

CONJUGATE POINTS OF FOURTH ORDER DIFFERENTIAL EQUATIONS WITH IMPULSES

KURT KREITH

*Department of Mathematics, University of California
Davis, California 95616 USA*

(Submitted by : F.V. Atkinson)

Abstract. Selfadjoint fourth order differential equations of the form

$$(p_2(t)y'')'' - (p_1(t)y')' + p_0(t)y = 0$$

are considered in a generalized context which allows for distributional terms in the coefficients. Criteria are established for the existence and non-existence of conjugate points for such equations.

1. Introduction. The classical existence theory for differential equations has, in a variety of contexts, been extended to allow for solutions which satisfy the underlying equation in only a generalized sense. Such theories of "generalized differential equations" can be developed (more or less equivalently) in terms of novel theories of integration [5], the introduction of measures [7] or via the theory of distributions [1]. (See also the very useful survey [9]).

While there is a substantial literature dealing with existence, uniqueness and stability for generalized differential equations, qualitative theories of oscillation, conjugacy and asymptotic behavior have tended to be pursued in a more classical context. Thus, in the case of selfadjoint fourth order equations of the form

$$(p_2(t)y'')'' - (p_1(t)y')' + p_0(t)y = 0 \tag{1.1}$$

the seminal paper of Leighton and Nehari [6] requires that p_2 , p_1 and p_0 be of class C'' , C' and C , respectively, on the interval under consideration. Most subsequent studies of such equations have involved similar limitations.

In the study of conjugate points, there are, of course, strong incentives for assuming sufficient regularity to enable one to bring powerful variational tools to bear. For second order Sturm-Liouville equations of the form

$$-(p_1(t)y')' + p_0(t)y = 0, \tag{1.2}$$

this means that $p_0(t)$ may include distributional terms corresponding to delta functions (leading to discontinuities in $y'(t)$) but not derivatives of delta functions (which lead to

Received September 17, 1987.

AMS(MOS) Subject Classifications: 34C10.

composition of continuous functions, the $y_j(t_1^+)$ depend continuously on the γ_j , as does the straight line segment of Γ connecting $(y_1(t_1^-), y_3(t_1^-))$ to $(y_1(t_1^+), y_3(t_1^+))$. This argument can now be applied for $t_1 < t < t_2$, with the $y_j(t_1^+)$ playing the same role as the γ_j in (α, t_1) . An inductive argument then yields the continuous dependence of Γ on the initial data γ_j for $0 \leq t < \infty$.

Remark. Whereas the Q_{ij} and R_{ij} correspond to discontinuities in the trajectories of (2.6), it is the Q_{ij} and q_{ij} which give rise to discontinuities in $y_1 = y$ and $y_2 = y'$, respectively. Thus, it is the presence of the Q_{ij} and q_{ij} which prevents us from developing a conjugacy theory for (2.6) based on the calculus of variations and the quadratic functional

$$J[y] = \int_{\alpha}^{\beta} (p_2 y''^2 + p_1 y'^2 + p_0 y^2) dt.$$

3. The saddle point case. It is well known [6] that the oscillatory behavior of solutions of $y^{iv} + p_0(t)y = 0$ depends in a fundamental way on the sign of $p_0(t)$. In the context of (2.1), the case $p_0(t) < 0$ corresponds to making the origin a saddle point for the force field associated with

$$y'' = x; \quad x'' = -p_0(t)y.$$

As shown in [3], familiar criteria for the existence of conjugate points for $y^{iv} + p_0(t)y = 0$ can be extended to (2.1) in terms of rather general properties of force fields which possess sufficiently strong saddle points at $(x, y) = (0, 0)$.

