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A DIFFERENTIABILITY RESULT FOR THE RELATIVE REARRANGEMENT

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(Submitted by: Roger Temam)

Motivation – Introduction. For solving the following plasmas physics model:

$$(P_1) \quad \begin{cases} -\Delta u + \lambda \dfrac{d^2 u_*}{ds^2}(\beta(u)(x)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

R. Temam [10] proposes to formulate the problem (P_1) as the Euler equations of the following variational problem

$$(P_2) \text{ Minimize} \qquad J(v) = \int_\Omega |\nabla v|^2\, dx + \lambda \int_{\Omega^*} |\frac{dv_*}{ds}|^2\, ds - 2\int_\Omega fv\, dx$$

over the set $K = \{v \in H_0^1(\Omega),\ v_* \in H^1(\Omega^*)\}$. Here, $\Omega^* = (0, \operatorname{meas}\Omega)$, v_* is the decreasing rearrangement of v.

But, from the results of Sperner [9], R. Temam and the author [8], it is known that if $u \in W^{1,p}(\Omega)$ then u_* is only in $W^{1,p}_{\text{loc}}(\Omega^*)$. More precisely, it is proved in [8] (see also [7]) that if Ω belongs to a class of "smooth" sets Σ_i, then

$$|k(\cdot)\frac{du_*}{ds}|_{L^p(\Omega^*)} \le Q(\Omega)\cdot|\nabla u|_{L^p(\Omega)}. \tag{1}$$

So it is natural to ask the following questions:

(Q_1) Does the inequality (1) hold for a class of sets other than Σ_i?
(Q_2) Does the smoothness of the domain really interfere in the regularity of u_*?
(Q_3) In view of using the set K, can we remove the singularity at $s = 0$ or $s = |\Omega|$ for u_*?

In the first section, we will answer the question (Q_1) by proving that if inequality (1) is valid for all u in $W^{1,1}(\Omega)$, then Ω belongs to Σ_i.

In the second section, we will exhibit some counterexamples showing the necessity of the smoothness of the domain. We begin the last section by showing that if Ω is in a class of sets Σ_i, u in $W^{1,p}(\Omega)$, $p > N$ then u_* is in $W^{1,q}(\Omega^*)$ for $1 \le q < 1/(1 + \frac{1}{p} - \frac{1}{N})$. We show by counterexample that result is sharp for the class of sets Σ_i.

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From this result, one can see that K is "small enough" since in deriving the Euler equation to the problem (P_2), one has to be able to say that $u+\epsilon v \in K$ for all $v \in C_0^\infty(\Omega)$ which is not true. This is one of the difficulties in the method proposed by Temam. Another difficulty in his approach is the use of the continuity of the map $u \to u_*$ which is actually an open problem.

For these reasons, I propose to study directly the variational Euler equation associated to the problem P_2 (or P_1). This equation can be formally written as:

$$(P_3) \quad \text{for all } v \in C_0^\infty(\Omega), \quad \int_\Omega \nabla v \cdot \nabla u \, dx + \lambda \int_{\Omega^*} \frac{dv_{*u}}{ds}\frac{du_*}{ds}\, ds - \int_\Omega f v \, dx = 0\,.$$

Again, this formulation brings up the question of differentiability not only for u_* but also for the more general concept of relative rearrangement v_{*_u}. Notice also that this formulation brings up the question Q_3. So in the last section, we will derive a differentiability result for the relative rearrangement v_{*_u}, first when u is a simple connected function; then for a larger class of function u, we use special approximations of u and v_{*_u}, to derive a differentiability result.

CONTENTS

0. Some useful definitions and results.

0.1 Relative rearrangement ([3], [4], [5], [6]). We will use only the Lebesgue measure on $\mathbb{R}^N$ and for a measurable set E, we will denote by $|E|$ its measure. If u is a measurable function defined on an open bounded set Ω, then u_* denotes the decreasing rearrangement of u defined on $\bar{\Omega}^* = [0, |\Omega|]$, that is, the generalized inverse of the distribution function $\mu(t) = |u > t|$.

Let Ω be an open bounded set of $\mathbb{R}^N$, u a measurable function on Ω and v in $L^1(\Omega)$. We define the function w on $\overline{\Omega^*} = [0, |\Omega|]$ by setting

$$w(s) = \begin{cases} \displaystyle\int_{u>u_*(s)} v(x)\, dx & \text{if } |u = u_*(s)| = 0 \\ \displaystyle\int_{u>u_*(s)} v(x)\, dx + \int_0^{s-|u>u_*(s)|} (v|_{P(s)})(\sigma)\, d\sigma & \text{if } |u=u_*(s)| \neq 0. \end{cases} \tag{0.1}$$

Here, $P(s) = \{u = u_*(s)\}$, $v|_{P(s)}$ is the restriction of v to the set $P(s)$.

Theorem 0.1. (see [3], [4]). *Let v be in $L^p(\Omega)$, $1 \le p \le +\infty$. Then,*

(i) $w \in W^{1,p}(\Omega^*)$

(ii) $|dw/ds|_{L^p(\Omega^*)} \le |v|_{L^p(\Omega)}$.

Definition 0.1. *The function dw/ds is called the relative rearrangement of v with respect to u and is denoted v_{*_u}.*

We will also use the notion of mean value operators introduced in [3]. In particular, we will use the following lemma (see [3], [6]).

Lemma 0.1. *Let u, v be two measurable functions from Ω into $\mathbb{R}$, v in $L^p(\Omega)$ $(1 < p \le +\infty)$ and g in $L^q(\Omega^*)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, the mean value operator $M_{u,v}$ satisfies*

$$M_{u,v}(g) \in L^q(\Omega) \quad \text{and} \quad \int_{\Omega^*} g v_{*_u}\, d\sigma = \int_\Omega M_{u,v}(g) v\, dx. \tag{0.2}$$

Definition 0.2. *We say that an open set $\Omega \subset \mathbb{R}^N$ $(N > 1)$ satisfies a relative isoperimetric inequality if there exists a constant $Q > 0$ depending only on Ω such that for every measurable set $E \subset \Omega$*

$$\min\left(|E|^{1-1/N}, (|\Omega| - |E|)^{1-1/N}\right) \le Q \cdot P_\Omega(E). \tag{0.3}$$

Definition 0.3. *A bounded open set Ω belongs to the class $\mathcal{J}_\alpha$ if there exists a constant $M \in (0, |\Omega|)$ such that*

$$\mathsf{U}_\alpha(M) = \sup \frac{|\mathcal{T}|^\alpha}{P_\Omega(\mathcal{T})} < +\infty,$$

where the supremum is taken over all open bounded set $\mathcal{T} \subset \Omega$ such that $\partial\mathcal{T} \cap \Omega$ is a manifold of class C^∞ and $|\mathcal{T}| \le M$.

