Differential and Integral Equations, Volume 1, Number 4, October 1988, pp. 423-432.

SINGULAR SELF-ADJOINT STURM-LIOUVILLE PROBLEMS

A.M. KRALL† AND A. ZETTL‡

Mathematics and Computer Science Division, Argonne National Laboratory
9700 S. Cass Ave., Argonne, Illinois 60439 USA

(Submitted by : W.N. Everitt)

Abstract. Singular self-adjoint boundary conditions for Sturm-Liouville (S-L) problems are characterized. We believe this characterization is (i) new, (ii) simpler and more explicit than the well known characterization, and (iii) it is an exact parallel of the regular case.

1. Introduction. There are two fundamental classes of boundary value problems for the Sturm-Liouville (S-L) expression

$$My = \frac{1}{w}[-(py')' + qy] \quad \text{on} \quad I = (a, b), \quad -\infty \le a < b \le \infty; \tag{1.1}$$

i.e., regular and singular. In both cases, the boundary conditions required to obtain self-adjoint realizations of M are well known (and have been known for over a century). For details, see the book by Naimark [1968]. In the regular case, these conditions can be interpreted as linear combinations of the values of the function y and its quasi-derivative py' at the end points a and b. Such a representation is not possible at a singular end point c, say, because $y(c)$ and $(py')(c)$ do not exist even in a limiting sense, in general. The known characterization of the singular self-adjoint boundary conditions involves the sesquilinear form associated with M and elements of the maximal domain. In this paper, we show that the characterization of the singular self-adjoint boundary conditions is identical to that in the regular case provided that y and py' are replaced by certain Wronskians involving y and two linearly independent solutions of $My = 0$.

Notation and basic assumptions. The real valued Lebesgue measurable functions p, q and w are assumed to satisfy the following basic conditions:

$$p^{-1},\, q,\, w \in L_{\text{loc}}(I), \quad w(t) > 0 \text{ a.e.} \tag{1.2}$$

These conditions are assumed to hold throughout this paper. The local integrability conditions of (1.2) are necessary and sufficient for arbitrary initial value problems, at any point c in I of the equation $My = \lambda wy$, $\lambda \in C$, to have unique solutions.

Received November 10, 1987.
This work was partially supported by the Applied Mathematical Sciences subprogram of the office of Energy Research, U.S. Department of Energy, under contract W-31-109-Eng-38.
†Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802 USA.
‡Department of Mathematics, Northern Illinois University, DeKalb, Illinois 60115 USA.
AMS(MOS) Subject Classifications: 34B20, 34B25.

The end point a is regular if it is finite and

$$p^{-1},\ q,\ w \in L[a,\ a+\epsilon] \quad \text{for some } \epsilon > 0. \tag{1.3}$$

Similarly, the end point b is regular if (1.3) holds with the interval $[a,\ a+\epsilon]$ replaced by $[b-\epsilon,\ b)$. An end point is called singular if it is not regular. Thus, a ia singular if it is either infinite or finite and (1.3) fails to hold for one or more of p^{-1}, q, w. Note that a can be regular even when $p(a) = 0$; $p(x) = \sqrt{x}$ is regular at $a = 0$. Also, p, q, or w may fail to be bounded in the neighborhood of a regular point. An important distinction between a regular end point and a singular end point is the fact that at a regular end point c, all initial value problems $y(c) = \alpha$, $(py')(c) = \beta$, α, $\beta \in C$, have unique solutions. This is not true when c is singular (Everitt and Race, 1978).

For the convenience of the reader and for clarity of exposition, we state the characterization of regular self-adjoint two-point S-L boundary conditions

$$AY(a) + BY(b) = 0 \tag{1.4}$$

where $Y = (y,\ py')^t$, t for transpose, and $A = (a_{ij})$, $B = (b_{ij})$ are 2×2 matrices over C, the complex number field.

