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PERIODIC BOUNDARY VALUE PROBLEM FOR SOME DUFFING EQUATIONS

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Abstract. We consider the periodic problem for the forced Duffing equation

$$u'' + cu' + g(t,u) = 0,$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

where g is 2π-periodic in t. We study existence of solution under the hypothesis that $c \neq 0$ and

$$\limsup_{|u|\to\infty} \left| \frac{g(t,u)}{u} \right| \leq 1 + c^2 .$$

We also consider conditions under which the set of solutions is an R_δ .

1. Some existence results. In recent years much work has been done concerning the existence of periodic solutions to the Duffing equation

$$u'' + cu' + g(t,u) = 0, \qquad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{1}$$

(cf. [2–11, 13, 14, 16]).

In this section we study a new existence result, assuming that $g(t,u)$ is a Carathéodory function (i.e., continuous in u and measurable in t), 2π-periodic in t, which in addition satisfies the following conditions.

(g_1) For each $R > 0$ there exists a function $a \in L^2(0,2\pi)$ such that for all $(t,u) \in [0,2\pi] \times [-R,R]$,

$$|g(t,u)| \leq a(t).$$

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(g_2) There exist functions b, $c \in L^2(0, 2\pi)$ such that

$$g(t,u) \geq b(t) \quad \text{if } t \in [0, 2\pi) \text{ and } u \geq 0,$$

$$g(t,u) \leq c(t) \quad \text{if } t \in [0, 2\pi) \text{ and } u \leq 0.$$

As in some of the above mentioned papers, an important role is played by the function

$$\gamma(t) = \limsup_{|u| \to \infty} \frac{g(t,u)}{u} \tag{2}$$

where the limit is supposed to be uniform with respect to $t \in [0, 2\pi]$.

We start by stating a result which we use later and whose proof is given in [6].

Lemma 1. *Let the Carathéodory function $g(t,u)$ satisfy (g_1) and (g_2). Suppose that $\gamma(t)$ in (2) is bounded. Then, for each $\epsilon > 0$ there exist Carathéodory functions $g_0(t,u)$, $g_1(t,u)$ and $d \in L^2(0, 2\pi)$ such that*

$$g(t,u) = g_0(t,u) \cdot u + g_1(t,u)$$

and for all $(t,u) \in [0, 2\pi] \times \mathbf{R}$,

$$0 \leq g_0(t,u) \leq \gamma(t) + \epsilon, \qquad |g_1(t,u)| \leq d(t).$$

We shall need also the following simple lemma about linear problems with periodic conditions. By $\| \ \|_2$ we denote the usual norm of $L^2(0, 2\pi)$.

Lemma 2. *Let $c \neq 0$. There exists a number $\epsilon > 0$ with the following property: Given any measurable, 2π-periodic function $s(t)$ such that $0 \leq s(t) \leq 1 + c^2 + \epsilon$, any nontrivial solution of*

$$\begin{gathered} u'' + cu' + s(t)u = 0 \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{gathered} \tag{3}$$

is everywhere different from zero.

Proof: Suppose first that $0 \leq s(t) \leq 1 + c^2$ in (3). We claim that (3) has nontrivial solutions if and only if $s(t) \equiv 0$ (so that solutions are constant). In fact from (3) we get

$$\int_0^{2\pi} u'^2 \, dt = \int_0^{2\pi} su^2 \, dt,$$

$$(1 + c^2) \int_0^{2\pi} u'^2 \, dt \leq \int_0^{2\pi} (u''^2 + c^2 u'^2) \, dt = \int_0^{2\pi} s^2 u^2 \, dt \tag{4}$$

where we have used the inequality

$$\int_0^{2\pi} z^2 \, dt \leq \int_0^{2\pi} z'^2 \, dt$$

valid for 2π-periodic functions $z(t)$ such that $\int_0^{2\pi} z = 0$. It follows that

$$(1 + c^2) \int_0^{2\pi} su^2 \, dt \leq \int_0^{2\pi} s^2 u^2 \, dt. \tag{5}$$

Now, if $u \not\equiv 0$ and $s \neq 0$ we cannot have $s(t) \equiv 1+c^2$. Suppose that s takes only the values 0 and $1+c^2$. Then, $u(t)$ cannot be of the form $C_1 \cos t + C_2 \sin t + C_3$, so that the first inequality in (4) is strict and the same occurs in (5); hence, $s(t)$ must take values different from zero and $1+c^2$ on a set of positive measure. This implies

$$\int_0^{2\pi} s^2 u^2 \, dt < (1+c^2) \int_0^{2\pi} s u^2 \, dt$$

contradicting (5).