Definition 0.4. *We define Σ_i as a class of sets which are open, bounded connected in $\mathbb{R}^N$ and which satisfy the relative isoperimetric inequality.*

Observe that $\Sigma_i \subset \mathcal{J}_{(N-1)/N}$. The following lemma is proved in [2].

Lemma 0.1. (see [2]). *A bounded domain having the cone property belongs to the class $\mathcal{J}_{(N-1)/N}$.*

Definition 0.5. *We denote $\mathcal{J}_\alpha^c$ all the elements of $\mathcal{J}_\alpha$ which are connected.*

Using Lemma 1.12 in [8] and the fact that $P_\Omega(E) = P_\Omega(\Omega - E)$, we deduce easily the:

Proposition 0.1.

$$\Sigma_i = \mathcal{J}^c_{(N-1)/N}.$$

The following theorem is also proved in [8] (see Theorem 1).

Theorem 0.2. (see [8]). *Suppose that $\Omega \in \Sigma_i$ and let u be in $W^{1,p}(\Omega)$, $1 \le p \le +\infty$. Then,*

$$\text{(i)} \qquad u_* \in W^{1,p}_{\text{loc}}(\Omega^*), \quad \Omega^* = (0, |\Omega|) \tag{0.4}$$

$$\text{(ii)} \qquad \left[\int_{\Omega^*} \bar{k}(\sigma)^p \left|\frac{du_*}{ds}\right|^p d\sigma\right]^{1/p} \le Q \big| |\nabla u|_{*_u} \big|_{L^p(\Omega^*)} \le Q |\nabla u|_{L^p(\Omega)}$$

where $\bar{k}(\sigma) = \min(\sigma^{1-1/N}, (|\Omega)-\sigma)^{1-1/N})$ and $|\nabla u|_{*_u}$ is the relative rearrangement of $|\nabla u|$.

1. Equivalence between an analytical property and geometrical property. In this paragraph, we propose to prove the converse of Theorem (0.2).

Theorem 1. *Suppose that for all $u \in W^{1,1}(\Omega)$, we have $u_* \in W^{1,1}_{\text{loc}}(\Omega^*)$ and inequality (0.4) of Theorem (0.4) holds. Then,*

(ii) *Ω is connected*

(iii) *Ω satisfies a relative isoperimetric inequality.*

The proof of the first part can be done easily by arguing by contradiction, while the second part of the theorem needs the following lemma.

Lemma 1. *Let Ω be an open bounded connected set and $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$. Then,*

(i) *For almost every $\theta \in (\text{ess inf}_\Omega, \text{ess sup}_\Omega\, u) = I(u)$*

$$\mu'(\theta)\frac{du_*}{ds}(\mu(\theta)) = 1$$

(ii) *If $E \subset \Omega^* = (0, |\Omega|)$ and $|E| = 0$. Then, $\{t \in I(u), \mu(t) \in E\} = \mu^{-1}(E)$ is also of measure zero.*

Proof: The proof of part (i) is the same as in (Lemma 1.3 [8]). The proof of part (ii) is the same as in (Lemma 1.5 [8]).

We also need the following approximation Lemma ([2]).

Lemma 2. [[2], p. 300, Theorem 6.1]. *Let $E \subset \Omega$ such that $P_\Omega(E) < +\infty$. Then there exists a sequence u_m such that:*

a) *$u_m \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$, $0 \le u_m \le 1$;*

b) *$\lim_{m\to+\infty} \int_\Omega |\nabla u_m|\, dx = P_\Omega(E)$;*

c) *$\forall\delta > 0$, $\exists m(\delta)$, $\forall m \ge m(\delta)$, $\| X_E - X_{\{u_m \ge \theta\}} \|_{L^1(\Omega)} \le \delta^{1/2}$ for all $\theta \in [\delta^{1/2}, 1-\delta^{1/2}]$.*

Here, X_A denotes the characteristic function of a set A.

Proof of Part (ii) of Theorem 1: Let us consider for $0 < \theta < 1$, and $0 < h < 1-\theta$, the Lipschitz function:

$$S_{\theta,h}(z) = \begin{cases} 0 & \text{if } z \le \theta, \\ 1 & \text{if } z \ge \theta + h \end{cases} \tag{1.1}$$

and affine elsewhere in $[\theta, \theta + h]$.

Consider also a set $E \subset \Omega$, such that $P_\Omega(E) < +\infty$ and $u_m \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ defined as in Lemma 2. Then, $v_m = S_{\theta,h}(u_m)$ is in $W^{1,1}(\Omega)$ and $v_{m^*} = S_{\theta,h}(u_{m^*})$ is in $W^{1,1}_{\text{loc}}(\Omega^*)$ (by assumption). From inequality (0.4), we then have

$$-\int_{\Omega^*} \bar{k}(\sigma)\frac{dv_{m^*}}{ds} \leq Q\int_\Omega |\nabla v_m|\,dx. \tag{1.2}$$

That is,

$$-\frac{1}{h}\int_{\theta<u_m\leq\theta+h} \bar{k}(\sigma)\frac{du_{m^*}}{ds}\,d\sigma \leq Q\frac{1}{h}\int_{\theta<u_m\leq\theta+h}|\nabla u_m|\,dx. \tag{1.3}$$

When h goes to zero, relation (1.3) becomes

$$\frac{d}{d\theta}\int_{u_{m^*}>\theta}\bar{k}(\sigma)\frac{du_{m^*}}{d\sigma}\,d\sigma \leq -Q\frac{d}{d\theta}\int_{u_m>\theta}|\nabla u_m|\,dx \quad \text{for a.e. } \theta. \tag{1.4}$$