Theorem 1. *Assume both end points a and b are regular. Then the boundary value problem consisting of the equation*

$$-(py')' + qy = \lambda wy \tag{1.5}$$

with boundary conditions (1.4) is self-adjoint if and only if the following two conditions hold: (i) the two equations in (1.4) are linearly independent; i.e., the rank of the 2×4 matrix $(A : B) = 2$, and (ii)

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21} \tag{1.6}$$

$$a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} = b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12} \tag{1.7}$$

$$a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} = b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22}\,. \tag{1.8}$$

Proof: This can be found in any "good" book on differential operators (e.g. Naimark, 1968). Conditions (1.6), (1.7) and (1.8) can be stated more compactly using matrix notation as follows: $AJA^* = BJB^*$ with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $MJM^* = 0$ with $M = (A : B)$ and $J = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$, J as above.

Remark. Note that (1.7) and (1.8) hold whenever the matrices A and B are both real and (1.6), in this case, reduces to

$$\det A = \det B\,. \tag{1.9}$$

The special case $\det A = 0 = \det B$ of (1.9) contains the very popular separated boundary conditions

$$a_{11}y(a) + a_{12}(py')(a) = 0 \tag{1.10}$$

$$b_{21}y(b) + b_{22}(py')(b) = 0. \tag{1.11}$$

The special case (1.9) also contains the periodic ($A = I = -B$) and the anti-periodic ($A = I = B$) cases

$$y(a) = y(b), \qquad (py')(a) = (py')(b) \tag{1.12}$$

$$y(a) = -y(b), \quad (py')(a) = -(py')(b) \tag{1.13}$$

2. Singular boundary conditions. The boundary conditions, if any, required for (1.5) at a singular end point depend on the so-called limit-point (LP) or limit-circle (LC) classification of the end point.

Assume that a and b are singular end points. For any α, β in the open interval (a,b) and any $\lambda \in C$, the conditions (1.2) imply that any solution y of (1.5) is in $L^2_w(\alpha,\beta)$. However, such a y may or may not be in $L^2_w(a,b)$. If y is in $L^2_w(a,\beta)$ for some β in (a,b), then this is true for all β in (a,b). If for some β in (a,b) all solutions of (1.5) are in $L^2_w(a,\beta)$, then we say that M is in the limit-circle case at a, or simply that a is LC. Otherwise, M is in the limit-point case at a or a is LP. Similarly, b is LC means that all solutions of (1.5) are in $L^2_w(\alpha,b)$, $a < \alpha < b$. This classification is independent of λ in (1.5) (Naimark [1968]). Otherwise, b is LP. The limit-point, limit-circle terminology is used for historical reasons.

It is well known (Naimark [1968]) that no boundary condition is needed at a limit-point end point in order to get a self-adjoint realization of (1.5). On the other hand, a boundary condition is needed for each limit-circle end point. Note that we said "for" each LC end point rather than "at" each LC end point. If both end points are LC, then two conditions are needed. One or both of these may be linked together; i.e., it is, in general, not possible to give one condition at one end point and the second at the other end point. Just as in the regular case there are separated boundary conditions and there are nonseparated or linked ones.

If both end points are LP, then no boundary conditions are necessary; i.e., (1.5) is self-adjoint without any additional conditions. This means, as we will see below, that the minimal (maximal) operator associated with M in the space $L^2_w(I)$ is itself self-adjoint and has no proper self-adjoint extensions (restrictions).

To describe the singular self-adjoint boundary conditions if at least one end point is singular and LC, we need some more notation.

For any functions f, g which are absolutely continuous on all compact subintervals of I, let $W(f,g) = fpg' - gpf'$.

