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TANGENT DIRECTIONS FOR A CLASS OF NONLINEAR EVOLUTION EQUATIONS

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Abstract. A class of nonlinear evolution equations involving locally Lipschitzian term is discussed. By using a global linearization iterative method, the existence of tangent directions to the set determined by nonlinear evolution equations is proved in two cases: without and with the control function involved. This result is then applied to the problem of optimal control with nonlinear evolution equation as constraint.

Introduction. The problem of tangent directions to the sets determined by some equation is a very important problem in optimization and optimal control. In the famous results obtained by Dubovitskii, Milyutin, Girsanov (cf. [6]), Joffe, Tikhominov (cf. [7]) and others, considering only one equality constraint in the form of ordinary differential equation, the proofs of the existence of tangent directions are based on the Lusternik theorem. The Lusternik theorem is also applied to optimization and optimal control problems with more than one equality constraint as in [8], [9], [10], [11], [12].

In [1], some generalization of the Lusternik theorem is proved by using the method of contractor directions under essentially weaker assumptions about differentiability than the Lusternik theorem requires. This generalization is applied to the problems of optimization and optimal control in [1], [13], [14] and [15].

However, none of these results (i.e., Lusternik theorem and its generalization) are applicable to the problems of optimal control with nonlinear evolution equation as constraint because of too strong assumptions that these results require.

In [3] and [5] the global linearization iterative method from [2] is applied to prove the existence of tangent directions for nonlinear evolution equation and quasilinear evolution equation, respectively. An application to the optimal control problem of quasilinear evolution equation as constraint is also considered.

In this paper, the results from [3] are extended to a more general class of nonlinear evolution equation with locally Lipschitzian term and the existence of tangent directions to the set determined by this equation is proved. Next, as an application, the local extremum principle for the optimal control problem with these general nonlinear evolution equation as constraint is proved.

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1. Existence of solution of nonlinear evolution equation. The existence of solution of nonlinear evolution equation with Lipschitzian term is proved in [2]. However application of these theorems to the case of existence of tangent direction necessitate formulating them in a bit different form.

Let $Z \subset Y \subset X$ be Banach spaces with norms $\|\cdot\|_Z \geq \|\cdot\|_Y \geq \|\cdot\|_X$.

A.0) We assume that there exist positive constants C, $\bar{s}$ with $0 < \bar{s} < 1$ such that

$$\|x\|_Y \leq C\|x\|_X^{1-\bar{s}}\|x\|_Z^{\bar{s}}. \tag{1.1}$$

Given $0 < b$, denote by $C(0,b;X)$ the Banach space of all continuous functions $x = x(t)$ defined on the interval $[0,b]$ with values in X and the norm

$$\|x\|_{\infty,X} = \sup_t [\|x(t)\|_X : 0 \leq t \leq b] . \tag{1.2}$$

In the same way, the norms $\|y\|_{\infty,Y}$ and $\|z\|_{\infty,Z}$ are defined for Y and Z.

Let $C'(o,b,X)$ stand for the vector space of all continuously differentiable functions from $[0,b]$ to X. Let the function

$$x_0 \in C(0,b,Z) \cap C'(0,b,X). \tag{1.3}$$

Let W_0 be an open ball in Y with center $x_0(0)$ and radius $r_0 > 0$. Put $V_0 = W_0 \cap Z$ and let V_1 be closure of V_0 in Y. Let $F : [0,b] \times V_1 \to X$ be a nonlinear mapping $f : [0,b] \times V_1 \to X$ be a nonlinear mapping locally Lipschitzian with respect to x.

Consider the following Cauchy problem:

$$\frac{dx}{dt} + F(t,x) + f(t,x) = 0, \quad 0 \leq t \leq b, \tag{1.4}$$

$$x(0) = \xi_0 , \tag{1.5}$$

where $\xi_0 = x_0(0)$. Let G be the set of functions x in $C(0,b;V_0(\|\cdot\|_Z)) \cap C'(0,b;X)$ with $x(0) = \xi_0 \in Z$ and $\|x - x_0\|_{\infty,Y} < r_0$. We assume that the mapping F is differentiable in the following sense:

For each $(t,x) \in [0,b] \times G$ there exists a linear operator $F'(t,x)$ such that

$$\epsilon^{-1}\|F(\cdot,x+\epsilon h) - F(\cdot,x) - \epsilon F'(\cdot,x)h\|_{\infty,X} \to 0$$

as $\epsilon \to 0^+$, where $h \in C(0,b;Z) \cap C'(0,b;X)$.

The operator f is differentiable at the point x_0 in the following sense:

For each $t \in [0,b]$ there exists a linear operator $f'(t,x_0)$ such that

$$\epsilon^{-1}\|f(\cdot,x_0+\epsilon h) - f(\cdot,x_0) - \epsilon f'(\cdot,x_0)h\|_{\infty,X} \to 0$$

as $\epsilon \to 0^+$ where $h \in C(0,b;Z) \cap C'(0,b;X)$.

Let us introduce a nonlinear mapping in the form

$$Px(t) = \frac{dx}{dt} + F(t,x) + f(t,x). \tag{1.6}$$

In order to prove the existence of the solution for (1.4), (1.5) we need to make the following assumptions:

A.1) The functions F, f are continuous in the following sense:

$$\|x_n - x\|_{\infty,Y} \to 0 \quad \text{implies} \quad \|F(\cdot, x_n) - F(\cdot, x)\|_{\infty,X} \to 0$$

as $n \to \infty$ and

$$\|x_n - x\|_{\infty,Y} \to 0 \quad \text{implies} \quad \|f(\cdot, x_n) - f(\cdot, x)\|_{\infty,X} \to 0.$$

