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A NONLINEAR SYSTEM FOR PHASE CHANGE WITH DISSIPATION

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Abstract. A nonlinear system modeling a phase change problem with dissipation is investigated. The model is derived with the thermodynamical theory of continuum mechanics; it includes among other features superheating and supercooling effects or irreversible phase changes. One of the equations is doubly nonlinear, with a nonlinearity on the time derivative of one of the unknowns and the resulting system is non-monotone in $(L^2)^2$ or in $(H^{-1})^2$. Uniqueness of the solution is proved using the accretivity of the system in $(L^1)^2$. The existence of a solution is shown through a regularization of one the nonlinearities. In this case, such a method permits to weaken the customary assumption of the L^1-framework.

1. Introduction In this paper, existence and uniqueness of the solution of a nonlinear system modelling some dissipative phase change phenomena is established. The system investigated is the following:

$$c(x)\frac{d\theta}{dt} + \frac{d\chi}{dt} - \operatorname{div}(k(x)\nabla\theta) = 0 \quad \text{in } \Omega \times (0,T) \tag{1.1}$$

$$\frac{d\chi}{dt} + \partial\phi(x, \frac{d\chi}{dt}) + \partial\psi(x,\chi) \ni \theta \quad \text{in } \Omega \times (0,T) \tag{1.2}$$

$$\theta = 0 \quad \text{on } \partial\Omega \times (0,T),$$

$$\theta(t=0) = \theta_0, \quad \chi(t=0) = \chi_0 \quad \text{in } \Omega,$$

where $\partial\phi(x,\cdot)$ and $\partial\psi(x,\cdot)$ are the subdifferentials of two convex functions $\psi(x,\cdot)$ and $\phi(x,\cdot)$ on $\mathbb{R}$.

Section 2 is devoted to the introduction of the physical problem under investigation. Also the usual techniques of thermodynamics are used to derive the model (1.1) and (1.2) in this section. Thermodynamical considerations allow one to make precise some properties

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of $\phi(x,\cdot)$ and $\psi(x,\cdot)$ (this is achieved by using the method developed by [4], [8], [9]). The equations (1.1)-(1.2) are shown to cover some usual models for phase changes with dissipation; for example, superheating and supercooling models [16] or irreversible phase change models [4], [8].

In Section 3, existence and uniqueness of the solution of the nonlinear system (1.1)-(1.2) is shown. This system is not monotone on $(L^2(\Omega))^2$, but we show that it is accretive in $(L^1(\Omega))^2$, which provides uniqueness. For this existence part, we use regularization and L^2-techniques rather than nonlinear semi-groups theory in L^1 as developed in [3], [6], although this could lead to an existence result.

2. Modeling the physical problem. As usual, the equations for the model are the energy conservation equation and the constitutive law linking the variables. For this type of problems, the variables are the absolute temperature field T (which is strictly positive) and the proportion χ of the one of the two phases into which the medium tends to transform when the temperature T is greater than a fixed temperature T_0 referred to as the phase change temperature. The equations are derived through the usual techniques of continuum mechanics. For completeness's sake, we now briefly recall these techniques.

The first equation is the usual energy conservation law. It reads as

$$\frac{de}{dt} + \operatorname{div} q = 0, \tag{2.1}$$

where $e(T,\chi)$ is the internal energy and q is the heat flux. We have assumed that there is no volumic rate of heat production; the presence of such a term would give a non-zero right-hand side in (2.1) which would lead to very few modifications to the following analysis, without change in the result.

Obviously, one needs two more equations relating q, T and χ. Furthermore, one needs the expression of e as a function of T and χ. These relations are obtained by looking at the thermodynamics of the physical problem. In the framework of thermodynamics of continuum mechanics, and especially in the standard material approach, the medium is defined by two potentials $\Psi(x,T,\chi)$ and $\phi(x,d\chi/dt)$, respectively, referred to as the free energy and the internal dissipation potential [9], [10]. The function $\phi(x,\cdot)$ is assumed to be convex, positive with $\phi(x,0)=0$.