In seeking to extend such criteria to systems of the form (2.6), we single out trajectories $\Gamma(v_0)$ determined by initial conditions of the form

$$y_1(\alpha) = y_1'(\alpha) = 0; \quad y_3(\alpha) = 1; \quad y_4(\alpha) = v_0. \quad (3.1)$$

The existence of a conjugate point for (2.6) then depends on being able to choose an "initial horizontal velocity" v_0 for which the associated trajectory $\Gamma(v_0)$ satisfies (2.4) and (2.5) for some $t > \alpha$. Because solutions of (2.6) have been shown to correspond to continuous trajectories which depend continuously on initial data, we shall be able to bring the techniques of [3] to bear in establishing criteria for the existence of such generalized conjugate points.

Denoting the open quadrants of the (x, y) -plane by I, II, III and IV, the following conditions are shown in [3] to ensure the existence of a conjugate point $\eta(\alpha)$ for (2.1):

- (A) If, for some $t_0 \geq \alpha$ the quantities $y(t_0)$, $y'(t_0)$, $x(t_0)$ and $x'(t_0)$ are all nonnegative (but not all zero), then $y(t)$, $y'(t)$, $x(t)$ and $x'(t)$ are all positive for $t > t_0$.
- (B) No trajectory $\Gamma(v_0)$ can remain in II for arbitrarily large values of t .
- (C) No trajectory in I satisfies
 - (i) $x(t) \downarrow x_0 \geq 0$ and $y(t) \uparrow \infty$ as $t \rightarrow \infty$, or
 - (ii) $y(t) \downarrow y_0 \geq 0$ and $x(t) \uparrow \infty$ as $t \rightarrow \infty$,
 nor can any trajectory in I tend to a finite limit point (x_0, y_0) in the closure of I as $t \rightarrow \infty$.
- (D) No trajectory can go directly from II to I to II.

To reformulate these hypotheses for (2.6), it is only necessary to replace y , y' , x and x' by y_1 , y_2 , y_3 and y_4 , respectively, and to regard I, II, III and IV as quadrants in the (y_1, y_3) -plane.

Theorem 3.1. *Suppose the coefficients $a(t)$, $b(t)$ and $c(t)$ are such that (2.1) satisfies hypotheses (A)–(D) above for some $\alpha \geq 0$. If the constants Q_{ij} , q_{ij} , R_{ij} , r_{ij} are nonnegative for $1 \leq i \leq n$, $1 \leq j \leq 4$, then (2.6) has a generalized conjugate point $\eta(\alpha)$.*

Proof: We begin by showing that the trajectories $\Gamma(v_0)$ of (2.6) satisfy appropriately modified conditions corresponding to (A)–(D). Because (B) and (C) are asymptotic in nature, they will not be influenced by a finite sequence of impulses. Also, while (D) is not fully satisfied as stated, (unless one hypothesizes that $R_{i2} = r_{i2} = 0$ for $1 \leq i \leq n$), a finite number of crossings from II to I to II are, in fact, allowable in [3]. Thus, (A) is the crucial “saddle point” condition which remains to be established for (2.6). But, if y_1, y_2, y_3 and y_4 are all positive for some t_0 , then the effect of the impulses corresponding to nonnegative coefficients in (2.6) can only serve to increase these quantities and thereby to reinforce the inequalities required in (A). Recalling now that the trajectories $\Gamma(v_0)$ are continuous and vary continuously with v_0 , it follows as in [3] that the sets

$$V_1 = \{v : \Gamma(v) \text{ enters } III \cup IV\}$$

and

$$V_2 = \{v : \Gamma(v) \text{ remains in the open upper half plane for all } t > \alpha\}$$

are open sets and that the trajectory $\Gamma(v_0)$ corresponding to

$$v_0 = \sup\{v : v \in V_1\}$$

lies in the closure of $I \cup III$ but does not enter $III \cup IV$. This shows that (2.6) has the desired conjugate point η which is characterized by (2.4) and (2.5) and realized by $\Gamma(v_0)$.