Let $\{x_n\} \subset G$ be a Cauchy sequence in $C(0,b;Y)$ and let $\{h_n\} \subset C(0,b;Y) \cap C'(0,b;X)$ be bounded in $C(0,b;Y)$. Then, $\epsilon_n \to 0$ implies

$$\epsilon_n^{-1} \|F(\cdot, x_n + \epsilon_n h_n) - F(\cdot, x_n) - \epsilon_n F'(\cdot, x_n) h_n\|_{\infty,X} \to 0$$

as $n \to \infty$. There exists a constant q_0 such that

$$\|f(\cdot, x + \epsilon h) - f(\cdot, x)\|_{\infty,X} \le q_0 \epsilon \|h\|_{\infty,X} .$$

A.2) There exists a constant $C_0 > 0$ with the following property. For $x \in G$ any function g such that $\|g\|_{\infty,X} < \infty$ if h is a solution of the equation

$$\frac{dh}{dt} + F'(t,x)h + g = 0, \quad 0 \le t < b, \quad h(0) = 0,$$

then

$$\|h\|_{\infty,X} \le bC_0 \|g\|_{\infty,X} .$$

A.3) For $x \in G$ the linearized equation

$$\frac{dz}{dt} + F'(t,x)z + F(t,x) - F'(t,x)x + f(t,x) = 0, \tag{1.7}$$

$0 \le t \le b$, $z(0) = 0$, admits smooth approximate solutions of order (μ, ν, σ) with $0 \le \mu < 1$ in the sense of the following definition.

Definition 1.1. (Altman [2]) *Let $\mu > 0$, $\nu > 0$, $\sigma > 0$ be given numbers. Then the linearized equation (1.7) admits smooth approximate solutions of order (μ, ν, σ) if there exists a constant $M > 0$ which has the following property. For every $x \in G$, $K > 1$ and $Q > 1$, if $\|x\|_{\infty,Z} < K$ then there exists a residual (error) function y and a function z such that*

$$\begin{aligned} \|z\|_{\infty,Z} &\le MQK^{\nu} , \\ \|y\|_{\infty,X} &\le MQ^{-\mu}K^{\sigma} , \end{aligned} \tag{1.8}$$

and

$$\frac{dz}{dt} + F'(t,x)z + F(t,x) - F'(t,x)x + f(t,x) + y = 0, \; 0 \le t \le b, \; z(0) = \xi_0 . \tag{1.9}$$

Let us consider $x \in G$ and z be a solution of the equation (1.9) and put $z = x + h$. Then h is a solution of the equation

$$\frac{dh}{dt} + F'(t,x)h + Px + y = 0, \;\; 0 \le t \le b, \;\; h(0) = 0 ,$$

where P is defined by (1.6). Under these assumptions we can construct an iterative method of contractor directions in a similar way as in [2]. The difference will lie only in the initial point of iterative process. In our process it will be a function $x_0(t)$ introduced by (1.3) (in [2] $x_0(t) \equiv x_0$, where x_0 is a fixed point of the space X). Next, let us assume that x_0 is given by (1.3) and $x_1, x_2, \dots, x_n \in G$ are known and satisfy the following induction assumptions for all indices $i \le n$

$$\|x_i\|_{\infty,Z} < A\exp(\alpha(1-q)t_i) = K_i,$$

and

$$\|Px_i\|_{\infty,X} \le \|Px_0\|_{\infty,X}\exp(-(1-q)t_i),$$

where $\alpha > 1$ and A are subject to condition

$$\alpha(1-q)-1>0, \quad \alpha > [\mu(1-\nu)-\sigma]^{-1}, \quad \mu(1-\nu)-\sigma>0, \tag{1.10}$$

and

$$M(2M)^{1/\mu}(\bar{q}P_0)^{-1/\mu} < A^{1-\nu-\sigma/\mu}[\alpha(1-q)-1], \tag{1.11}$$

with $P_0 = \|Px_0\|_{\infty,X}$. The above induction assumptions are true for all n in view of Lemma 1.2 from [2].

Let z_n be a solution of equation (1.9) with $x = x_n$, $y = y_n$. Let $Q = Q_n$ be such that

$$2MQ^{-\mu}K_n^{\sigma} < \bar{q}P_0\exp(-(1-q)t_n).$$

We put $z_n = x_n + h_n$ such that h_n is a solution of equation

$$\frac{dh_n}{dt} + F'(t,x_n)h_n + Px_n + y_n = 0, \quad 0 \le t \le b, \ h_n(0) = 0.$$

Now, with $0 < \epsilon_n \le 1$ to be determined, put

$$x_{n+1} = x_n + \epsilon_n h_n, \ \ t_{n+1} = t_n + \epsilon_n, \ \ t_0 = 0. \tag{1.12}$$

Under the above assumptions, by using the iterative method of contractor directions in [2], the following existence theorem is proved

Theorem 1.1. (Altman [2]) *In addition to the hypothesis* (A.0) to (A.3), *suppose that conditions (1.10) and (1.11) are satisfied and b' is such that*

$$[(1-q)\delta]^{-1}\exp[(1-q)\delta]C[b'C_0(1+\bar{q})P_0]^{1-\bar{s}}[\alpha(1-q)A]^{\bar{s}} < r_0, \tag{1.13}$$

where $P_0 = \|Px_0\|_{\infty,X}$ and $\bar{s}$ satisfies condition

$$\delta = 1-(1+\alpha)\bar{s} > 0. \tag{1.14}$$

Then equation (1.4), with b replaced by b', has a solution x, and

$$\|x_n - x\|_{\infty,Y} \to 0 \quad \text{as} \quad n \to \infty,$$

where $\{x_n\}$ is determined by (1.12) and $\|x_n - x_0\|_{\infty,Y} < r_0$ for all n.

Remark 1.1. In our case, the proof will be analogous, only instead of $x_0(t) \equiv x_0$, where x_0 is some point of the space X, the function $x_0(t)$ given by (1.3) will be discussed.