The dependence of $\psi(x,\cdot)$ and $\phi(x,\cdot)$ on the space variable x allows one to consider an inhomogeneous medium. The evolutions of variables are then restricted to satisfy the following relations:

$$e(x,T,\chi) = \Psi(x,T,\chi) - T\frac{\partial\Psi}{\partial T}(x,T,\chi), \tag{2.2}$$

$$-\frac{\partial\Psi}{\partial T}(x,T,\chi) = \frac{\partial\phi}{\partial(d\chi/dt)}\Big(x,\frac{d\chi}{dt}\Big). \tag{2.3}$$

If relations (2.2) and (2.3) hold and if

$$q\nabla\frac{1}{T} \geq 0, \tag{2.4}$$

it can be shown (see [8]) that the second law of thermodynamics is satisfied.

To insure (2.4) we use the classical Fourier law

$$q = -k(x)\nabla T \tag{2.5}$$

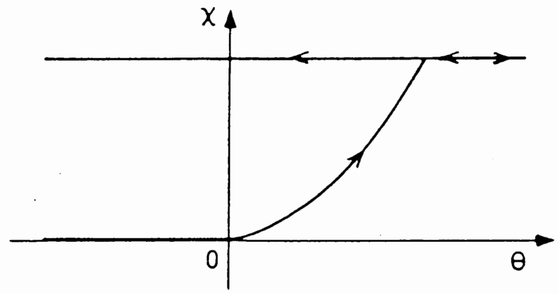

Figure 1. Irreversible phase change.

where the heat conductivity $k(x)$ is positive and may depend on the space variable x.

For the problem under consideration, the following choices of $\Psi(x,T,\chi)$ and $\phi(x, d\chi/dt)$ are convenient to obtain the desired behavior of the variables (see [4], [8] for details),

$$\Psi(x,T,\chi) = -c(x)T\mathrm{Log}\,T + T\psi(x,\chi) - \frac{L}{T_0}(T-T_0)\chi, \tag{2.6}$$

$$\Phi\left(x,\frac{d\chi}{dt}\right) = \frac{1}{2}\left(\frac{d\chi}{dt}\right)^2 + \phi\left(x,\frac{d\chi}{dt}\right), \tag{2.7}$$

where $c(x)$ is the heat capacity of the medium and L is the latent heat of the phase change transformation.

With expressions (2.6)-(2.7), the equations of the models (2.1), (2.2), (2.3), (2.4) and (2.5) become

$$c(x)\frac{d\theta}{dt} + L\frac{d\chi}{dt} - \mathrm{div}\,(k(x)\nabla\theta) = 0, \tag{2.8}$$

$$\frac{d\chi}{dt} + \partial\phi\left(x,\frac{d\chi}{dt}\right) + \partial\psi(x,\chi) \ni \theta, \tag{2.9}$$

where θ is equal to $T - T_0$. In (2.9), the term $\partial\psi$ should read as $T\partial\psi$, which is permissible in the few examples described below; in a general case, this simplification is merely an approximation of the general term.

The model has to be completed with boundary and initial conditions which will be stated later.

Let us give some examples of behavior corresponding to special choices for $\phi(x,\cdot)$ and $\psi(x,\cdot)$.

• If $\phi(x,\cdot)$ is set to be identically equal to 0 and $\psi(x,\cdot)$ is the indicator function $I_{[0,1]}$ of the closed interval $[0,1]$, (2.9) leads to

$$\frac{d\chi}{dt} + \partial I(\chi) \ni \theta. \tag{2.9$_i$}$$

The model (2.8) and $(2.9)_i$ describes some superheating and supercooling effects and has been investigated, for example, in [16].