Now, suppose that a number $\epsilon > 0$ as stated in the lemma does not exist. Then we can choose bounded functions $s_n(t)$ and solutions $u_n(t)$ of

$$u_n'' + c u_n' + s_n(t) u_n = 0$$

$$u_n(0) = u_n(2\pi), \qquad u_n'(0) = u_n'(2\pi)$$

so that $u_n(t_n) = 0$ for some $t_n \in [0, 2\pi]$, $\|u_n\|_2 + \|u_n'\|_2 = 1$ and $0 \leq s_n(t) \leq 1 + c^2 + 1/n$. Passing to convenient subsequences we may assume that $u_n \to u$ in $H^2(0, 2\pi)$ weakly, $s_n \to s$ in $L^\infty(0, 2\pi)$ weak* and $t_n \to t_0$. Then, u satisfies $\|u\|_2 + \|u'\|_2 = 1$,

$$u'' + cu' + s(t) u = 0,$$

u is periodic, and $u(t_0) = 0$. However, $0 \leq s(t) \leq 1 + c^2$ and this contradicts the fact that u should be some nonzero constant. Hence the proof is complete.

We now present our main existence theorem about problem (1). The proof is similar to that of theorem 1 in [15].

Define

$$g^-(t) = \limsup_{u \to -\infty} g(t,u), \quad g_+(t) = \liminf_{u \to +\infty} g(t,u).$$

Theorem 1. *Let the Carathéodory function satisfy* (g_1) *and* (g_2) *and suppose that* $\gamma(t) \leq 1 + c^2$ *in (2). Then, if*

$$\int_0^{2\pi} g^-(t) \, dt < 0 < \int_0^{2\pi} g_+(t) \, dt, \tag{6}$$

problem (1) has at least one solution.

Proof: Choose a number $a \in (0, 1+c^2)$. We consider the family of homotopic problems

$$\begin{gathered} u'' + cu' + (1-\lambda) a u + \lambda g(t,u) = 0 \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \quad 0 < \lambda < 1. \end{gathered} \tag{7}$$

By well-known results of degree theory, the theorem will be proved if we show that solutions of (7) are bounded in $L^2(0, 2\pi)$ independently of λ. So suppose that there exist sequences $\lambda_n \in (0,1)$, $u_n(t)$ satisfying (7) and $\|u_n\|_2 \to +\infty$. Set $v_n(t) = u_n(t)/\|u_n\|_2$.

First we choose a decomposition of g,

$$g(t,u) = u g_0(t,u) + g_1(t,u)$$

with the properties stated in Lemma 1, where ϵ is a number as in Lemma 2. Writing (7) for λ_n, u_n we obtain

$$v_n'' + cv_n' + (1-\lambda_n)av_n + \lambda_n v_n g_0(t,u_n) + \frac{g_1(t,u_n)}{\|u_n\|_2} = 0,$$
$$v_n(0) = v_n(2\pi), \quad v_n'(0) = v_n'(2\pi).$$

Now from standard estimates it follows that we can choose a subsequence, still denoted v_n, such that

$$v_n \to v \quad \text{in } H^2(0,2\pi) \text{ weakly,}$$

$$g_0(t,u_n) \to s(t) \quad \text{in } L^\infty(0,2\pi) \text{ weak}^*,$$

$$\lambda_n \to \lambda\,.$$

The limit equation is

$$v'' + cv' + [(1-\lambda)a + \lambda s(t)]v = 0$$

and v is periodic. Since $0 \le (1-\lambda)a + \lambda s(t) \le 1 + c^2 + \epsilon$, it follows from Lemma 2 that $v \ne 0$ at every point. Since $v_n \to v$ uniformly, v_n has the same sign of v for n sufficiently large. To fix ideas suppose that $v_n > 0$ for large n. Then, $u_n(t) \to +\infty$ at every point, so that

$$\liminf_{n\to\infty} g(t,u_n(t)) \ge g_+(t), \qquad t \in [0,2\pi]. \tag{8}$$

Integrating (7) with $\lambda = \lambda_n$ and $u = u_n(t)$ we get

$$\int_0^{2\pi} g(t,u_n(t))\,dt \le 0.$$

Since (g_2) allows us to use Fatou's lemma, this inequality implies

$$\int_0^{2\pi} \liminf_{n\to\infty} g(t,u_n(t))\,dt \le 0$$

which, together with (8), contradicts the right-hand side of (6). In the same manner we can deal with the case where $v_n < 0$, and the proof is complete.