On the other hand, for almost every θ in (0,1),

$$\int_{u_{m^*}>\theta}\bar{k}(\sigma)\frac{du_{m^*}}{d\sigma}\,d\sigma = \int_0^{|u_m>\theta|}\bar{k}(\sigma)\frac{du_{m^*}}{d\sigma}\,d\sigma. \tag{1.5}$$

If we set $\mu_m(\theta) = |u_m > \theta|$, then relation (1.5) and Lemma 1 imply, for almost every $\theta \in (0,1)$,

$$\frac{d}{d\theta}\int_{u_{m^*}>\theta}\bar{k}(\sigma)\frac{du_m}{d\sigma}\,d\sigma = \mu_m'(\theta)\frac{du_{m^*}}{d\sigma}(\mu_m(\theta))k(\mu_m(\theta)) = k(\mu_m(\theta)). \tag{1.6}$$

Inserting this last relation in (1.4) and integrating with respect to θ, one has[1]

$$\int_0^1 k(\mu_m(\theta))\,d\theta \leq Q\int^\Omega |\nabla u_m|\,dx\,. \tag{1.7}$$

Let us prove that

$$\lim_{m\to+\infty}\int_0^1 \bar{k}(\mu_m(\theta))\,d\theta = \bar{k}(|E|). \tag{1.8}$$

For this purpose, let us denote $\bar{\mu}_m(\theta) = |u_m \geq \theta|$. Then,

$$\int_0^1 \bar{k}(\mu_m(\theta))\,d\theta = \int_0^1 \bar{k}(\bar{\mu}_m(\theta))d\theta. \tag{1.9}$$

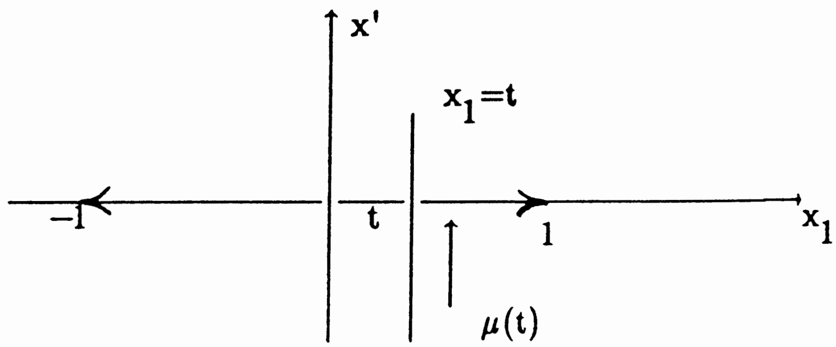
Let $\epsilon > 0$; then there exists a number $\delta : 0 < \delta < \inf(1,\epsilon)$ such that for all σ, σ', $|\sigma-\sigma'| \leq \delta^{1/2}$ we have $|\bar{k}(\sigma) - \bar{k}(\sigma')| \leq \epsilon$. From Lemma 2, there exists $m_\epsilon : \forall m \geq m_\epsilon$,

$$\big|\,|E| - |u_m \geq \theta|\,\big| \geq \delta^{1/2} \quad \text{for all } \theta \in (\delta^{1/2},\, 1-\delta^{1/2}).$$

Then,

$$\big|\bar{k}(|E|) - \bar{k}(\bar{\mu}_m(\theta))\big| \leq \epsilon \quad \text{for } \theta \in (\delta^{1/2},\, 1-\delta^{1/2}).$$

[1]The map $\theta \to \int_{u_m>0}|\nabla u_m|\,dx$ is absolutely continuous from Fleming-Rishel formula see [1].



An integration of this last relation leads to:

$$\left|\bar{k}(|E|) - \int_0^1 \bar{k}(\bar{\mu}_m(\theta))\,d\theta\right| \le C\epsilon^{1/2} \tag{1.10}$$

where C depends only on $\max_{\overline{\Omega^*}} \bar{k}$.

Taking the limit on (1.7) and using (1.8) and Lemma 2, one obtains for all measurable set $E \subset \Omega$:

$$\bar{k}(|E|) \le QP_\Omega(E).$$

Remark. One can prove that the converse theorem is true if we take Ω satisfying inequality (0.4) with $\bar{k}$ replaced by the weight $\bar{k}_\alpha(\sigma) = \min(\sigma^\alpha, (|\Omega|) - \sigma)^\alpha)$.

2. Counterexample proving the necessity of the smoothness of the domain.

Consider the following domain:

$$D_\alpha = \Big\{(x_1, x_2, \cdots, x_N) \in \mathbb{R}^N,\ \Big(\sum_{k=2}^N x_k^2\Big)^{1/2} < (1-x_1^2)^\alpha,\ -1 < x_1 < 1\Big\}. \tag{2.1}$$

For $\alpha > 0$, define

$$u(x_1, x') = x_1 \quad \text{for } (x_1, x') \in D_\alpha. \tag{2.2}$$

Proposition 1. *Let u and D_α be defined as in (2.1) and (2.2). Then,*

$$\frac{du_*}{ds}(s) \underset{s\to 0}{\sim} -\frac{C_\alpha}{s^\beta \alpha}$$

where

$$C_\alpha = \alpha_{N-1}^{-1} 2^{(1-N)\alpha}[\gamma_\alpha]^{\beta_\alpha}, \quad \beta_\alpha = \frac{(N-1)\alpha}{(N-1)\alpha+1}$$

$$\gamma_\alpha = \alpha_{N-1}\frac{2^{(N-1)^\alpha}}{(N-1)\alpha+1}, \quad \frac{du_*}{ds}(s) \underset{s\to|D_\alpha|}{\sim} -\frac{C_\alpha}{(|D_\alpha|-s)^\beta \alpha}.$$

Here, α_{N-1} denotes the measure of the unit ball in $\mathbb{R}^{N-1}$.