Example. 1. It is known [3] that the system (2.1) corresponding to

$$(p_2(t)y'')'' + p_0(t)y = 0$$

satisfies conditions (A)–(D) if $p_2(t) > 0$, $p_0(t) < 0$ and

$$\liminf_{t \rightarrow \infty} \left[t^2 \min \frac{1}{p_2(t)}, -p_0(t) \right] > \frac{1}{4}.$$

For such coefficients, we consider

$$(p_2(t)y'')'' + p_0(t)y = \delta'(t - 1)y + 3\delta(t - 1)y'.$$

Since $\delta'(t - 1)y = [\delta(t - 1)y]' - \delta(t - 1)y'$, this equation corresponds to the system

$$\begin{aligned} y'_1 &= y_2 \\ y'_2 &= \frac{1}{p_0(t)}y_3 \\ y'_3 &= y_4 + \delta(t - 1)y_1 \\ y'_4 &= -p_0(t)y_1 + 2\delta(t - 1)y_2. \end{aligned}$$

According to Theorem 3.1, this system has a generalized conjugate point.

4. The rotary case. In case $p(t) > 0$ for $0 \leq t < \infty$ the equation

$$y^{iv} + p(t)y = 0 \tag{4.1}$$

is disconjugate in the sense that nontrivial solutions of (4.1) cannot have more than one double zero in $[0, \infty)$ [6]. This phenomenon generalizes to (2.1) by noting that the condition $p(t) > 0$ corresponds to a rotary force field for the system

$$y'' = x; \quad x'' = -p(t)y.$$

Introducing polar coordinates

$$r^2 = x^2 + y^2; \quad \theta = \arctan y/x$$

into (2.1) yields [2]

$$\theta' = \frac{xy' - yx'}{x^2 + y^2} \quad (4.2)$$

and

$$(r^2\theta')' = bx^2 - cy^2 \quad (4.3)$$

or, using (2.2),

$$(r^2\theta')' = \frac{1}{p_2}x^2 + \left(p_0 + \frac{p_1''}{2} - \frac{p_1^2}{4p_2}\right)y^2. \quad (4.4)$$

If $p_0 \geq p_1^2/4p_2 - p_1''/2$ for all t , then (4.4) is nonnegative definite for $\alpha \leq t < \infty$ and the angular momentum of the particle of unit mass whose motion is described by (2.1) is nondecreasing. Since double zeros of $y(t)$ correspond to zeros of $r^2\theta'$, these conditions preclude the existence of conjugate points for (2.1).

Such disconjugacy criteria carry over readily to certain generalized fourth order equations. In particular, by specializing (2.6) to

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= ay_1 + by_3 + \sum_{i=1}^n q_i \delta(t - t_i) y_3 \\ y_3' &= y_4 \\ y_4' &= cy_1 + ay_3 + \sum_{i=1}^n r_i \delta(t - t_i) y_1 \end{aligned} \quad (4.5)$$

one can establish the following.

Theorem 4.1. *If the quadratic form*

$$Q(x, y) = b(t)x^2 - c(t)y_2$$

is nonnegative definite for $0 \leq t < \infty$ and if the impulses in (4.5) satisfy $q_i \geq 0$ and $r_i \leq 0$ for $1 \leq i \leq n$, then (4.5) is disconjugate on $[\alpha, \infty)$ for all $\alpha \geq 0$.

Proof: Defining $\theta(t) = \arctan y_1(t)/y_3(t)$ and $r(t) = \sqrt{y_1^2(t) + y_3^2(t)}$ we have

$$(r^2\theta')'(t) = Q(y_3, y_1) + \sum_{i=1}^n q_i \delta(t - t_i) y_3^2 - \sum_{i=1}^n r_i \delta(t - t_i) y_1^2.$$

Thus, $r^2\theta'$ is nondecreasing for $t_{i-1} < t < t_i$, while

$$(r^2\theta')(t_i^+) - (r^2\theta')(t_i^-) = \sum_{i=1}^n [q_i y_3^2(t_i^-) - r_i y_1^2(t_i^-)] \geq 0.$$

As a result, $r^2\theta'$ is nondecreasing for all $t > 0$, and this precludes the existence of classical or generalized conjugate points for any $\alpha \geq 0$.