• If $\phi(x,\cdot)$ is set to be the indicator function $I_{[0,+\infty)}$ of the interval $[0,+\infty)$ and $\psi(x,\cdot)$ is equal to $I_{[0,1]}$ as before, (2.9) becomes:

$$\frac{d\chi}{dt} + \partial I_{[0,+\infty)}\left(\frac{d\chi}{dt}\right) + \partial I_{[0,1]}(\chi) \ni \theta. \tag{2.9$_{ii}$}$$

The model (2.8)-$(2.9)_{ii}$ corresponds to irreversible phase changes because $(2.9)_{ii}$ forces $d\chi/dt$ to be positive which means that the medium can transform from phase 1 ($\{\chi = 0\}$) to phase 2 ($\{\chi = 1\}$) but the reverse is impossible. This behavior is illustrated in figure 1.

3. Existence and uniqueness result. Let Ω be a "regular" domain in $\mathbb{R}^N$, with boundary $\partial\Omega$, and $[0,T]$ be the interval of time t where the evolution of the medium occupying the domain Ω is to be studied (no confusion between T and the absolute temperature may occur since the absolute temperature will not be used explicitly in the sequel).

The boundary and initial conditions are

$$\begin{aligned} &\theta = 0 && \text{on } \partial\Omega, \\ &\theta(t=0) = \theta_0 && \text{in } \Omega, \\ &\chi(t=0) = \chi_0 && \text{in } \Omega, \end{aligned}$$

where θ_0 and χ_0 are given.

Assumptions on the data.

H1 The functions $c(x)$ and $k(x)$ belong to $C^1(\bar{\Omega})$ and $L^\infty(\Omega)$, respectively, and there exist two strictly positive constants such that

$$\begin{aligned} &\bar{c} \leq c(x) && \text{a.e. in } \Omega, \\ &\bar{k} \leq k(x) && \text{a.e. in } \Omega. \end{aligned}$$

H2 The functions $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are two normal convex integrands on $\Omega \times \mathbb{R}$ with values in $\mathbb{R} \cup \{+\infty\}$ ([7], [15]) such that

(i) $\phi(x,\cdot)$ is positive and $\phi(x,0) = 0$,

(ii) there exist two functions $a(x)$ in $L^2(\Omega)$ and $b(x)$ in $L^1(\Omega)$ such that for every $r \in \mathbb{R}$,

$$\psi(x,r) + a(x)r + b(x) \geq 0 \quad \text{a.e. in } \Omega.$$

(iii) for each $x \in \Omega$

. either $\sup D(\partial\phi(x,\cdot)) > 0$ or $\inf D(\partial\psi(x,\cdot)) \in D(\partial\psi(x,\cdot))$

and

. either $\inf D(\partial\phi(x,\cdot)) < 0$ or $\sup D(\partial\psi(x,\cdot)) \in D(\partial\psi(x,\cdot))$

where $D(\partial\phi(x,\cdot))$ and $D(\partial\psi(x,\cdot))$ denote, respectively, the domains of $\partial\phi(x,\cdot)$ and $\partial\psi(x,\cdot)$.

H3 The initial data θ_0 and χ_0 lie in $L^2(\Omega)$ and $\psi(x,\chi_0)$ belongs to $L^1(\Omega)$.

Remark 3.1. Assumption H2 iii) seems technical but is natural in this type of problems and is implied by the condition $D(\partial\phi) + D(\partial\psi) = \mathbb{R}$ (see [3] for a more general set-up).

We now prove the following existence and uniqueness result.

Theorem 3.1. *Under the assumptions H1, H2 and H3, there exists a unique solution* (θ,χ), *with the regularity*

$$\theta \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \tag{3.1}$$

and

$$\chi \in W^{1,2}(0,T;L^2(\Omega)), \tag{3.2}$$

of the problem

$$c(x)\frac{d\theta}{dt} + \frac{d\chi}{dt} - div\,(k(x)\nabla\theta) = 0 \quad in\ (0,T)\times\Omega, \tag{3.3}$$

$$\frac{d\chi}{dt} + \partial\phi(x,\frac{d\chi}{dt}) + \partial\psi(x,\chi) \ni \theta \quad a.e.\ in\ (0,T)\times\Omega, \tag{3.4}$$

$$\theta(t=0) = \theta_0 \quad in\ \Omega, \tag{3.5}$$

$$\chi(t=0) = \chi_0 \quad in\ \Omega. \tag{3.6}$$

Proof: We start with uniqueness.