Let θ and ϕ denote solutions of $My = 0$, satisfying

$$W(\theta,\phi)(x) = 1 \quad \text{for all} \quad x \in I. \tag{2.1}$$

Clearly, such θ and ϕ exist; e.g., they can be determined by the initial conditions:

$$\theta(c) = 1, \ \ (p\theta')(c) = 0, \ \ \phi(c) = 0, \ \ (p\phi')(c) = 1 \ \text{ for } c \text{ in } I.$$

Theorem 2. *Assume both end points a and b are singular and limit-circle. Consider the boundary value problem consisting of the equation*

$$-(py')' + qy = \lambda wy \quad on \quad I = (a,b) \tag{1.5}$$

with the boundary conditions

$$AY(a) + BY(b) = 0, \tag{2.2}$$

where $A = (a_{ij})$, $B = (b_{ij})$ *are* 2×2 *matrices over* C *and*

$$Y = (W(y,\theta)(a), W(y,\phi)(b))^t \,, \tag{2.3}$$

where

$$W(y,\theta)(a) = \lim_{x\to a^+} W(y,\theta)(x), \quad W(y,\phi)(b) = \lim_{x\to b^-} W(y,\phi)(x) \tag{2.4}$$

(the limits in (2.3) both exist because y, θ and ϕ are all in the maximal domain – see below for the definition of maximal domain). The boundary value problem consisting of the equation (1.5) with the boundary conditions (2.2) is self-adjoint if and only if conditions (i) and (ii) of Theorem 1 hold.

Proof: This will be given in the next section.

To illustrate Theorem 2, we consider the classical Legendre equation

Example. $I = (-1,1)$, $p(x) = 1 - x^2$, $q(x) = 0$, $w(x) = 1$, $-1 < x < 1$. Both end points are singular and LC. In this case, θ and ϕ can be given explicitly:

$$\theta(x) = 1, \quad \phi(x) = -\frac{1}{2}\log(\frac{1-x}{1+x}), \quad -1 < x < 1\,. \tag{2.5}$$

Thus,

$$W(y,\theta) = yp\theta' - \theta py' = -py' \tag{2.6}$$
$$W(y,\phi) = yp\phi' - \phi py' = y - \phi(py')\,. \tag{2.7}$$

Each of the following boundary conditions is self-adjoint; i.e., determines a self-adjoint boundary problem for the classical Legendre equation

$$My = -(py')' = \lambda y \quad \text{on} \quad (-1,1)\,. \tag{2.8}$$

I. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This gives

$$\lim_{x\to -1^+} (py')(x) = 0 = \lim_{x\to 1^-} (py')(x). \tag{2.9}$$

The classical Legendre polynomials are the eigenfunctions of (2.8), (2.9). There are many equivalent formulations of the boundary conditions (2.9) (see Kaper, Kwong, and Zettl [1984]). Another well known condition, equivalent to (2.9), is the requirement that both

$$\lim_{x\to -1^+} y(x) \quad \text{and} \quad \lim_{x\to 1^-} y(x) \tag{2.10}$$

exist and are finite. Let $W(y,\theta)(a) = \lim_{x\to a^+} W(y,\theta)(x)$ and define $W(y,\theta)(b)$, $W(y,\phi)(a)$ and $W(y,\phi)(b)$ similarly.

II. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ yields

$$\lim_{x\to -1^+} (y - \phi py')(x) = 0 = \lim_{x\to 1^-} (y - \phi py')(x)\,. \tag{2.11}$$

III. I and II are special cases of the separated singular conditions

$$a_{11}W(y,\theta)(a) + a_{12}W(y,\phi)(a) = 0, \tag{2.12}$$

$$b_{21}W(y,\theta)(b) + b_{22}W(y,\phi)(b) = 0. \tag{2.13}$$

Here, a_{11}, a_{12}, b_{21}, b_{22} can be any real numbers as long as not both of a_{11}, a_{12} and not both of b_{21}, b_{22} are zero.