Proof: We have

$$\mu(t) = |u > t| = \alpha_{N-1}\int_t^1 (1-x_1^2)^{(N-1)\alpha}\,dx_1. \tag{2.3}$$

Since Ω is connected and $u \in C^\infty(\bar\Omega)$, then u_* is in $C(\bar\Omega^*)\cap W^{1,\infty}_{\rm loc}(\Omega^*)$ (see [8]). Moreover, u has no flat region, so we can deduce that

$$s = \alpha_{N-1}\int_{u_*(s)}^{1}(1-x_1^2)^{(N-1)\alpha}\,dx, \qquad \text{for all } s\in\overline{\Omega^*}. \tag{2.4}$$

Since $u_*(0)=1$, we deduce from (2.4) that

$$\frac{du_*}{ds}(s) = -\frac{1}{\alpha_{N-1}(1-u_*^2(s))^{(N-1)\alpha}} \underset{s\to 0}{\sim} -\frac{1}{\alpha_{N-1}2^{(N-1)\alpha}(1-u_*(s))^{(N-1)\alpha}}$$

and

$$s \underset{s\to 0}{\sim} \alpha_{N-1}2^{(N-1)\alpha}\int_0^{1-u_*(s)} t^{(N-1)\alpha}\,dt = \alpha_{N-1}\frac{2^{(N-1)\alpha}}{(N-1)\alpha+1}(1-u_*(s))^{(N-1)\alpha+1}.$$

Using these two last relations, we find

$$\frac{du_*}{ds}(s) \underset{s\to 0}{\sim} -\frac{C_\alpha}{s^{\beta_\alpha}}$$

with $C_\alpha = \alpha_{N-1}^{-1}2^{(1-N)\alpha}[\gamma_\alpha]^{\beta_\alpha}$

$$\gamma_\alpha = \alpha_{N-1}\frac{2^{(N-1)\alpha}}{(N-1)\alpha+1}, \quad \beta_\alpha = \frac{(N-1)\alpha}{(N-1)\alpha+1}.$$

Using the following relations

$$|D_\alpha| - s = \alpha_{N-1}\int_{-1}^{u_*(s)}(1-x_1^2)^{(N-1)\alpha}\,dx_1$$

the same argument leads to

$$\frac{du_*}{ds}(s) \underset{s\to|D|_\alpha}{\sim} -\frac{C_\alpha}{(|D_\alpha|-s)^{\beta_\alpha}}.$$

Corollary 1. *If $\alpha > 1$ then,*

$$\min(s^{1-1/N},\ (|D_\alpha|-s)^{1-1/N})\frac{du_*}{ds} \notin L^p(\Omega^*)$$

for $p \geq ((N-1)\alpha+1)/(N-1)(\alpha-1)$.

Notice that here $u \in C^\infty(\bar\Omega)$.

Proof: It is a direct consequence of Proposition 1.

3. Differentiability of the relative rearrangement.

3.0 Unweighted inequality for u_*.

Theorem 2. *Let Ω be in Σ_i and $u \in W^{1,p}(\Omega) N < p \leq +\infty$. Then,*

(i) $$u_* \in W^{1,q}(\Omega^*) \text{ with } 1 \leq q < \frac{1}{1+(1/p)-(1/N)} = q_0.$$

Moreover,

(ii) $$\left|\frac{du_*}{ds}\right|_{L^q(\Omega^*)} \leq Q \cdot C(N,p)|\Omega|^{\beta(N,p)} \left| |\nabla u|_{*_u} \right|_{L^p(\Omega^*)} \leq Q \cdot C(N,p)|\Omega|^{\beta(N,p)} |\nabla u|_{L^p(\Omega)}$$

with

$$C(N,p) = \frac{4}{[1-(1-\frac{1}{N})pq/(p-q)]^\nu}, \quad \nu = \frac{1}{p} - \frac{1}{q},$$

$$\beta(N,p) = \frac{1}{N} + \frac{1}{q} - \frac{1}{p} - 1, \quad Q = Q(\Omega).$$

Proof: Let $u \in W^{1,p}(\Omega)$, $p > N$ and $1 \leq q < q_0$. Since $du_*/ds \in L^q_{loc}(\Omega^*)$ (see Theorem 0.2 or [8]) we have to prove that $du_*/ds \in L^q(\Omega^*)$.

$$\left[\int_0^{|\Omega|} \left|\frac{du_*}{ds}\right|^q ds\right]^{1/q} = \left[\int_0^{|\Omega|} (\bar{k})^{-q} \left|\bar{k}\frac{du_*}{ds}\right|^q ds\right]^{1/q}. \tag{3.0}$$

From Hölder inequality and the estimate (0.4) in Theorem (0.2), (3.0) is less or equal to

$$\left[\int_0^{|\Omega|} (\bar{k})^{-q}(\sigma)^{(1-q/p)} d\sigma\right]^{p/q(p-q)} \left[\int_0^{|\Omega|} \left|\bar{k}\frac{du_*}{ds}\right|^p ds\right]^{1/p} \leq C(N,p)|\Omega|^{\beta(N,p)} Q \left| |\nabla u|_{*_u} \right|_{L^p(\Omega^*)}$$

with $C(N,p)$ and $\beta(N,p)$ given by (ii) Theorem 2. ∎

The exponent q_0 in Theorem 2 is sharp for the class of sets Σ_i. In fact, consider the given example in §2, relations (2.1) and (2.2).

Proposition 2. *Let $\alpha = 1$, $u(x_1, x') = x_1$. Then,*

$$\frac{du_*}{ds} \in L^q(\Omega^*) \quad \textit{for } 1 \leq q < \frac{N}{N-1} \textit{ and}$$

$$\frac{du_*}{ds} \notin L^q(\Omega^*) \quad \textit{for } q \geq \frac{N}{N-1}.$$

Proof: It is a direct consequence of proportional.

Remark. When I began to look for the class of functions u such that the first derivative of u_* does not have singularity at $s = 0$ and $s = |\Omega|$, I first looked at the class $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. However, one can find functions u in $W^{1,p}(B_1) \cap C(\bar{B}_1)$ where B_1 is the unit ball of $\mathbf{R}^N$, $p \leq N$ such that $u_* \notin W^{1,p}(B_1^*)$.

A first application of Theorem 2 is the differentiability of the relative rearrangement v_{*_u}.