From now on we will be interested in a special case of problem (1), namely

$$\begin{aligned} &u'' + cu' + g(u) = h(t) \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{aligned} \tag{9}$$

where g is continuous in $\mathbf{R}$ and $h \in L^2(0,2\pi)$. In addition to notation already introduced, we denote by $\| \|_\infty$ the usual sup norm in the space $C([0,2\pi])$ of continuous functions in $[0,2\pi]$. If $u \in L^2(0,2\pi)$, we write

$$Pu = \frac{1}{2\pi}\int_0^{2\pi} u(t)\,dt$$

so that P is the orthogonal projection onto the space of constant functions.

Assume that $c \neq 0$ in (9). Suppose further that g and h are such that the following condition holds.

(g_3). There exists $M > 0$ such that

$$g(u) > Ph \quad \text{if } u \geq M, \qquad g(u) < Ph \quad \text{if } u \leq -M.$$

It then follows from the proof of Theorem 6.3 [14, p. 205] that there exists a constant $M' \geq M$, depending only on c, M and h, such that any solution of (9) satisfies

$$\|u\|_\infty \leq M'\,. \tag{10}$$

Therefore if we truncate $g(u)$ to obtain the bounded, continuous function

$$G(u) = \begin{cases} g(-M') & \text{if } \; u \leq -M', \\ g(u) & \text{if } \; |u| \leq M' \\ g(M') & \text{if } \; u \geq M' \end{cases}$$

and noting that G satisfies (g_3) with the same constant M, we obtain the following result which will be used in what follows.

Proposition 1. *If $c \neq 0$ and (g_3) holds, the set of solutions of (9) is the same as the set of solutions of*

$$\begin{gathered} u'' + cu' + G(u) = h(t) \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi)\,. \end{gathered} \tag{11}$$

In particular, (9) has at least one solution.

Remark 1. The last assertion in Proposition 1 is also a special case of Theorem 6.3 [14, p. 205]. We could obtain it as well from the estimate (10) and applying Theorem 1 to problem (11).

Remark 2. It follows from the proof of Theorem 1 in [9] that Proposition 1 is true if $c = 0$ provided that g satisfies, in addition to (g_3), the growth restriction

$$|g(u)| \leq \gamma|u| + K, \quad u \in \mathbf{R},$$

for some constants $\gamma \in [0, 1)$ and $K > 0$.

It may also be interesting to note that we can obtain multiplicity results for a certain class of forcing terms if $c \neq 0$. Consider the case where $h(t) = a$ is a constant. Then, obviously the only solutions of (9) are the real roots of the equation $g(u) = a$. To obtain a simple version of this result for nonconstant h, write (9) in the form

$$\begin{gathered} u'' + cu' + g(u) = a + e(t) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{gathered} \tag{12}$$

where $a \in \mathbf{R}$ and $e(t)$ is continuous and $Pe = 0$. For simplicity we suppose that $g(u) = a$ has at least three roots; more precisely, we assume that g satisfies:

(g_4) There exists $M > 0$ such that

$$g(u) > a \quad \text{if } u \geq M,$$

$$g(u) < a \quad \text{if } u \leq -M.$$

(g_5) There exist numbers $K_1 < K_2$ such that

$$g(K_2) < a < g(K_1).$$

Note that (g_4) is a restatement of (g_3).

Theorem 2. *Suppose $c \neq 0$. Let $e(t)$ be continuous, 2π-periodic and $Pe = 0$. If g is a continuous function satisfying (g_4), (g_5), then there exists a number $\delta > 0$ such that, if $\|e\|_\infty < \delta$, the problem (12) has at least three solutions.*

Proof: Any solution u of (12) satisfies

$$|u(t) - u(s)| \leq \sqrt{2\pi}\,\|e\|_2\,|c|^{-1}, \quad t, s \in [0, 2\pi]. \tag{13}$$

Choose $\delta > 0$ such that

$$g(K_1) > a + \delta, \quad g(K_2) < a - \delta \tag{14}$$

and

$$\begin{aligned} g(z) > a \quad &\text{if} \quad |z - K_1| < \epsilon, \\ g(z) < a \quad &\text{if} \quad |z - K_2| < \epsilon, \end{aligned} \tag{15}$$

where $\epsilon = 2\pi\delta|c|^{-1}$.

Now, (12) has a solution $u(t)$ such that $K_1 \leq u(t) \leq K_2$; $0 \leq t \leq 2\pi$, since from (14) it follows that K_1 (respect. K_2) is a subsolution (supersolution) of (12) if $\|e\|_\infty < \delta$. To obtain a second solution we truncate g to obtain another continuous function $\tilde{g}$ such that

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \geq K_2 \\ g(K_2) & \text{if } x \leq K_2. \end{cases}$$

Because of Proposition 1 the modified problem (with g replaced for $\tilde{g}$) has at least a solution $v(t)$. Then using (13) and (15) we conclude that $v(t) > K_2$ for all $t \in [0, 2\pi]$, so that $v(t)$ is in fact a solution of (12). In the same way we obtain a third solution $w(t)$ such that $w(t) < K_1$.