Simple examples of nonseparated singular boundary conditions are the analogues of the periodic and antiperiodic cases:

IV.

$$W(y,\theta)(a) = W(y,\theta)(b), \quad W(y,\phi)(a) = W(y,\phi)(b) \tag{2.14}$$

$$\lim_{x\to -1}(py')(x) = \lim_{x\to +1}(py')(x), \quad \lim_{x\to -1}(y - \phi py')(x) = \lim_{x\to +1}(y-\phi py')(x) \qquad (2.14)$$

V.

$$W(y,\theta)(a) = -W(y,\theta)(b), \quad W(y,\phi)(a) = -W(y,\phi)(b) \tag{2.15}$$

$$\lim_{x\to -1^+}(py')(x) = -\lim_{x\to +1^-}(py')(x), \quad \lim_{x\to -1^+}(y - \phi py')(x) = \lim_{x\to +1^-}(y-\phi py')(x)\,.$$

Theorem 3. *(a) Assume the left end point a is regular and the right end point b is singular and LC. Then all self-adjoint boundary conditions for the equation*

$$-(py')' + qy = \lambda wy \quad \text{on} \quad I = (a,b) \tag{1.5}$$

can be described as follows:

$$AY(a) + BY(b) = 0 \tag{2.16}$$

but where

$$Y(a) = (y, py')^t(a) \tag{2.17}$$

$$Y(b) = (W(y,\theta),\, W(y,\phi))^t\,(b) \tag{2.18}$$

and the matrices A, B satisfy the conditions (i) and (ii) of Theorem 1.

(b) If a is singular and LC and b is regular, then let

$$Y(a) = (W(y,\theta),\, W(y,\phi))^t\,(a)$$

$$Y(b) = (y,\, py')^t(b)$$

and the rest is the same as in Theorem 3(a).

Theorem 4. *Assume one end point is LP and the other is either regular or singular LC.*

(a) Suppose a is LP. Then the conclusion of Theorem 2 holds with $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$Y(b) = (y,\, py')^t(b) \quad \text{if } b \text{ is regular}$$

$$Y(b) = (W(y,\theta),\, W(y,\phi))^t\,(b) \quad \text{if } b \text{ is singular LC.}$$

(b) If b is LP and a is regular or singular LC, then the conclusion of Theorem 2 holds with $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$Y(a) = (y,\, py')^t(a) \quad \text{if } a \text{ is regular}$$

$$Y(a) = (W(y,\theta),\, W(y,\phi))^t\,(a) \quad \text{if } a \text{ is singular LC.}$$

3. Proofs and the bridge to the operator theoretic characterization. Let $H = L_w^2(I)$, let $y^{[1]} = py'$, the quasi-derivative of y. The maximal domain D is defined by

$$D = \left\{ y \in H : y,\, y^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(I) \text{ and } w^{-1}My \in H \right\}.$$

The maximal operator T is defined by

$$Ty = w^{-1}My, \quad y \in D. \tag{3.1}$$

It is well known (Naimark [1968]) that D is dense in H. Hence, it has a uniquely defined adjoint. Let

$$T_0 = T^* \quad \text{and} \quad D_0 = \operatorname{domain} T_0 .$$

The operator T_0 is called the minimal operator of M on I. Of critical importance to the description of boundary conditions is the sesquilinear form $[y, z]$, sometimes called bilinear concomitant, given by

$$[y, z] = y\overline{z^{[1]}} - \bar{z}y^{[1]}, \quad y,\, z \in D. \tag{3.2}$$

Observe that Green's formula holds:

$$\int_\alpha^\beta \left\{ M(y)\bar{z} - y\overline{M(z)} \right\} = [y, z](\beta) - [y, z](\alpha) = [y, z]_\alpha^\beta ; \quad y,\, z \in D, \quad \alpha,\, \beta \in I. \tag{3.3}$$

For any $y,\, z \in D$, $\lim_{\beta \to b^-} [y, z](\beta)$ and $\lim_{\alpha \to a^+} [y, z](\alpha)$ exist and are denoted by $[y, z](b)$ and $[y, z](a)$, respectively.