3.1. Differentiability of v_{*_u} when u is a simple connected function.

Definition 1. *We will say that a function $u : \Omega \to \mathbf{R}$ is a simple connected function if:*

(i) *u is simple; that is, u assumes only a finite number of different values on Ω.*

(ii) *If $t_1, \cdots, t_n$ are all distinct values of u, then the interior of $\{u = t_k\}$, $k = 1, \cdots, n$ belongs to $\Sigma_i - \{\phi\}$.*

Theorem 3. *Let Ω be in Σ_i, v in $W^{1,p}(\Omega)$, $p > N$ and let u be a simple connected function on Ω. Then, there exists a finite sequence $\{a_k\}_{k=0,\cdots,n+1}$, $0 = a_{n+1} < a_n < \cdots < a_1 < a_0 = |\Omega|$ such that*

(i) v_{*_u} *belongs to* $W^{1,q}(a_{j+1}, a_j)$, $j = 0, \cdots, n$, $1 \le q < \dfrac{1}{1 + (1/p) - (1/N)}$. *Moreover,*

(ii) v_{*_u} *belongs to* $BV(\Omega^*)$ *and the total variation* $\int_{\Omega^*} |\frac{dv_{*_u}}{ds}|$ *satisfies*

$$\int_{\Omega^*} |\frac{dv_{*_u}}{ds}| \le \sum_{k=0}^{n-1} |\sup_{G_k} v - \inf_{G_{k+1}} v| + \sum_{k=1}^{n} \big[\sup_{G_k} v - \inf_{G_k} v\big].$$

Here, G_k denotes a plateau of u.

Proof: Let us set $G_k = \{u = t_k\}$, $t_0 < t_1 < \cdots < t_n$ and denote

$$a_k = \sum_{j=k}^{n} |G_j|, \quad a_n = |G_n|, \quad a_{n+1} = 0.$$

From the definition of the relative arrangement (see formula 0.1), if v is in $L^1(\Omega)$, then for all $s \in \Omega^*$

$$\begin{aligned} v_{*_u}(s) &= \big(v|_{G_k}\big)_*(s - a_{k+1}), \quad a_{k+1} \le s < a_k, \quad k = 1, \cdots, n-1, n \\ v_{*_u}(s) &= \big(v|_{G_0}\big)_*(s - a_1), \quad a_1 \le s \le a_0. \end{aligned} \tag{3.1}$$

Here, $\big(v|_{G_k}\big)_*$ is the decreasing rearrangement of the restriction of v to G_k.

From this last relation and Theorem 2, we get the statement (i). As for the second statement (ii), notice that if $0 \le s_0 < s_1 < \cdots < s_M \le |\Omega|$ is an arbitrary finite sequence in $\bar{\Omega}^*$ then:

$$\sum_{k=0}^{M-1} |v_{*_u}(s_{k+1}) - v_{*_u}(s_k)| \le \sum_{k=0}^{n-1} |\sup_{G_k} v - \inf_{G_{k+1}} v| + \sum_{k=0}^{n} \int_{a_{k+1}}^{a_k} |\frac{d}{ds}(v|_{G_k})_*(s - a_{k+1})|\, ds \tag{3.2}$$

by Theorem 2

$$\int_{a_{k+1}}^{a_k} |\frac{d}{ds}(v|_{G_k})_*(s - a_{k+1})|\, ds = \sup_{G_k} v - \inf_{G_k} v$$

we then get (ii).

Remark. Using Theorem 2, the last term in Relation (3.2) is less or equal to

$$C(N,p)|\Omega|^{\beta(N,p)} \sup_k Q(G_k) \sum_{k=0}^{n} |\nabla v|_{L^p(G_k)}.$$

As we see from this last theorem, we need to know the behavior of $Q(\Omega)$ when $|\Omega|$ changes. So, we introduce

3.2. A few estimates for the constant Q.

Proposition 3. *Let Ω be in Σ_i. Then, $N^{-1}\alpha_N^{-1/N} \leq Q$ where α_N is the measure of the unit ball.*

Proof: Consider a ball $\overline{B(x_0,R)} \subset \Omega$, $0 < |B(x_0,R)| \leq |\Omega|/2$. By the Definition (0.4) for Σ_i

$$Q \geq \frac{|B(x_0,R|^{1-1/N}}{P_\Omega(B(x_0,R))} = \frac{\alpha_N^{1-1/N}R^{N-1}}{N\alpha_N R^{N-1}} = N^{-1}\alpha_N^{-1/N}.$$

∎

For an upper bound to Q, we begin by a particular case.

Let T_{λ,x_0} be a transformation which is a composition of a translation and an homothetic (say $T_{\lambda,x_0}(x_0) = \lambda(x-x_0)+y_0$ for all $x \in \mathbb{R}^N$). Let us set $E_\lambda = T_{\lambda,x_0}(E)$. Then, $|E_\lambda| = \lambda^N|E|$ and $P_{\Omega_\lambda}(E_\lambda) = \lambda^{N-1}P_\Omega(E)$. From this observation, one can deduce that:

Proposition 4. *For all Ω_0 in Σ_i and for all transformation T_{λ,x_0}, we have*

$$T_{\lambda,x_0}(\Omega_0) \in \Sigma_i \quad \text{and} \quad Q(\Omega_0) = Q(T_{\lambda,x_0}(\Omega_0)).$$

For instance, if Ω_0 is the unit cube of $\mathbb{R}^N$ then, for all cube $\Omega, Q(\Omega) = Q(\Omega_0)$. We recover here also the fact that any ball $\Omega, Q(\Omega) = Q(\Omega_0)$ where Ω_0 is the unit ball.

The following general estimate can be sometimes useful

Theorem 4.

$$Q(\Omega) \leq 2\sup \frac{|u-\bar{u}|_{L^{N|(N-1)(\Omega)}}}{|\nabla u|_{L^1(\Omega)}}. \tag{3.3}$$

Here, $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx$ and the supremum is taken on all over the function u in $W^{1,1}(\Omega)$, $u \neq 0$, the set Ω is in Σ_i.

Remark. If Ω is a convex domain satisfying a cone property, then using an integral representation of $u \in C^1(\bar{\Omega})$ and a Sobolev inequality, one can prove that there exists a constant C_0 depending only on the cone which determines the "cone property" of Ω, such that for all u in $W^{1,1}(\Omega)$

$$|u-\bar{u}|_{L^{N/(N-1)}(\Omega)} \leq \big(C_0 + (\frac{\alpha_N}{|\Omega|})^{1-1/N}(\operatorname{diam}\Omega)^N\big)|\nabla v|_{L^1(\Omega)}.$$

So, $Q(\Omega) \leq 2[C_0 + (\frac{\alpha_N}{|\Omega|})^{1-1/N}(\operatorname{diam}\Omega)^N]$.