Uniqueness. System (3.3)-(3.4) is equivalent to the following system

$$c(x)\frac{d\theta}{dt} - \operatorname{div}(k(x)\nabla\theta) + (id + \partial\phi(x,\cdot))^{-1}(\theta - \partial\psi(x,\chi)) \ni 0 \quad \text{in } \Omega\times(0,T). \tag{3.7}$$

$$\frac{d\chi}{dt} - (id + \partial\phi(x,\cdot))^{-1}(\theta - \partial\psi(x,\chi)) \ni 0 \quad \text{a.e. in } \Omega\times(0,T), \tag{3.8}$$

where $(id + \partial\phi(x,\cdot))^{-1}$ denotes the inverse with respect to r of the monotone operator on $\mathbb{R}$, $r \mapsto r + \partial\phi(x,r)$ (resolvent of $\partial\phi(x,\cdot)$).

Due to assumption H2 (which implies that $r \mapsto \phi(x,r)$ is convex and *l.s.c* on $\mathbb{R}$ for almost every x in Ω), the graph $(id + \partial\phi(x,\cdot))^{-1}$ is a single valued contraction defined on the whole $\mathbb{R}$.

Let (θ_1,χ_1) and (θ_2,χ_2) be two solutions of the system (3.7)-(3.8) with the regularity (3.1)-(3.2). The Lipschitz function $sg_\epsilon(r)$ is defined on $\mathbb{R}$ by

$$sg_\epsilon(r) = \begin{cases} 1 & \text{if } r \geq \epsilon, \\ r/\epsilon & \text{if } -\epsilon \leq r \leq \epsilon, \\ -1 & \text{if } r \leq -\epsilon. \end{cases}$$

Subtracting the equations corresponding to (θ_1,χ_1) and (θ_2,χ_2), multiplying them respectively by $sg_\epsilon(\theta_1-\theta_2)$ (which lies in $L^2(0,T;H_0^1(\Omega))$), and $sg_\epsilon(\chi_1-\chi_2)$ and integrating over $(0,t)$, we obtain

$$\begin{aligned} &\int_0^t \left\langle c(x)\frac{d(\theta_1-\theta_2)}{dt},\ sg_\epsilon(\theta_1-\theta_2)\right\rangle ds + \int_0^t\int_\Omega \frac{d(\chi_1-\chi_2)}{dt} sg_\epsilon(\chi_1-\chi_2)\,dx\,ds + \\ &\int_0^t\int_\Omega k(x) sg'_\epsilon(\theta_1-\theta_2)|\nabla(\theta_1-\theta_2)|^2\,dx\,ds + \int_0^t\int_\Omega B_\epsilon(s,x)\,ds\,dx = 0. \end{aligned} \tag{3.9}$$

Here, $B_\epsilon(s,x)$ is defined by

$$\begin{aligned} B_\epsilon(s,x) = &\left[(id + \partial\phi(x,\cdot))^{-1}(\theta_1 - \partial\psi(x,\chi_1)) - (id + \partial\phi(x,\cdot))^{-1}(\theta_2 - \partial\psi(x,\chi_2))\right] \\ &\times \left[sg_\epsilon(\theta_1-\theta_2) - sg_\epsilon(\chi_1-\chi_2)\right], \end{aligned}$$

and lies in $L^1((0,T)\times\Omega)$ by equation (3.8) and the regularity (3.2) of χ_1 and χ_2. In the definition of $B_\epsilon(s,x)$, $\partial\psi(x,\chi_1)$ and $\partial\psi(x,\chi_2)$ denote the sections of the sets $\partial\psi(x,\chi_1)$ and $\partial\psi(x,\chi_2)$ for which the relations (3.8) are satisfied for χ_1 and χ_2.