It is well known that T_0 is a closed symmetric (unbounded and not necessarily self-adjoint) operator in H and $T_0^* = T$ (see Naimark [1968]). Any self-adjoint extension of T_0 is a self-adjoint restriction of T and vice versa:

$$T_0 \subset S = S^* \subset T_0^* = T.$$

Thus, any self-adjoint extension of T_0 or self-adjoint restriction S of T is determined by its domain $D(S)$. We call such domains $D(S)$ self-adjoint domains.

For $\lambda \in C$, the set of complex numbers, let R_λ denote the range of $T_0 - \bar{\lambda}E$, E being the identity operator on H; $N_\lambda = R_\lambda^\perp$ and let

$$N^+ = N_i, \quad N^- = N_{-i}, \quad i = \sqrt{-1},$$

$d^+ =$ dimension of N^+, $d^- =$ dimension of N^-. The spaces N^+, N^- are called the deficiency spaces of T_0 and d^+, d^- are called the deficiency indices of T_0. These are related to the equation

$$My = -(py')' + qy = \lambda wy \quad \text{on} \quad I \tag{1.5}$$

as follows:

$$N_\lambda = \left\{ y \in H : T_0^* y = Ty = w^{-1}My = \lambda y \right\}.$$

Thus, N^+, N^- consist of the solutions of the equation (1.5) which lie in the space $H = L_w^2(I)$ for $\lambda = +i$ and $\lambda = -i$, respectively. Hence, d^+, d^- are the number of linearly independent solutions of (1.5) which are in the space H for $\lambda = +i$ and $\lambda = -i$, respectively. It is clear that

$$0 \le d^+ = d^- \le 2.$$

We denote the common value by d and call d the deficiency index of M on I.

A few basic facts needed later are summarized in

Proposition 1.

(a) $D_0 = \{y \in D : [y,z]_a^b = 0 \text{ for all } z \in D\}$.

(b) *If* $c = a$ *or* $c = b$ *is an LP end point, then* $[y,z](c) = 0$ *for all* y, z *in* D.

(c) *If an end point* c *is regular, then for any solution* y, y *and* $y^{[1]}$ *are continuous at* c.

(d) *If* a *and* b *are both regular, then for any* α, β, γ, δ *in* C *there exists a function* y *in* D *satisfying*

$$y(a) = \alpha,\ \ y^{[1]}(a) = \beta,\ \ y(b) = \gamma,\ \ y^{[1]}(b) = \delta.$$

(e) *If* a *is regular and* b *is singular, then a function* y *in* D *is in* D_0 *if and only if the following two conditions are satisfied:*

(i) $y(a) = 0$ *and* $y^{[1]}(a) = 0$.

(ii) $[y,z](b) = 0$ *for all* z *in* D

(similarly, for the case when a *is singular and* b *is regular).*

Proof: See Naimark [1968].

Next, we summarize the known characterization of the self-adjoint domains.

Proposition 2. *If the operator* S *with domain* $D(S)$, $D_0 \subset D(S) \subset D$ *is a self-adjoint extension of the minimal operator* T_0 *with deficiency index* d, *then there exist* $\Psi_1, \ldots, \Psi_d$ *in* $D(S) \subset D$ *satisfying the following conditions:*

(i) $\Psi_1, \ldots, \Psi_d$ *are linearly independent modulo* D_0;

(ii) $[\Psi_j,\ \Psi_k]_a^b = 0,\ \ j,\ k = 1, \ldots, d$;

(iii) $D(S)$ *consists of all* y *in* D *satisfying* $[y,\ \Psi_j]_a^b = 0$, $j = 1, \ldots, d$.

Conversely, given $\Psi, \ldots, \Psi_d$ *in* D *which satisfy (i) and (ii), the set* $D(S)$ *defined by (iii) is a self-adjoint domain.*

Proof: See Naimark [1968, Theorem 4, pp. 75-76].

Remark. When $d = 0$, conditions (i), (ii) and (iii) are vacuous. In this case, the minimal operator T_0 is itself self-adjoint and has no proper self-adjoint extensions. This case occurs only when both end points are LP. When $d > 0$, conditions (iii) are "boundary conditions" and (i) and (ii) are the conditions on the "boundary conditions" which determine self-adjoint domains.