The proof of Theorem 4 uses the following approximation lemma.

Lemma 2. (see [2]). *Let $\mathcal{O}$ be an open set such that $\partial\mathcal{O} \cap \Omega$ is a manifold of class C^∞. Then, there exists a sequence u_m with the property that:*

1. *u_m is in $W^{1,\infty}(\Omega)$*
2. *$u_m = 0$ in $\Omega \backslash \mathcal{O}$*
3. *u_m is in $[0,1]$ in Ω*
4. *For any compactum $K \subset \mathcal{O}$, $\exists m_k$, $u_m(x) = 1$ $\forall m \geq m_k$, $\forall x \in K$*
5. $\limsup_{m\to+\infty}\int_\Omega |\nabla u_m|\,dx = P_\Omega(\mathcal{O})$.

Proof of Theorem 4: Let

$$C_1 = \sup_{u \in W^{1,1}(\Omega)} \frac{|u - \bar{u}|_L N|(N-1)_{(\Omega)}}{|\nabla u|_{L^1(\Omega)}}\,. \tag{3.4}$$

Consider $\mathcal{O}$ an open set in Ω such that $\partial\mathcal{O} \cap \Omega$ is a manifold of class C^∞ and consider also the sequence u_m defined in Lemma 2. Then, from (3.4):

$$\left(\int_\Omega |u_m - \bar{u}_m|^{N/(N-1)}\right)^{1-1/N} \leq C_1 \int_\Omega |\nabla u_m|\, dx. \tag{3.5}$$

Using the properties 2 and 4 of Lemma 2, one obtains

$$\limsup_{m \to +\infty} \bar{u}_m = \frac{|\mathcal{O}|}{|\Omega|}\,. \tag{3.6}$$

The Relation (3.5), (3.6) and property 5 of Lemma 2 yield

$$\left(\int_\mathcal{O} \left|1 - \frac{|\mathcal{O}|}{|\Omega|}\right|^q dx\right)^{1/q} \leq C_1 P_\Omega(\mathcal{O}), \quad q = \frac{N}{N-1}\,. \tag{3.7}$$

(3.7) implies for, $|\mathcal{O}| \leq |\Omega|/2$,

$$\frac{1}{2}|\mathcal{O}|^{1/q} \leq \frac{|\Omega|\,|\mathcal{O}|}{|\Omega|}|\mathcal{O}|^{1/q} \leq C_1 P_\Omega(\mathcal{O}). \tag{3.8}$$

By approximation lemma (see [2], p. 300), we deduce that for all $E \subset \Omega$, $|E| \leq \frac{|\Omega|}{2}$, $|E|^{1-1/N} \leq 2C_1 P_\Omega(E)$. This implies that the best constant $Q(\Omega) \leq 2C_1(\Omega)$. ∎

As one expects, to get a differentiability result for v_{*_u}, one has to take an approximation of u, say u_n, such that $v_{*_{u_n}}$ converges to v_{*_u} in an appropriate sense and u_n is a simple connected function. To insure the convergence of $v_{*_{u_n}}$, the sequence u_n must be chosen carefully. Thus, we introduce the following approximation lemmas.

3.3 Special approximation of u and v_{*_u}. In this paragraph, we denote by $P_t(u) = \{u = t\}$ a plateau of u (u is measurable) $P(u) = \cup_{t \in D_u} P_t(u)$, D_u is at most countable. If D_u is countable, then we set

$$D_u = \{t_0, t_1, \cdots, t_k, \cdots\}, \quad t_i \neq t_j, \quad D_u^m = \{t_k,\ k \leq m\}.$$

We also assume that D_u does not contain the values $-\infty$ and $+\infty$.

Lemma 3. *Let u be a measurable function on Ω. For any sequence of simple functions $\{\bar{u}_n\}$ converging to u almost everywhere, one can associate a sequence of simple functions $\{u_n\}$ such that:*

(i) *u_n converge to u almost everywhere on $\Omega/P(u)$*

(ii) *For all $t \in D_u \cap$ range (u_n), $\{u_n = t\} = \{u = t\}$, $|u_n > t| = |u > t|$.*

Proof: Set $u_0 = u\backslash_{\Omega|P(u)}$ (restriction of u on $\Omega\backslash P(u)$) and $u_{0n} = \bar{u}_n\backslash_{\Omega/P(u)}$. Then, u_{0n} converge to u_0 almost everywhere. If the set range $(u_{0n}) \cap D_u$ is non empty, then we redefine u_{0n} as follows:

For each $t_r \in$ range $(u_{0n}) \cap D_u$, consider an element t'_r in the dense set $\mathbb{R} - D_u$ such that $t'_r > t_r$ and $|t'_r - t_r| < 1/2^{n+1}$. Then, replace t_r by t'_r. Such a process will provide a new sequence, still denoted u_{0n}, such that range $(u_{0n}) \cap D_u = \phi$.

Let $t \in D_u$ and consider u_1^{nt} the smallest value in the range of u_{0n} such that $u_1^{nt} > t$, also consider u_2^{nt} the greatest value in the range of $u_{0n} : u_2^{nt} < t$. If the set $I_n^t = \{u_{0n} < t\} \cap \{u_0 > t\}$ is non void, then redefine u_{0n} on I_n^t by setting $x \in I_n^t$, $u_{0n}(x) = u_1^{nt}$. If the set $J_n^t = \{u_{0n} > t\} \cap \{u_0 \le t\}$ is non void, then redefine u_{0n} on J_n^t by setting $x \in J_n^t$, $u_{0n}(x) = u_2^{nt}$. Such a process will provide a new sequence, still denote u_{0n}, such that

$$\text{for all } t \in D_u : \{u_{0n} > t\} = \{u_0 > t\}$$

and u_{0n} converge to u_0 almost everywhere. If D_u is finite then define u_n by

$$u_n(x) = \begin{cases} u_{0n}(x) & \text{for } x \in \Omega\backslash P(u) \\ u(x) = t & \text{for } x \in P_t(u) \end{cases} \tag{3.9}$$