Example. Consider the equation

$$y^{iv} - (p_1(t)y')' + p_0(t)y = 0 \quad (4.6)$$

where $p_1(t)$ satisfies

$$p_1(t) = \begin{cases} -t & 0 \leq t < 1 \\ t - 2 & 1 < t < \infty \end{cases}$$

and $p_0(t) > t^2/4$. Then, in the sense of distributions, $p_1''(t) = 2\delta(t-1)$ and according to (2.2), equation (4.6) corresponds to a system of the form (4.5) for which

$$a(t) \equiv 0; \quad b(t) \equiv 1; \quad c(t) = \frac{1}{4}p_1^2(t) - p_0(t); \quad r_1 = -1.$$

Since our hypotheses assure that $p_0(t) > \frac{1}{4}p_1^2(t)$ for all t and since $r_1 < 0$, equation (4.6) is disconjugate by Theorem 4.1.

5. Generalization. Certain aspects of this discussion lend themselves to direct generalizations. In [2], it is shown that nonselfadjoint fourth order differential equations

$$(p_2(t)y'' - s_2(t)y')'' - (p_1(t)y' - s_1(t)y)' + p_0(t)y = 0 \quad (5.1)$$

can be represented as systems of the form

$$\begin{aligned} y'' &= a(t) + b(t)x \\ x'' &= c(t)y + d(t)x \end{aligned} \quad (5.2)$$

in which the inequality $a(t) \neq d(t)$ reflects the nonselfadjointness of (5.1). Systems of this type can also be subject to impulses and used to study conjugacy properties of generalized differential equations corresponding to (5.1).

Conjugate points, as defined by (2.3) or by (2.4) and (2.5) can also be replaced by various focal points, such as the left focal point characterized (in the classical case) by

$$y(\alpha) = y'(\alpha) = 0 = (p_2y'')(t) = (p_2y'')'(t).$$

These can also be interpreted in terms of trajectories in the (x, y) -plane and analyzed by the techniques of §3 and §4.

Finally, one can consider the possibility of a countable set of impulses corresponding to terms such as $\sum_{i=1}^{\infty} Q_{ij}\delta(t-t_i)y_j$ in (2.6). For the "rotary case" considered in §4, such a generalization is immediate. However, in the case of the saddle points, the generalization of conditions (B) and (C) of [3] would have to be addressed.

REFERENCES

- [1] A. Halanay and D. Wexler, *Qualitative Theory of Systems with Impulses*, Editura Acad. Rep. Soc. Romania, Bucarest, 1968.
- [2] K. Kreith, *A nonselfadjoint dynamical system*, Proc. Edinburgh Math. Soc. 19 (1974), 77-87.

- [3] K. Kreith, *A dynamical criterion for conjugate points*, Pacific J. Math. 58 (1975), 123-132.
- [4] K. Kreith, *Sturm-Liouville oscillations in the presence of strong impulses*, Colloq. Math. Soc. Janos Bolyai 47 (1984), 573-589.
- [5] J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czech. Math. J. 82 (1957), 418-449.
- [6] W. Leighton & Z. Nehari, *On the oscillation of solutions of selfadjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. 89 (1958), 325-377.
- [7] S.G. Pandit & S.G. Deo, *Differential Systems Involving Impulses*, Lecture Notes in Math. 954, Springer Verlag, New York, 1952.
- [8] W.T. Reid, *Generalized linear differential systems*, J. Math. Mech. 8 (1959), 705-726.
- [9] S. Schwabik, *Generalized Differential Equations*, Naklad. Ceskoslovenske Akad. Ved, Prague, 1985.
- [10] W.M. Whyburn, *On selfadjoint ordinary differential equations of the fourth order*, Amer. J. Math. 52 (1930), 171-196.