For $f,\ g \in \mathrm{AC}_{\mathrm{loc}}(I)$ let

$$W(f,\ g) = fpg' - gpf'\,. \tag{3.4}$$

Choose solutions θ and ϕ of $My = 0$ satisfying

$$W(\theta, \phi)(x) = 1 \ \text{ for all } x \in I\,. \tag{3.5}$$

Lemma 1. (Fulton [1977], Littlejohn and Krall [1986]). *For any* y, z *in* D *we have*

$$\begin{aligned}[y,z] &= (W(\bar z,\theta),\, W(\bar z,\phi)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W(y,\theta) \\ W(y,\phi) \end{pmatrix} \\ &= \overline{W}(z,\phi)W(y,\theta) - \overline{W}(z,\theta)W(y,\phi) \\ &= \det \begin{bmatrix} W(y,\theta),\, W(y,\phi) \\ W(\bar z,\theta),\, W(\bar z,\phi) \end{bmatrix}. \end{aligned}$$

Proof: From (3.4) and (3.5), we get

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta & \phi \\ p\theta' & p\phi' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta & p\theta' \\ \phi & p\phi' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

Note that

$$[y,z] = (\bar{z}, p\bar{z}') \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ py' \end{pmatrix}.$$

Now, replace $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by (3.6) and simplify.

Lemma 2. *Given α, β, γ, δ in C, there exists a $\Psi \in D \backslash D_0$ such that*

$$\alpha = W(\Psi,\theta)(a), \quad \beta = W(\Psi,\phi)(a), \quad \gamma = W(\Psi,\theta)(b), \quad \delta = W(\Psi,\phi)(b). \tag{3.7}$$

Furthermore, Ψ can be taken to be a linear combination of θ and ϕ near each end point. These linear combinations may be different at different end points.

Proof: First, we establish the special case $\gamma = 0 = \delta$. Choose c, d, such that $a < c < d < b$. Set $\Psi_1 = \beta\theta - \alpha\phi$ on $(a, c]$. Then,

$$W(\Psi_1,\theta) = \beta W(\theta,\theta) - \alpha W(\phi,\theta) = \alpha \quad \text{on } (a,c] \text{ and hence also at } x = a.$$
$$W(\Psi_1,\phi) = \beta W(\theta,\phi) - \alpha W(\phi,\phi) = \beta \quad \text{on } (a,c] \text{ and hence also at } x = a.$$

Now, continue ϕ_1 from c to d such that Ψ_1 and $\Psi_1^{[1]}$ are absolutely continuous on $[c,d]$ and $\Psi_1(d) = 0 = \Psi_1^{[1]}(d)$. Then, set $\Psi_1(x) = 0$ for $d < x < b$. With $\Psi = \Psi_1$ we have that (1) holds when $\gamma = 0 = \delta$.

Similarly, we construct Ψ_2 such that (3.7) holds for $\Psi = \Psi_2$ when $\alpha = 0 = \beta$. Setting $\Psi = \Psi_1 + \Psi_2$ we have that (1) holds. From the construction it is clear that Ψ_1 and Ψ_2 and hence Ψ are in D. Note that $[y,\Psi] = [y,\Psi_1] + [y,\Psi_2]$. Now, $[y,\Psi_1](a) = \bar{\beta}[y,\theta](a) - \bar{\alpha}[y,\phi](a) \neq 0$ when y is either θ or ϕ unless both of α and β are zero. Hence, by Theorem 1, $\Psi_1 \in D \backslash D_0$. Similarly, it follows that $[y,\Psi_2](b) \neq 0$ for all y in D, showing that $\Psi_2 \in D \backslash D_0$. Hence, $\Psi \in D \backslash D_0$.

Below we show how Theorems 2, 3 and 4 follow from Proposition 2 and Lemmas 1 and 2. The cases $d = 0, 1, 2$ are considered separately.