2. Set of solutions. The solution to problem (9) is not necessarily unique. Hence, it becomes an important question to study the structure of the set of solutions of (9), which we denote by $S(h)$.

Under the conditions of Proposition 1 (or if $c = 0$ and Remark 2 applies) we know that $S(h)$ is compact in $H^2(0, 2\pi)$. However, it need not be connected; see [5], example 11.

We are going to study some conditions under which the set $S(h)$ is an R_δ. We recall that an R_δ is a set homemorphic to the intersection of a decreasing sequence of compact absolute retracts. Any R_δ is acyclic and, in particular, it is nonempty, compact, and connected but there exist compact and connected sets which cannot be continuous images of R_δ's [12]. We shall use the following classical result due to Aronszajn [1].

Theorem 3. *Let K be a closed, convex and bounded set in a Banach space E with norm $\|\ \|$. Let $T : E \to E$ be a compact map such that $T(K) \subset K$. Suppose that for every $\epsilon > 0$ there exists a compact map $T_\epsilon : E \to E$ such that*

- **(a)** $\|T_\epsilon u - Tu\| < \epsilon$ *for every $u \in K$,*
- **(b)** *there exists $\rho > 0$ such that for every $\epsilon > 0$, $(I - T_\epsilon)$ maps K bijectively onto a set containing $B(0, \rho) = \{u \in E : \|u\| \leq \rho\}$.*

Then the set of fixed points of T is an R_δ.

In order to apply this result we need a uniqueness lemma. In the proof we make use of the linear operator

$$L : D(L) \subset L^2(0,2\pi) \to L^2(0,2\pi),$$

where

$$D(L) = \left\{u \in H^2(0,2\pi) : \ u(0) = u(2\pi), \ u'(0) = u'(2\pi)\right\} .$$

and

$$Lu = u'' + cu' .$$

The range of L, $R(L)$, consists of L^2-functions having zero mean-value. Recall also that

$$(Lu, u) \geq -\frac{1}{1+c^2}\|Lu\|_2^2 , \quad u \in D(L),$$

where $(\cdot,\cdot)$ denotes the usual scalar product of $L^2(0,2\pi)$.

Lemma 3. *Let g satisfy (g_3). Suppose that g is strictly increasing and there exists $k > 0$ such that*

$$|g(u) - g(v)| \leq k|u - v|, \quad \textit{for every } u, v \in \mathbf{R}. \tag{16}$$

If $k < 1 + c^2$, then (9) has at most one solution.

Proof: Let u, v be solutions. Then, $L(u-v) + g(u) - g(v) = 0$. Now, multiplying this equation by $u - v$ and integrating on $[0, 2\pi]$ we obtain

$$\begin{aligned} 0 &= \big(L(u-v),\, u-v\big) + \int_0^{2\pi} [g(u(t)) - g(v(t))]\big(u(t) - v(t)\big)\, dt \\ &\geq \frac{-1}{1+c^2}\|L(u-v)\|_2^2 + \int_0^{2\pi} |g(u(t)) - g(v(t))| \cdot |u(t) - v(t)|\, dt \\ &\geq \Big(\frac{1}{k} - \frac{1}{1+c^2}\Big) \int_0^{2\pi} [g(u(t)) - g(v(t))]^2\, dt\,. \end{aligned}$$

Since, now $k < 1 + c^2$ we get $u(t) = v(t)$ for every $t \in [0, 2\pi]$.

Remark 3. Note that for g increasing, but not strictly, the result may fail [5]. On the other hand, it is obvious that (16) implies

$$|g(u)| \leq k|u| + |g(0)|, \qquad u \in \mathbf{R}.$$

Now, we are in a position to prove

Theorem 4. *Suppose g is increasing and satisfies (g_3) and (16) with $k < 1 + c^2$. then, $S(h)$ is an R_δ in $E \equiv C([0, 2\pi])$.*

Proof: By Proposition 1 and Remarks 2 and 3, $S(h)$ is the set of solutions of (11). It is well known that this set is the set of fixed points of the compact operator $T : E \to E$ defined by

$$Tu = Pu + H(I - P)Nu + \delta PNu$$

where $H : R(L) \to R(L)$ is the partial inverse of L, N is the Niemytski operator corresponding to the function $h - G(u)$ and δ is an arbitrary, fixed nonzero number. Now choose A, $B > 0$ large enough, and $\eta > 0$ small enough so that

$$\big\|H(I-P)Nu\big\|_\infty \leq B, \qquad \forall\, u \in E, \tag{17}$$

and

$$|z| \leq B+1 \Rightarrow G(A+z) \geq h_0 + \eta, \quad G(-A+z) \leq h_0 - \eta. \tag{18}$$