Remark 1.2. In the proceeding of the proof the following estimates are obtained (cf. [3])

$$\sum_{n=0}^{\infty} \epsilon_n \|h_n\|_{\infty,Y} \le [(1-q)\delta]^{-1}\exp[(1-q)\delta]C[b'C_0(1+\bar{q})P_0]^{1-\bar{s}}[\alpha(1-q)A]^{\bar{s}} \tag{1.15}$$

$$\|h_n\|_{\infty,X} \leq bC_0(1+\bar{q})\|Px_0\|_{\infty,X}\exp(-1(1-q)t_n) \tag{1.16}$$

$$\sum_{n=0}^{\infty} \epsilon_n \exp(-\delta(1-q)t_n) \leq [\delta(1-q)]^{-1}\exp(\delta(1-q))\,. \tag{1.17}$$

Remark 1.3. Theorem 1.1 shows only one example of the existence theorem by using global linearization iterative method (GLIMI) based on the method of contractor directions. Other existence theorem can be introduced by using combinations of smoothing operators and elliptic regularization as in [2], Chapter 8, Theorem 1.1 and also by using elliptic regularization without smoothing operators, as in [2], Chapter 9, Theorem 1.1. In all these methods, in our case the proof will be analogous with the replacement of function $x_0(t) \equiv x_0$ by the function $x_0(t)$ given by (1.3).

2. Existence of tangent directions. Existence of tangent directions for nonlinear evolution equation without Lipschitzian term has been proved in [3] and for quasilinear evolution equation, in [5].

The existence Theorem 1.1, as well as theorems mentioned in Remark 1.2, can be applied to prove existence of tangent directions to the set defined by general nonlinear evolution equation with Lipschitzian term; i.e., in the form (1.4)-(1.5).

Definition 2.1. *A vector $h \in Y$ is a tangent direction to the set Q at the point x_0 if there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ there exists $x(\epsilon) \in Q$ satisfying conditions*

$$x(\epsilon) = x_0 + \epsilon h + r(\epsilon) \quad \text{and} \quad \epsilon^{-1}\|r(\epsilon)\| \to 0 \text{ as } \epsilon \to o^+.$$

In our problem

$$Q = \left\{x \in C(0,b;V_0(\|\cdot\|_Z)) \cap C^1(0,b;X) : Px = 0,\, x(0) = \xi_0\right\}, \tag{2.1}$$

where P is defined by (1.6), ξ_0 is some element of the space Z. We shall prove

Theorem 2.1. *Let*
1. *x_0 be a solution of the Cauchy problem (1.4), (1.5), i.e., $x_0 \in Q$;*
2. *the hypothesis of Theorem 1.1 be satisfied with function x_0 replaced by $x_0 + \epsilon h$;*
3. *there exist constants $c_1 = c_1(h)$ and $c_2 = c_2(h)$ such that*

$$\|F(\cdot, x_0+\epsilon h) - F(\cdot, x_0) - \epsilon F'(\cdot, x_0)h\|_{\infty,X} \leq c_1\epsilon^2\,, \tag{2.2}$$

$$\|f(\cdot, x_0+\epsilon h) - f(\cdot, x_0) - \epsilon f'(\cdot, x_0)h\|_{\infty,X} \leq c_2\epsilon^2\,; \tag{2.3}$$

then any h satisfying the equation

$$\frac{dh}{dt} + F'(t,x_0)h + f'(t,x_0)h = 0 \tag{2.4}$$

is a tangent direction to the set Q at the point x_0.

Proof: In view of Definition 2.1, we have to prove that, for $x_0 \in Q$, from the fact that $h \neq 0$ satisfies equation (2.4) follows that there exists a function $r(\epsilon)$ such that $x(\epsilon) = x_0 + \epsilon h + r(\epsilon)$ is a solution of the Cauchy problem (1.4), (1.5) and $\epsilon^{-1}\|r(\epsilon)\| \to 0$ as $\epsilon \to 0$.

Let $P_0(\epsilon) = \|P(x_0 + \epsilon h)\|_{\infty,X}$. By using equation (2.4) and the fact that $x_0 \in Q$ we have

$$\begin{aligned} P_0(\epsilon) &= \|P(x_0 + \epsilon h\| \\ &= \|d(x_0 + \epsilon h)/dt + F(\cdot, x_0 + \epsilon h) + f(\cdot, x_0 + \epsilon h)\| \\ &= \|d(x_0 + \epsilon h)/dt + F(\cdot, x_0 + \epsilon h) + f(\cdot, x_0 + \epsilon h) - dx_0/dt \\ &\quad - F(\cdot, x_0) - f(\cdot, x_0) - \epsilon(dh/dt + F'(\cdot, x_0) + f'(\cdot, x_0)h)\| \\ &\leq \|F(\cdot, x_0 + \epsilon h) - F(\cdot, x_0) - \epsilon F'(\cdot, x_0)h\| + \|f(\cdot, x_0 + \epsilon h) - f(\cdot, x_0) - \epsilon f'(\cdot, x_0)h\| . \end{aligned} \tag{2.5}$$

In view of assumption 3, (2.5) implies the estimates

$$P_0(\epsilon) \leq c_1\epsilon^2 + c_2\epsilon^2 = c\epsilon^2,$$

where $c = c_1 + c_2$. Then

$$[P_0(\epsilon)]^{1-\bar{s}} \leq c^{1-\bar{s}}\epsilon^{2(1-\bar{s})} \tag{2.6}$$

and

$$0 \leq \frac{1}{\epsilon}[P_0(\epsilon)]^{1-\bar{s}} \leq c^{1-\bar{s}}\epsilon^{1-2\bar{s}} . \tag{2.7}$$

From condition (1.14) of Theorem 1.1, it follows that $1 - 2\bar{s} > 0$ (since $\alpha > 1$). Then (2.7) implies that

$$\frac{1}{\epsilon}[P_0(\epsilon)]^{1-\bar{s}} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.8}$$