Case 1. $d = 0$. In this case, both end points are LP and the minimal operator T_0 is itself self-adjoint and has no proper self-adjoint extensions.

Case 2. $d = 1$. In this case, one end point must be LP and the other either regular or LC.

2(a). Assume a is LP and b is regular. In this case, (iii) becomes

$$[y,\Psi]_a^b = [y,\Psi](b) = y(b)(p\Psi')(b) - \Psi(b)(py')(b) = 0. \tag{3.8}$$

If b is regular, then $\Psi(b)$ and $\Psi^{[1]}(b)$ can take on arbitrary values and so (3.8) can be rewritten as

$$b_{11}y(b) + b_{12}y^{[1]}(b) = 0. \tag{3.9}$$

From (i), we have that not both b_{11} and b_{12} can be zero since this would imply, by Proposition 1e, that $\Psi \in D_0$. Condition (ii) becomes

$$b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12} = 0. \tag{3.10}$$

Since b_{11} can be taken to be real, (3.9) just means that both b_{11} and b_{12} must be real. To summarize, we can say that if a is LP and b is regular, then the self-adjoint "boundary conditions" are all of the form (3.9) with b_{11} and b_{12} real and not both zero.

Similarly, if a is regular and b is LP, then the self-adjoint "boundary conditions" are all of the form

$$a_{11}y(a) + a_{12}y^{[1]}(a) = 0$$

with a_{11} and a_{12} real and not both zero.

(2b) Assume a is LP and b is LC. Using Lemma 1 and Proposition 1, we can express (iii) as

$$[y, \Psi]_a^b = [y, \Psi](b) = \left[W(\overline{\Psi}, \phi)W(y, \theta) - W(\overline{\Psi}, \theta)W(y, \phi)\right](b) = 0. \tag{3.11}$$

Set

$$b_{11} = W(\overline{\Psi}, \phi)(b), \ \ b_{12} = -W(\overline{\Psi}, \theta)(b). \tag{3.12}$$

Note that for fixed θ and ϕ a given $\Psi \in D$ determines b_{11} and b_{12} by (3.12). Conversely, by Lemma 2, given b_{11}, b_{12} in C, there exists a $\Psi \in D$ such that (3.12) holds. Thus, the "boundary condition" (iii) can be expressed as

$$b_{11}W(y, \theta)(b) + b_{12}W(y, \phi)(b) = 0. \tag{3.13}$$

Again, by (i), b_{11} and b_{12} cannot both be zero.

With the identification (3.12), condition (ii) again becomes (3.10) and reduces to requiring both b_{11} and b_{12} to be real.

In summary, we can say that if the end point a is LP and b is LC, then all self-adjoint domains are determined by "boundary conditions" of the form

$$b_{11}W(y, \theta)(b) + b_{12}W(y, \phi)(b) = 0$$

where b_{11} and b_{12} are real and not both zero.

Remark 1 Assume a is LP. Comparing (3.13) with (3.9), note that when $y(b)$ is replaced by $W(y, \theta)(b)$ and $y^{[1]}(b)$ is replaced by $W(y, \phi)(b)$, then the singular case when the end point b LC is an exact parallel of the case when b is regular.

Similarly, when a is LC and b is LP, all self-adjoint domains are determined by the "boundary conditions"

$$a_{11}W(y, \theta)(a) + a_{12}W(y, \phi)(a) = 0$$

where a_{11}, a_{12} are real and not both are zero.

Remark 2 If b is regular, then

$$W(y, \theta)(b) = y(b)(p\theta')(b) - \theta(b)(py')(b) = y(b)$$
$$W(y, \phi)(b) = y(b)(p\phi')(b) - \phi(b)(py')(b) = y^{[1]}(b)$$

if θ and ϕ are determined by the initial conditions $\phi(b) = -1$, $\phi^{[1]}(b) = 0$, $\theta(b) = 0$, $\theta^{[1]}(b) = 1$.