Then $D_u \cap$ range $(u_n) = D_u$ (since range $(u_{0n}) \cap D_u = \phi$); this means $\{u_n = t\} = \{u = t\}$ for all $t \in D_u$, and

$$\begin{aligned} |u_n > t| &= |u_n > t| + |x \in P(u), u_n(x) > t| \\ &= |u_0 > t| + |x \in P(u), u(x) > t| = |u > t|. \end{aligned}$$

Notice that in this case u_n converges to u almost everywhere on Ω. If D_u is countable, $D_u = \{t_0, t_1, \cdots, t_k, \cdots\}$, $t_k \neq \mp\infty$ and $t_i \neq t_j$. Then for a fixed n consider $D_u^n = \{t_k,\ 0 \le k \le n\}$ and also a rearrangement of D_u^n, $t_{\sigma(0)} < t_{\sigma(1)} < \cdots < t_{\sigma(n)}$. One can find a sequence $t' \in \mathbb{R} - D_u$ such that:

$$t'_{-1} < t_{\sigma(0)} < t'_0 < t_{\sigma(1)} < t'_1 < \cdots < t'_{\sigma(n-1)} < t'_{n-1} < t_{\sigma(n)} < t'_n.$$

Let us now define u_n by

$$u_n(x) = \begin{cases} u_{0n}(x) & \text{if } x \text{ is in } \Omega\backslash P(u) \\ u(x) & \text{if } x \in P_t(u),\ t \in D_u^n \\ t'_k & \text{if } x \in P_t(u),\ t_{\sigma(k)} < t < t_{\sigma(k+1)},\ k = 0, \cdots, n-1 \\ t'_{-1} & \text{if } x \in P_t(u),\ t < t_{\sigma(0)} \\ t'_n & \text{if } x \in P_t(u),\ t > t_{\sigma(n)}\,. \end{cases} \tag{3.10}$$

Then, $D_u \cap$ range $(u_n) = D_u^n$ which insures

$$\{u_n = t\} = \{u = t\} \quad \text{for all } t \in D_u^n.$$

Moreover, for $t \in D_u^n$, $\exists j : t_{\sigma(j)} = t$. Then,

$$\begin{aligned} |u_n > t| = |u_n > t_{\sigma(j)}| &= |u_{0n} > t| + \sum_{\tau > t_{\sigma(j)}} |P_\tau(u)| \\ &= |u_0 > t| + |x \in P(u), u(x) > t_{\sigma(j)}| = |u > t|. \end{aligned}$$

Corollary 2. *Let u be in $L^1(\Omega)$ such that D_u remains in a bounded set of $\mathbb{R}$. Then there exists a sequence of simple functions u_n:*

(i) *u_n converges to u in $L^1(\Omega)$*

(ii) *For all $t \in D_u \cap \text{range}(u_n)$, $\{u_n = t\} = \{u = t\}$, $|u_n > t| = |u > t|$.*

Proof: Let a, b be such that $D_u \subset [a, b] \subset \mathbb{R}$. A slight modification in the construction of u_n in Lemma 3 leads to a sequence of simple functions such that:

$$|u_n(x)| \leq |u(x)| + \frac{1}{2^{n+1}} \quad \text{a.e. on } \Omega/P(u)$$

and

$$u_n(x) \in [a, b] \quad \text{for a.e. } x \in P(u)$$

$$\int_\Omega |u_n(x) - u(x)|\, dx = \int_{\Omega/P(u)} |u_n(x) - u(x)|\, dx + \int_{P(u)} |u_n(x) - u(x)|\, dx. \tag{3.11}$$

By Lebesgue's theorem the first term of relation (3.11) tends to zero when n goes to infinity, while the second terms can be written as:

$$\int_{P(u)} |u_n(x) - u(x)|\, dx = \int_{P(u) - P^n(u)} |u_n(x) - u(x)|\, dx. \tag{3.12}$$

Here, $P^n(u) = \bigcup_{t \in D_u^n} P_t(u)$, $P^n(u) \subset P^{n+1}(u) \subset \bigcup_{k \geq 0} P^k(u) = P(u)$. Knowing that $u_n(x)$ and $u(x)$ remains in $[a, b]$ for $x \in P(u)$, Relation (3.12) implies

$$\int_{P(u)} |u_n(x) - u(x)|\, dx \leq 2 \sup(|a|, |b|) \cdot |P(u) - P^n(u)|.$$

By the measure theory, this last term goes to zero when n increases to infinity.

Theorem 5. (Approximation of v_{*_u}). *Let u be a measurable function on Ω such that $\pm\infty \notin D_u$. Then there exists a sequence u_n of simple functions such that:*

(i) *u_n converge to u almost everywhere on $\Omega \backslash P(u)$ and almost everwhere on Ω if D_u is finite.*

(ii) *For all v in $L^p(\Omega)$, $1 < p \leq +\infty$, $v_{*_{u_n}} \underset{n \to +\infty}{\rightharpoonup} v_{*_u}$ in $L^p(\Omega^*)$ weakly if $p < +\infty$, weak $-^*$ if $p = +\infty$.*

(iii) *$v_{*_{u_n}}$ and v are equimeasurable.*

Proof: Let $g \in C(\overline{\Omega^*})$, $M_{u,v}(g)$ will denote the value of the mean value operator at g.

Consider the sequence $\{u_n\}$ constructed in Lemma 3. Then u_n satisfies (i). Moreover, we know that for all $t \in D_u^n = \text{range}\,(u_n) \cap D_u$, we have

$$\{u_n = t\} = \{u = t\} \quad \text{and} \quad |u_n > t| = |u > t|. \tag{3.13}$$

This relation (3.13) implies that for almost all x in $P^n(u) = \bigcup_{t \in D_u^n} P_t(u)$

$$M_{u_n,v}(g)(x) = M_{u,v}(g)(x). \tag{3.14}$$

Since $\lim_{n \to +\infty} u_n(x) = u(x)$ almost everywhere on $\Omega \backslash P(u)$ we then have almost everywhere in $\Omega \backslash P(u)$

$$\lim_{n \to +\infty} M_{u_n,v}(g)(x) = M_{u,v}(g)(x). \tag{3.15}$$

Combining (3.14) with (3.15), one gets, with the help of Lemma (0.1),

$$\lim_{n\to+\infty}\int_{\Omega^*} v_{*u_n}(s)g(s)\,ds = \int_\Omega M_{u,v}(g)v dx = \int_{\Omega^*} v_{*u} g\,ds \tag{3.16}$$

for all $g \in C(\overline{\Omega^*})$. Using Theorem 1 and density result, one can see that (3.16) remains true for all g in $L^{p'}(\Omega^*)$, $(1/p') + (1/p) = 1$.