Then, in view of (2.6) for sufficiently small ϵ, condition (1.13) of Theorem 1.1 is satisfied with P_0 replaced by $P_0(\epsilon)$. Thus, by virtue of this theorem with x_0 replaced by $x_0 + \epsilon h$, there exists a solution $x(\epsilon)$ of equation (1.4). Let us put $r(\epsilon) = x - x_0(\epsilon)$. Then $x(\epsilon) = x_0 + \epsilon h + r(\epsilon)$. It is easy to prove that $x(\epsilon)$ satisfies also condition (1.5), since the iterative sequence $\{x_n\}$ consists of functions belonging to G; i.e., such that $x_n(0) = \xi_0$ and $\|x_n - x(\epsilon)\|_{\infty,Y} \to 0$ as $n \to \infty$. In view of the definition of $\|\cdot\|_{\infty,Y}$ the last condition implies that

$$\forall t \in [0, b] \ \|x_n(t) - x(\epsilon)(t)\|_Y \to 0;$$

i.e., for $t = 0$, $x(\epsilon)(0) = \xi_0$. Now it is sufficient to prove that

$$\epsilon^{-1}\|r(\epsilon)\| \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.9}$$

From the iterative process in our case, it follows that

$$\|r(\epsilon)\| = \sum_{i=0}^{\infty} \epsilon_n \|h_n\| . \tag{2.10}$$

Then, in view of condition (1.15) from Remark 1.2 with P_0 replaced by $P_0(\epsilon)$ and (2.10), we get estimate

$$\epsilon^{-1}\|r(\epsilon)\| \leq ((1-q)\delta)^{-1} \exp((1-q)\delta)c(b'c_0(1+\bar{q}))^{1-\bar{s}}\epsilon^{-1}(P_0(\epsilon))^{1-\bar{s}}((\alpha(1-q)A)^{\bar{s}}. \tag{2.11}$$

In view of (2.8) the above estimate implies (2.9)

3. Nonlinear evolution equation with control function. Existence of solution. Let $Z \subset Y \subset X$ be Banach spaces satisfying the same conditions as before. Let us introduce a Banach space U and the space $L_\infty(0,b;U)$ of control functions $u = u(t)$, $0 \le t \le b$ with the norm

$$\|u\|_{\infty,U} = \text{ess sup}\{\|u(t)\|_U : 0 \le t \le b\}\,.$$

Given two nonlinear mappings $F(t,x,u)$ and $f(t,x,u)$. We will discuss the following Cauchy problem

$$\frac{dx}{dt} + F(t,x(t),u(t)) + f(t,x(t),u(t)) = 0, \;\; 0 \le t \le b \tag{3.1}$$

$$x(0) = \xi_0\,, \tag{3.2}$$

where $\xi_0 = x_0(0)$, $x_0(0)$ is introduced as in Section 1. Let $r_1 > 0$, u_0 be given control function. Denote by B the set $B = \{u : \|u - u_0\|_{\infty,U} < r_1\}$ and let G be defined as in Section 1. We assume that the mapping F is differentiable in the following sense: for each $(t,x,u) \in [0,b] \times G \times B$, there exist linear operators, $F_x(t,x,u)$ and $F_u(t,x,u)$ such that

$$\epsilon^{-1}\|F(\cdot,x+\epsilon h,u+\epsilon v) - F(\cdot,x,u) - \epsilon(F_x(\cdot,x,u)h + F_u(\cdot,x,u)v)\|_{\infty,X} \to 0 \tag{3.3}$$

as $\epsilon \to 0^+$, $h \in C(0,b;Z) \cap C^1(0,b;X)$ and $v \in B$.

The operator f is differentiable at the point (x_0,u_0) in the following sense: for each $t \in [0,b]$ there exist linear operators $f_x(t,x_0,u_0)$ and $f_u(t,x_0,u_0)$, such that

$$\epsilon^{-1}\|f(\cdot,x_0+\epsilon h,u_0+\epsilon v) - f(\cdot,x_0.u_0) - \epsilon(f_x(\cdot,x_0,u_0)h + f_u(\cdot,x_0,u_0)v\|_{\infty,X} \to 0 \tag{3.4}$$

as $\epsilon \to 0^+$, where $h \in C(0,b;Z) \cap C^1(0,b;X)$.

Let us introduce nonlinear mapping $(x,u) \to P(x,u)$ in the form

$$P(x,u)(t) = \frac{dx}{dt} + F(t,x(t),u(t)) + f(t,x(t),u(t))\,. \tag{3.5}$$

In order to prove the existence of the solution for (3.1) and (3.2) we need to make the following assumptions.

(B1) The functions F, f are continuous in the following sense:

$$\|x_n - x\|_{\infty,Y} \to 0 \quad \text{and} \quad \|u_n - u\|_{\infty,U} \to 0 \quad \text{as} \quad n \to \infty \quad \text{imply}$$

$$\|F(\cdot,x_n,u_n) - F(\cdot,x,u)\|_{\infty,X} \to 0 \quad \text{as} \quad n \to \infty,$$

$$\|x_n - x\|_{\infty,Y} \to 0 \quad \text{and} \quad \|u_n - u\|_{\infty,U} \to 0 \quad \text{as} \quad n \to \infty \quad \text{imply}$$

$$\|f(\cdot,x_n,u_n) - f(\cdot,x,u)\|_{\infty,X} \to 0 \quad \text{as} \quad n \to \infty,$$

Let $\{x_n\} \subset G$ be a Cauchy sequence in $C(0,b;Y)$ and let

$$\{h_n\} \subset C(0,b;Y) \cap C^1(0,b;X)$$

be bounded in $C(0,b;Y)$. Then $\epsilon_n \to 0$ implies

$$\epsilon_n^{-1}\|F(\cdot,x_n+\epsilon_n h_n,u_n+\epsilon_n v_n) - F(\cdot,x_n,u_n) - \epsilon_n(F_x(\cdot,x_n,u_n)h_n + F_n(\cdot,x_n,u_n)v_n)\|_{\infty,X} \to 0 \tag{3.6}$$

as $n \to \infty$ where $\{u_n\}$ is a Cauchy sequence and $\{v_n\}$ is bounded in the norm $\|\cdot\|_{\infty,U}$.