Thus, the case b regular is subsumed ("reduces" to ?) the singular case when b is LC. Note that when b is singular LP or LC

$$W(y, z)(b) = \lim_{x \to b}[ypz' - zpy'](x)$$

exists for any y, $z \in D$ but the separate terms ypz' and zpy' may not (and generally do not) have finite limits at b.

Case 3. $d = 2$. In this case, each end point is either regular or LC. Setting

$$\begin{aligned} a_{11} &= -W(\overline{\Psi}_1, \phi)(a), \ a_{12} = W(\overline{\Psi}_1, \theta)(a), \ b_{11} = W(\overline{\Psi}_1, \phi)(b), \ b_{12} = -W(\overline{\Psi}_1, \theta)(b) \\ a_{21} &= -W(\overline{\Psi}_2, \phi)(a), \ a_{22} = W(\overline{\Psi}_2, \theta)(a), \ b_{21} = W(\overline{\Psi}_2, \phi)(b), \ b_{22} = -W(\overline{\Psi}_2, \theta)(b) \end{aligned} \tag{3.14}$$

and proceeding as in case 2 above, we find that condition (iii) is equivalent to the equations

$$\begin{aligned} a_{11}W(y,\theta)(a) + a_{12}W(y,\phi)(a) + b_{11}W(y,\theta)(b) + b_{12}W(y,\phi)(b) = 0 \\ a_{21}W(y,\theta)(a) + a_{22}W(y,\phi)(a) + b_{21}W(y,\theta)(b) + b_{22}W(y,\phi)(b) = 0. \end{aligned} \tag{3.15}$$

Condition (i) is equivalent to the linear independence of the two equations (3.15), and (ii) reduces to the following three conditions:

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21} \tag{3.16}$$

$$a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} = b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12} \tag{3.17}$$

$$a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} = b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22}. \tag{3.18}$$

Remark. Note that (3.17) and (3.18) hold whenver the matrices $A = (a_{ij})$, $B = (b_{ij})$, $i, j = 1, 2$ are both real; in particular, whenever Ψ_1 and Ψ_2 are real, and (3.16), in this case, reduces to

$$\det A = \det \begin{bmatrix} W(\Psi_1, \Psi) & W(\Psi_1, \theta) \\ W(\Psi_2, \phi) & W(\Psi_2, \theta) \end{bmatrix} (a) = \det \begin{bmatrix} W(\Psi_1, \phi) & W(\Psi_1, \theta) \\ W(\Psi_2, \phi) & W(\Psi_2, \theta) \end{bmatrix} (b) = \det B. \tag{3.19}$$

The special case $\det A = 0 = \det B$ of (3.19) contains the separated singular boundary conditions case

$$\begin{aligned} a_{11}W(y,\theta)(a) + a_{12}W(y,\phi)(a) = 0, \\ b_{21}W(y,\theta)(b) + b_{22}W(y,\phi)(b) = 0. \end{aligned} \tag{3.20}$$

The basic conditions (1.2) guarantee that there are no singularities in the interior of the interval (a, b). We plan to study interior singularities in a subsequent paper.

REFERENCES

[1] W.N. Everitt & D. Race, *On necessary and sufficient conditions for the existence of caratheodory type solutions of ordinary differential equations*, Questiones Mathematicae 2 (1978), 507-512.

[2] C.T. Fulton, *Parametrizations of Titchmarsh's $m(\lambda)$-function in the limit circle case*, Trans. Amer. Math. Soc., 229 (1977), 51-63.

[3] H.G. Kaper, M.K. Kwong & A. Zettl, *Regularizing transformations for certain singular Sturm-Liouvile boundary value problems*, SIAM J. Math. Anal. 15 (1984), 957-963.

[4] L.L. Littlejohn & A.M. Krall, *Orthogonal polynomials and singular Sturm-Liouville systems*, Rocky Mountain J. Math., 16 (1986), 435-479.

[5] M.A. Naimark, *Linear Differential Operators: II*, New York, Ungar, 1968.