Since u_n is a simple function, formula (3.1) implies that

$$|v_{*u_n} > \sigma| = \sum_{k=0}^{m}\int_{a_{k+1}^n}^{a_k^n} H\big(((v|_{G_k^n})_*)(s - a_{k+1}^n) - \sigma\big)\,ds \tag{3.17}$$

(here, $G_k^n = \{u_n = t_k^n\}$, $t_0^n < \cdots < t_m^n$, (see Theorem (0.3) for the definition of a_{k+1}^n), H appropriate heaveside function. By change of variable in (3.17) and using the equimeasurability of the monotone rearrangement, one has

$$|v_{*u_n} > \sigma| = \sum_{k=0}^{m}\int_{G_k^n} H\big(v|_{G_k^n} - \sigma\big)\,dx = |v > \sigma|, \quad \text{for all } \sigma \in \mathbb{R}$$

the same proof holds for $|v_{*u_n} \le \sigma| = |v \le \sigma|$.

Remark.

(a) If $p = 1$, one has for all $g \in C_c(\Omega^*)$

$$\lim_{n\to+\infty}\int_{\Omega^*} v_{*u_n} g\,ds = \int_{\Omega^*} v_{*u} g\,ds$$

and $|v_{*u_n}|_{L^1(\Omega^*)} = |v|_{L^1(\Omega)}$.

The proof is the same as before, Lemma (0.1) for the mean value operators remains true in these conditions.

(b) As we notice in Corollary 2, we can choose u_n such that u_n converges to u in $L^1(\Omega)$ if $u \in L^1(\Omega)$ and D_u is bounded.

Theorem 5 will be the starting point for extending Theorem 3. We provide below a few results concerning the regularity of v_{*u} for a wider class of u. Further investigation will be done later.

3.4 Differentiability of v_{*u} for a larger class of u. Let u be as in Theorem 5 and let $v \in W^{1,p}(\Omega)$, $p > N$.

(H_1) Assume that the sequence u_n defined in Theorem 5 is a sequence of simple connected functions and that

(H_2) there are two constants A and B depending only on u, v and Ω such that

$$\sum_{k=0}^{m-1}\Big|\sup_{G_k^n} v - \inf_{G_{k+1}^n}\Big| \le A, \quad \sum_{k=0}^{m}\big[\sup_{G_k^n} v - \inf_{G_k^n} v\big] \le A$$

$$G_k^n = \{u_n = t_k^n\}, \quad t_0^n < \cdots < t_m^n$$

or

$$\sup_{k,n}\big[Q(G_k^n)\big] \le B$$

and

$$\sum_{k=0}^{m}|\nabla v|_{L^p(G_k^n)} \le B.$$

Theorem 6. *Assume* (H_1) *and* (H_2)*. Then,*

(i) $v_{*_u} \in BV(\Omega^*)$

(ii) $v_{*_{u_n}}$ *converge strongly to* v_{*_u} *in* $L^q(\Omega^*)$ *for all* $1 \leq q < +\infty$ *and* v_{*_u}, $v_{*_{u_n}}$, v *are equimeasurable.*

Proof: From Theorem 3 on the estimate of the total variation $\int_{\Omega^*} |\frac{d}{ds} v_{*_{u_n}}|$, we have

$$\int_{\Omega^*} |\frac{dv_{*_{u_n}}}{ds}| \leq A'$$

and $|v_{*_{u_n}}|_{L^q} = |v|_{L^q}$ (Theorem 5, (iii), for all $q \in [1, +\infty]$). Then, $v_{*_{u_n}}$ remains in a bounded set of $BV(\Omega^*)$, since $v_{*_{u_n}}$ converge to v_{*_u} weakly (by Theorem 5); we conclude that, $v_{*_{u_n}}$ converges to v_{*_u} weakly in $BV(\Omega^*)$. So, we get (i). By compactness, $v_{*_{u_n}} \to v_{*_u}$ in $L^\sigma(\Omega^*)$ (for $1 \leq \sigma < \frac{N}{N-1}$) strongly. Since

$$\int_{\Omega^*} |v_{*_{u_n}} - v_{*_u}|^q \, ds \leq (2 \sup |v|)^{q-\sigma} \int_{\Omega^*} |v_{*_{u_n}} - v_{*_u}|^\sigma \, ds$$

for $q \geq \sigma$, we deduce $v_{*_{u_n}}$ converges to v_{*_u} strongly in $L^q(\Omega^*)$ for all $q < +\infty$. Since $[v_{*_{u_n}}]_* = v_*$ implies $[v_{*_u}]_* = v_*$, then the equimeasurability. ∎

Naturally, the question is: what are those functions (u, v) satisfying (H_1) and (H_2). An investigation in this sense will be done later. As a conclusion, let us anticipate briefly the case $N = 1$. In this case, $Q = 1$ (see [9]) and if u is a monotone function, v is in $W^{1,1}(\Omega)$, $\Omega = (a, b)$, then for any sequence of simple functions $\{u_n\}$ (u_n is monotone as u), we have:

$$\sum_{k=0}^{m-1} \left| \sup_{G_k^n} v - \inf_{G_{k+1}^n} v \right| \leq \sum_{k=0}^{m-1} \int_{G_k^n \cup G_{k+1}^n} |\nabla v| \, dx \leq 2 \int_\Omega |\nabla v| \, dx = A.$$

Here, $G_k^n = \{u_n = t_k^n\}$. Thus we get:

Corollary 3. *Let* $N = 1$, $\Omega = (a, b)$*. Then, for all* v *in* $W^{1,1}(\Omega)$ *and* u *monotone on* (a, b) v_{*_u} *belongs to* $BV(0, b - a)$*.*

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