There exists a constant q_0 such that

$$\|f(\cdot, x+\epsilon h, u+\epsilon v) - f(\cdot, x, u)\|_{\infty,X} \le q_0\epsilon(\|x\|_{\infty,X} + \|u\|_{\infty,U}). \tag{3.7}$$

(B2) There exists a constant C_0 such that the following condition is satisfied. For $x \in G$, any function g, such that $\|g\|_{\infty,X} < \infty$, if (h, v) is a solution of the equation

$$\frac{dh}{dt} + F_x(t,x,u)h + F_u(t,x,u)v + g = 0, \quad 0 \le t \le b, \quad h(0) = 0, \tag{3.8}$$

then

$$\|h\|_{\infty,X} + \|u\|_{\infty,U} \le bC_0\|g\|_{\infty,X}.$$

(B3) For $x \in G$, the linearized equation

$$\frac{dz}{dt} + F_x(t,x,u)z + F_u(t,x,u)v + F(t,x,u) - F_x(t,x,u)x + f(t,x,u) = 0, \quad 0 \le t \le b, \quad z(0) = \xi_0 \tag{3.9}$$

admits approximate solution of order (μ, ν, σ) with $0 \le \mu < 1$ in the sense of following definition

Definition 3.1. *Let $\mu > 0$, $\nu > 0$, $\sigma \ge 0$ be given numbers. Then the linearized equation (3.9) admits approximate solution of order (μ, ν, σ) if there exists a constant $M > 0$ with the following property. For every $x \in G$, $u \in B$, $k > 1$, $Q > 1$, if $\|x\|_{\infty,Z} < k$ and $u \in B$ then there exist z, v and y such that*

$$\|z\|_{\infty,Z} < MQk^{\nu}$$
$$\|v\|_{\infty,U} < \infty$$
$$\|y\|_{\infty,X} \le MQ^{-\mu}k^{\sigma}$$

and

$$\frac{dz}{dt} + F_x(t,x,u)z + F_u(t,x,u)v + F(t,x,u) - F_x(t,x,u)x + f(t,x,u) + y = 0 \tag{3.10}$$

where $0 \le t \le b$, $z(0) = \xi_0$.

Let us consider some $x \in G$ and $u \in B$ and let z, v be a solution of equation (3.10) and put $z = x + h$. Then h is a solution of the equation

$$\frac{dh}{dt} + F_x(t,x,u)h + F_u(t,x,u)v + P(x,u) + y = 0 \qquad 0 \le t \le b, \quad h(0) = 0, \tag{3.11}$$

where P is defined by (3.5).

Now we can construct an iterative method of contractor directions for this problem in a similar way as in Section 1. Let us assume that $x_0, x_1, \ldots, x_n \in G$ and $u_0, u_1, \ldots, u_n \in B$ where $u_i = u_i(t)$, $0 \le t \le b$ and $t_0 = 0$, $t_1, t_2, \ldots, t_n$ are known. Then we find a solution (z_n, v_n) of the equation (3.10) with $x = x_n$, and $y = y_n$. Next let $z_n = x_n + h_n$. We put

$$x_{n+1} = x_n + \epsilon_n h_n \tag{3.12}$$
$$t_{n+1} = t_n + \epsilon_n \tag{3.13}$$
$$u_{n+1} = u_n + \epsilon_n v_n. \tag{3.14}$$

We shall consider the iterative method of contractor directions under the following induction assumptions for all $i \leq n$

$$\|x_i\|_{\infty,Z} < A\exp(\alpha(1-q)t_i) = k_i \tag{3.15}$$

$$u_i \in B \tag{3.16}$$

$$\|P(x_i,u_i)\|_{\infty,X} \leq P_0\exp(-(1-q)t_i) \tag{3.17}$$

$$\|v_i\|_{\infty,U} \leq bC_0(1+\bar{q})P_0\exp(-(1-q)t_n). \tag{3.18}$$

The induction assumption (3.15), (3.17) are true for all n in view of Lemma 1.2 from [2] with function $P(x)$ replaced by $P(x,u)$. The condition (3.18) is true for all n according to the same procedure as in Theorem 1.1, where equation (1.16) is true with h_n replaced by v_n.

Then we shall prove the following existance theorem.

Theorem 3.1. *Let us assume that (B1)-(B3) and all assumptions of Theorem 1.1 are satisfied. Then equation (3.1) with b replaced by b' has a solution (x,u) such that*

$$\|x_n - x\|_{\infty,Y} \to 0, \quad \|u_n - u\|_{\infty,U} \to 0 \tag{3.19}$$

as $n \to \infty$; $\{x_n\}$, $\{u_n\}$ are determined by (3.12), (3.14) and

$$\|x - x_0\|_{\infty,Y} < r_0, \quad \|u - u_0\|_{\infty,U} < r_1. \tag{3.20}$$

Proof: The proof of (3.19) for $\{x_n\}$ is the same as in Theorem 1.1. For $\{u_n\}$, (3.19) follows from estimates (3.18) in the same way as the convergence of sequence $\{x_n\}$. The result

$$\|P(x_n,u_n)\| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.21}$$

which is auxiliary in the proof, can be obtained as in Theorem 1.1, with function $P(x)$ replaced by $P(x,u)$ given by (3.5).

The fact that (x,u) is a solution of equation (3.1): i.e, $P(x,u) = 0$ follows from condition (B1) for functions F and f. In fact,

$$\begin{aligned}\|P(x_n,u_n) - P(x,u)\| \leq \|\frac{dx_n}{dt} - \frac{dx}{dt}\| &+ \|F(\cdot,x_n,u_n) - F(\cdot,x,u)\| \\ &+ \|f(\cdot,x_n,u_n) - f(\cdot,x,u)\|.\end{aligned} \tag{3.22}$$

Using (3.19) and (B1), we get that

$$\|P(x_n,u_n) - P(x,u)\| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Combining (3.21) and (3.22), we obtain $P(x,u) = 0$.