On the other hand, since G is a Lipschitz function, we can fix $\delta > 0$ so that the function

$$u \to u - \delta G(u+z)$$

is increasing for any fixed z. With this choice of A, B and δ we claim that the closed, bounded, convex subset of E

$$K = \{u = u_0 + u_1 \in E : |u_0| \leq A, \quad \|u_1\|_\infty \leq B+1\}$$

(where we have written $u_0 = Pu$ and $u_1 = u - u_0$) is mapped into itself by T. Indeed, let $u \in K$. Because of (17) we have only to show that

$$|Pu + \delta PNu| = \left| u_0 + \delta\Big(h_0 - \frac{1}{2\pi}\int_0^{2\pi} G(u(t))\,dt\Big)\right| \leq A. \tag{19}$$

To this purpose, observe that, for each $t \in [0, 2\pi]$ we have, in consequence of (18),

$$A - \delta(G(A + u_1(t)) - h_0) \leq A - \delta\eta,$$

$$-A + \delta(h_0 - G(-A + u_1(t)) \geq -A + \delta\eta,$$

and it follows from our choice of δ that the function

$$s \to s + \delta(h_0 - G(s + u_1(t)))$$

maps the interval $[-A, A]$ into $[-A+\delta\eta,\ A-\delta\eta]$. Therefore, (19) holds in the stronger form, for $u \in K$:

$$|Pu + \delta PNu| \leq A - \delta\eta. \tag{20}$$

To apply Aronzajn's Theorem it remains to see that T and K, constructed above, satisfy the remaining assertions, namely that there exists a family of compact maps $T_\epsilon (\epsilon > 0$, small) and a positive number ρ such that

- **(a)** $\|(T - T_\epsilon)u\|_\infty \leq \alpha(\epsilon)$, $u \in E$, and $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$.
- **(b)** $I - T_\epsilon$ is one-to-one,
- **(c)** $(I - T_\epsilon)K \supset B(0, \rho)$.

For each $\epsilon > 0$, let $G_\epsilon(u) = G(u) + \epsilon \arctan u$ and consider the corresponding Niemytski operator

$$N_\epsilon(u) = h - G_\epsilon(u)$$

and set

$$T_\epsilon = P + H(I-P)N_\epsilon + \delta P N_\epsilon.$$

We claim that this family of operators has the properties **(a)**, **(b)**, **(c)**.

Proof of (a): From the definition of T and T_ϵ, it follows immediately that there exists a constant $\alpha > 0$ such that, for every $u \in E$,

$$\|Tu - T_\epsilon u\|_\infty \leq \alpha\epsilon.$$

Proof of (b): Suppose that $u, v \in E$ are such that $u - T_\epsilon u = v - T_\epsilon v$. Then we obtain

$$u - v = P(u - v) + H(I - P)(N_\epsilon u - N_\epsilon v) + \delta P(N_\epsilon u - N_\epsilon v)$$

from which it follows that $P(N_\epsilon u - N_\epsilon v) = 0$ and

$$L(u - v) = N_\epsilon u - N_\epsilon v\,.$$

Now, as in the proof of Lemma 3, from the fact that G_ϵ is strictly increasing and has Lipschitz constant $k + \epsilon$, we conclude that, for ϵ sufficiently small, $u = v$.

Proof of (c): We shall show that $(I - T_\epsilon)(K)$ contains all functions $w = w_0 + w_1$ such that $|w_0| \le \delta\eta$ and $\|w_1\|_\infty \le 1/2$. In fact, if w satisfies these conditions and we consider the operator

$$u \to T_\epsilon(u) + w,$$

we see that it maps K into K for $\epsilon > 0$ small. This is so because, for small $\epsilon > 0$,

$$\|H(I - P)N_\epsilon u\| \le B + \frac{1}{2}, \qquad u \in E$$

and clearly, δ can be chosen so that the functions

$$s \to s - \delta G_\epsilon(s + z)$$

are increasing independently of ϵ. Then, we obtain (20) with N_ϵ instead of N and it follows that $u \in K$ implies

$$|P(T_\epsilon(u) + w)| \le A\,.$$

Therefore, it suffices to invoke Schauder's fixed point theorem to see that $u = T_\epsilon u + w$ has a solution in K, and the proof is complete.

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