambda_0 + 1 \right) \\ &= \frac{\lambda_0^n}{\lambda_0+1}\left((\lambda_0-1)^2\theta^2 + 2(\lambda_0-1)\theta + \lambda_0 + 1 \right) \\ &\geq \lambda_0^n > 0. \end{aligned}$$

This is a contradiction and the proof is complete.

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