The first part of (3.20) follows from estimates (1.15), and the second part from (3.18) in view of estimates (1.17), where $\delta = 1$, $r_1 = bC_0(1+\bar{q})P_0(1-q)^{-1}\exp(1-q)$.

Remark 3.1. A similar result as in Theorem 3.1 can be obtained by using elliptic regularization with smoothing operators or without smoothing operators to construct approximate solution.

4. Existence of tangent directions. The existence of tangent directions for nonlinear evolution equation without the Lipschitzian term and for quasilinear evolution equation with control function involved has been proved in [3] and [5], respectively.

Now, Theorem 3.1 will be applied to prove the existence of tangent directions to the set given by general nonlinear evolution equation with Lipschitzian term; i.e., by (3.1)-(3.2).

Let

$$Q = \left\{ (x,u) : \frac{dx}{dt} + F(t,x(t),u(t)) + f(t,x(t),u(t)) = 0, \ \ 0 \le t \le b, \ \ x(0) = \xi_0 \right\}. \tag{4.1}$$

Definition 4.1. *A vector (h,v) is a tangent direction to the set Q at the point (x_0,u_0) if there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ there exists $(r(\epsilon), w(\epsilon)) \in Q$ satisfying conditions*

$$x(\epsilon) = x_0 + \epsilon h + r(\epsilon), \quad \text{and} \quad u(\epsilon) = u_0 + \epsilon v + w(\epsilon), \tag{4.2}$$

where

$$\epsilon^{-1}\|r(\epsilon)\|_{\infty,Y} \to 0, \quad \epsilon^{-1}\|w(\epsilon)\|_{\infty,U} \to 0 \quad \text{as} \quad \epsilon \to 0^+. \tag{4.3}$$

We shall prove

Theorem 4.1. *Let*

1. (x_0,u_0) be a solution of the Cauchy problem (3.1), (3.2); i.e, $(x_0,u_0) \in Q$

2. the hypothesis of Theorem 3.1 be satisfied with pair (x_0,u_0) replaced by $(x_0+\epsilon h, u_0+\epsilon v)$

3. there exist constants $c_1 = c_1(h,v)$ and $c_2 = c_2(h,v)$ such that

$$\|F(\cdot,x_0+\epsilon h,u_0+\epsilon v) - F(\cdot,x_0,u_0) - \epsilon(F_x(\cdot,x_0,u_0)h + F_u(\cdot,x_0,u_0)v)\|_{\infty,X} \le c_1\epsilon^2, \tag{4.4}$$

and

$$\|f(\cdot,x_0+\epsilon h,u_0+\epsilon v) - f(\cdot,x_0,u_0) - \epsilon(f_x(\cdot,x_0,u_0)h + f_u(\cdot,x_0,u_0)v)\|_{\infty,X} \le c_2\epsilon^2. \tag{4.5}$$

Then, any (h,v) satisfying the equation

$$\frac{dh}{dt} + F_x(t,x_0,u_0)h + f_x(t,x_0,u_0)h + F_u(t,x_0,u_0)v + f_u(t,x_0,u_0)v = 0, \tag{4.6}$$

where $0 \le t \le b$, $h(0) = 0$, is a tangent direction to the set Q at the point (x_0,u_0).

Proof: In view of Definition 4.1, we must prove the fact that $(h,v) \ne 0$ satisfies equation (4.2), implies that $(x_0+\epsilon h+r(\epsilon), u_0+\epsilon v+w(\epsilon))$ is a solution of the Cauchy problem (3.1), (3.2) with $r(\epsilon)$ and $w(\epsilon)$ satisfying conditions (4.3). Let $P_0(\epsilon) = \|P(x_0+\epsilon h, u_0+\epsilon v)\|_{\infty,X}$, where operator P is given by (3.5). From equation (4.6) and assumption 1, we have

$$\begin{aligned}
P_0(\epsilon) &= \|P(x_0+\epsilon h, u_0+\epsilon v)\|_{\infty,X} \\
&= \|\frac{d(x_0+\epsilon h)}{dt} + F(\cdot,x_0+\epsilon h,u_0+\epsilon v) + f(\cdot,x_0+\epsilon h,u_0+\epsilon v)\|_{\infty,X} \\
&= \|\frac{d(x_0+\epsilon h)}{dt} + F(\cdot,x_0+\epsilon h,u_0+\epsilon v) + f(\cdot,x_0+\epsilon h,u_0+\epsilon v) \\
&\quad - \frac{dx_0}{dt} - F(\cdot,x_0,u_0) - f(\cdot,x_0,u_0) - \epsilon(\frac{dh}{dt} + F_x(\cdot,x_0,u_0)h \\
&\quad + F_u(\cdot,x_0,u_0)v + f_x(\cdot,x_0,u_0)h + f_u(\cdot,x_0,u_0)v)\| \\
&\le \|F(\cdot,x_0+\epsilon h,u_0+\epsilon v) - F(\cdot,x_0,u_0) - \epsilon F_x(\cdot,x_0,u_0)h - \epsilon F_u(\cdot,x_0,u_0)v\| \\
&\quad + \|f(\cdot,x_0+\epsilon h,u_0+\epsilon v) - f(\cdot,x_0,u_0) - \epsilon f_x(\cdot,x_0,u_0)h - \epsilon f_u(\cdot,x_0,u_0)v\|.
\end{aligned} \tag{4.7}$$

Then, by using assumption 3, (4.7) implies that

$$P_0(\epsilon) \le c_1\epsilon^2 + c_2\epsilon^2 = c\epsilon^2 \quad \text{where} \quad c = c_1 + c_2\,. \tag{4.8}$$

Then, in the same way as in the proof of Theorem 3.1, we can get conditions (2.6), (2.7) and (2.8). Then, in view of (2.6), condition (1.13) of Theorem 1.1 is satisfied with P_0 replaced by $P_0(\epsilon)$ for sufficiently small ϵ. Thus, we can apply Theorem 3.1 with pair (x_0, u_0) replaced by $(x_0+\epsilon h, u_0+\epsilon v)$ and we get that there exists a pair $(x(\epsilon), u(\epsilon))$ satisfying equation (3.1). In the same way as in the proof of Theorem 2.1, we shall show that $x(\epsilon)$ satisfies also condition (3.2). Now it is enough to prove conditions (4.3).

From the iterative process, it follows that

$$\|r(\epsilon)\| = \sum_{n=0}^{\infty} \epsilon_n \|h_n\|, \quad \|w(\epsilon)\| = \sum_{n=0}^{\infty} \epsilon_n \|v_n\| . \tag{4.9}$$

Combining condition (1.13) (with P_0 replaced by $P_0(\epsilon)$) and (4.8), we get estimates (2.11), which in view of (2.8), implies the first condition of (4.3).

The second part of (4.3) can be obtained similarly. Combining (3.18) (with P_0 replaced by $P_0(\epsilon)$) and (4.9), we get

$$\|w(\epsilon)\| \le bc_0(1+\bar{q})P_0(\epsilon) \sum_{n=0}^{\infty} \epsilon_n \exp(-(1-q)t_n) .$$

Applying estimates (1.17) with $\delta = 1$ to the last inequality and dividing by ϵ, we have

$$\epsilon^{-1}\|w(\epsilon)\| \le bC_0(1+\bar{q})\epsilon^{-1}P_0(\epsilon)(1-q)^{-1}\exp(1-q) . \tag{4.10}$$

In view of (4.8), $P_0(\epsilon)/\epsilon \to 0$; then (4.10) implies the second part of (4.3).

5. An application: The extremum principle. The results of Section 4 can be applied to problems of optimal problem with set Q given by (4.1) as constraint. Let us consider the following optimal control problem.

Minimize the functional

$$I(x,u) = \int_0^b f^\circ(t, x(t), u(t))\, dt \tag{5.1}$$

under the constraints

$$\frac{dx}{dt} = F(t, x(t), u(t)) + f(t, x(t), u(t)), \tag{5.2}$$

$$x(0) = \xi_0, \tag{5.3}$$

$$u(\cdot) \in V, \tag{5.4}$$

where $x(\cdot) \in C(0,b;Y) \cap C^1(0,b;X)$,

$$V = \{u(\cdot) \in L_\infty(0,b;U) : u(t) \in W \subset U \ \text{ for } \ t \in [0,b] \text{ a.e. } \} ,$$

X, Y, L_∞, U are Banach spaces introduced in Sections 1 and 3, ξ_0, as previously, is a fixed element of the space Z. Let (x_0, u_0) be a solution of the problem (5.1)-(5.4).

We shall assume that

C.1. functions F and f satisfy all the assumptions of Theorem 4.1

C.2. function f° is differentiable in the neighborhood N of (x_0, u_0) in the following sense: there exist operators $f_u^\circ(\cdot, x, u)$ such that

$$\epsilon^{-1}|f^\circ(\cdot, x+\epsilon h, u) - f^\circ(\cdot, x, u) - \epsilon f_x^\circ(\cdot, x, u)h| \to 0$$
$$\epsilon^{-1}|f^\circ(\cdot, x, u+\epsilon v) - f^\circ(\cdot, x, u) - \epsilon f_u^\circ(\cdot, x, u)v| \to 0$$

as $\epsilon \to 0$, $h \in C(0, b; Z) \cap C^1(0, b; X)$, $v \in L_\infty(0, b; U)$

C.3. for any $(x, u) \in N$ one of the following conditions is satisfied: $f_x^\circ(\cdot, x, u)$ is continuous with respect to u, i.e.,

$$\|u_n - u\|_{\infty,U} \to 0 \ \text{ implies } \ |f_x^\circ(\cdot, x, u_n) - f_x^\circ(\cdot, x, u)\| \to 0 \ \text{ as } \ n \to \infty$$

$f_u^\circ(\cdot, x, u)$ is continuous with respect to x, i.e.,

$$\|x_n - x\|_{\infty,Y} \to 0 \ \text{ implies } \ |f_u^\circ(\cdot, x_n, u) - f_u^\circ(\cdot, x, u)| \to 0 \ \text{ as } \ n \to \infty$$

C.4. f°, f_x°, f_u° are measurable with respect to t for any (x, u)

C.5. f° satisfies the Lipschitz condition with respect to (x, u) in the following sense: there exists a constant k such that for any (x, u), $(x', u') \in N$

$$|f^\circ(\cdot, x, u) - f^\circ(\cdot, x', u')| \leq k\left(\|x - x'\|_{\infty,Y} + \|u - u'\|_{\infty,U}\right)$$

C.6. the set W is convex, closed and posseses a nonempty interior in U

C.7. the equation

$$\frac{dh}{dt} = F_x(t, x_0, u_0)h + f_x(t, x_0, u_0)h + F_u(t, x_0, u_0)v + f_u(t, x_0, u_0)v, \quad h(0) = 0 \tag{5.5}$$

has a solution $h \in C(0, b; Y) \cap C^1(0, b; X)$ for every $v \in B$

The problem of minimizing (5.1) under constraints (5.2)-(5.4) we shall call problem I.

Denote $S = C(0, b; Y) \cap C^1(0, b; X)$. By making use of the Dubovitskii-Milyutin Theorem (cf. [6] Sec. 6) and Theorem 4.1, we shall prove a local extremum principle for problem I.

Theorem 5.1. (Local extremum principle). *Let (x_0, u_0) be an optimal process for problem I and assumptions C.1-C.7 be satisfied. If there exist $\lambda_0 > 0$ and function $\psi(t)$ not all zero satisfying equation*

$$\frac{d\psi}{dt} = \lambda_0 f_x^\circ(t, x_0, u_0) - F_x^*(t, x_0, u_0)\psi(t) - f_x^*(t, x_0, u_0)\psi(t), \quad \psi(b) = 0 \tag{5.6}$$

then the maximum condition holds

$$\int_0^b \left(\lambda_0 f_u^\circ(t, x_0, u_0) - F_u^*(t, x_0, u_0)\psi(t) - f_u^*(t, x_0, u_0)\psi(t),\ u - u^\circ(t)\right) dt \geq 0 \tag{5.7}$$

for all $0 \leq t \leq b$, $u \in V$.

Proof: Let us define the following sets:

$$Z_1 = \{(x, u) \in S \times L_\infty(0, b; U) : u(\cdot) \in V\} \tag{5.8}$$

$$Z_2 = \left\{(x, u) \in S \times L_\infty(0, b; U) : \frac{dx}{dt} = F(t, x, u) + f(t, x, u), \ \ x(0) = \xi_0\right\}. \tag{5.9}$$

In view (5.8) and (5.9), our problem can be expressed in the form: minimize the functional $I(x, u)$ under the constraint $(x, u) \in Z_1 \cap Z_2$.

In order to use the Dubovitskii-Milyutin Theorem, we have to find

C_0 the cone of decrease of the functional I at the point (x_0, u_0);
C_1 feasible cone to the set Z_1 at the point (x_0, u_0);
C_2 tangent cone to the set Z_2 at the point (x_0, u_0);

and dual cones C_0^* and C_1^*.

Let us consider the functional $I(x,u)$. In view of assumption C.2-C.4, we have that there exists derivative of $I(x,u)$ in any direction (h,v) such that

$$I'(x_0,u_0)(h,v)=\int_0^b \left(f_x^\circ(t,x_0,u_0)h+f_u^\circ(t,x_0,u_0)v\right)\,dt\,. \tag{5.10}$$

In view of assumption C.5, functional $I(x,u)$ also satisfies the Lipschitz condition with constant bk since

$$\begin{aligned}|I(x,u)-I'(x',u')| &\le b|f^\circ(\cdot,x,u)-f^\circ(\cdot,x',u')|\\ &\le bk\left(\|x-x'\|+\|u-u'\|\right)\,.\end{aligned} \tag{5.11}$$

In view of (5.10) and (5.11), we have that all the assumptions of Theorem 7.3 from [6] are satisfied. Then $I(x,u)$ regularly decreases at (x_0,u_0) and

$$C_0=\left\{(h,v)\in S\times L_\infty(0,b;U):\int_0^b \left(f_x^\circ(t,x_0,u_0)h+f_u^\circ(t,x_0,u_0)v\right)\,dt<0\right\}\,. \tag{5.12}$$

Then, in view of Theorem 10.2, from [6]

$$C_0^*=\left\{f_0\in(S\times L_\infty(0,b;U))^*:f_0(h,v)=-\lambda_0\int_0^b \left(f_x^\circ(t,x_0,u_0)h+f_u^\circ(t,x_0,u_0)v\right)\,dt\right\}\,. \tag{5.13}$$

Now let us consider constraint Z_1. In view of the assumption C.6 and the definition of the norm $\|\cdot\|_{\infty,U}$ (cf. Section 3), it is easy to prove that the set V is also convex, closed and possesses a nonempty interior in the topology of the space $L_\infty(0,b;U)$ generated by the norm $\|\cdot\|_{\infty,U}$. Then, in view of Theorem 8.2, from [6], we get

$$C_1=\{(h,v)\in S\times L_\infty(0,b;U):v=\lambda(u-u_0),\ \ u\in\operatorname{int}V,\ \lambda>0\}\,, \tag{5.14}$$

and applying Theorem 10.5 from [6] we have

$$C_1^*=\{f_1(h,v)\in(S\times L_\infty(0,b;U))^*:f_1(h,v)=f_1'(v)\}\,, \tag{5.15}$$

where f_1' is a functional supporting the set V at the point u_0. Let us consider Z_2. From assumption C.1 we can apply Theorem 4.1 and we get inclusion

$$\begin{aligned}\Big\{(h,v)\in S\times L_\infty(0,b;U):\ &\frac{dh}{dt}+F_x(t,x_0,u_0)h+f_x(t,x_0,u_0)h\\ &+F_u(t,x_0,u_0)v+f_u(t,x_0,u_0)v=0,\ \ 0\le t\le b,\ \ h(0)=0\Big\}\subset C_2\,.\end{aligned} \tag{5.16}$$

Now we can use the Dubovitskii-Milyutin Theorem from [6]. In view of its proposition, there exist functionals $f_i\in C_i^*$, $i=0,1,2$, not all zero, such that the Euler-Lagrange condition holds; i.e.,

$$f_0(h,v)+f_1(h,v)+f_2(h,v)=0, \tag{5.17}$$

for all $(h,v)\in S\times L_\infty(0,b;U)$.

Let us consider (5.17) for the pairs (h,v) satisfying condition (5.5). Then, from (5.16), $f_2(h,v)=0$ and using (5.13) and (5.15) we have

$$f_1(h,v)=f_1'(v)=\lambda_0\int_0^b \left(f_x^\circ(t,x_0,u_0)h+f_u^\circ(t,x_0,u_0)v\right)\,dt\,. \tag{5.18}$$

Now proceeding analogously as in [6], [4]; i.e., integrating by parts and using (5.5) and (5.6), we get

$$f_1'(v) = \int_0^b (-F_u^*(t, x_0, u_0)\psi(t) - f_u^*(t, x_0, u_0)\psi(t) + \lambda_0 f_u^\circ(t, x_0, u_0),\ v)\ dt\,, \tag{5.19}$$

where f_1' is a functional supporting the set V at the point u_0.

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