Using the positivity of k (assumption H1) and (3.5)-(3.6), equality (3.9) leads to

$$\int_\Omega c|\theta_1-\theta_2|_\epsilon(t)\,dx + \int_\Omega |\chi_1-\chi_2|_\epsilon(t)\,dx + \int_0^t\int_\Omega B_\epsilon(s,x)\,dx\,ds \leq 0,$$

where $|\cdot|_\epsilon$ is the primitive of sg_ϵ with zero value at zero.

Passing to the limit as ϵ goes to 0 in the above inequality, and using the assumption H1 again, gives

$$\bar{c}\|\theta_1 - \theta_2\|_{L^1(\Omega)}(t) + \|\chi_1 - \chi_2\|_{L^1(\Omega)}(t) + \int_0^t \int_\Omega B(s,x)\,dx\,ds \leq 0,$$

where $B(s,x)$ is obtained by replacing in $B_\epsilon(s,x)$, $sg_\epsilon(r)$ by the function sg_0 defined by

$$sg_0(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

To prove uniqueness of the solution, it is then sufficient to show that $B(s,x)$ is non-negative almost everywhere in $(0,T)\times\Omega$. The subset of $(0,T)\times\Omega$ where $B(s,x)$ is not trivially equal to zero is investigated in the following.

– On the subset of $(0,T)\times\Omega$ where either $(\theta_1(s,x)-\theta_2(s,x))(\chi_1(s,x)-\chi_2(s,x)) < 0$ or $\theta_1 = \theta_2$, we have, due to the monotonicity of $\partial\psi(x,\cdot)$ on $\mathbb{R}$,

$$A(s,x) = [\theta_1 - \partial\psi(x,\chi_1) - (\theta_2 - \partial\psi(x,\chi_2))][sg_0(\theta_1-\theta_2) - sg_0(\chi_1-\chi_2)] \geq 0.$$

As $(id + \partial\phi(x,\cdot))^{-1}$ is single valued and monotone, we can deduce that

$$B(s,x) = A(s,x)\frac{(id+\partial\phi(x,\cdot))^{-1}(\theta_1 - \partial\phi(x,\chi_1)) - (id+\partial\phi(x,\cdot))^{-1}(\theta_2 - \partial\psi(x,\chi_2))}{\theta_1 - \partial\psi(x,\chi_1) - (\theta_2 - \partial\psi(x,\chi_2))}$$

is non-negative for almost all (s,x) in $(0,T)\times\Omega$.

– On the subset of $(0,T)\times\Omega$ where $\chi_1(s,x) = \chi_2(s,x)$, by using equations (3.8), we have,

$$B(s,x) = (\frac{d\chi_1}{dt} - \frac{d\chi_2}{dt})sg_0(\theta_1 - \theta_2).$$

But the regularity (3.2) of χ_1 and χ_2 implies that $(d\chi_1/dt) = (d\chi_2/dt)$ for almost all (s,x) in the set where $\chi_1 = \chi_2$, so that $B(s,x) = 0$ almost everywhere on this set. This concludes the proof of uniqueness.

Remark 3.2. In fact, our proof of uniqueness shows that the nonlinear operator associated with (3.7)-(3.8) is accretive in the Banach space $L^1(\Omega)\times L^1(\Omega)$ endowed with the norm

$$\|(\theta,\chi)\| = \|c\theta\|_{L^1(\Omega)} + \|\chi\|_{L^1(\Omega)}.$$

Furthermore, it can be shown that this nonlinear operator is m-accretive in $L^1(\Omega)\times L^1(\Omega)$ which can provide the existence of a unique weak solution to (3.7)-(3.8) using the theory of nonlinear semi-groups in general Banach spaces [6]. Moreover, following the techniques developed in [3], the weak solution may be proved to be a strong solution (i.e., with the regularity (3.1)-(3.2)) by deriving some $L^2_{loc}(0,T;L^2(\Omega))\times L^2((0,T)\times\Omega)$ estimates on the time derivatives $d\theta/dt$ and $d\chi/dt$ of the weak solution.

Our technique is totally different and relies on a regularization of the problem (3.7)-(3.8) together with estimates for the regularized solution $(\theta_\lambda,\chi_\lambda)$ in the natural spaces appearing in (3.1)-(3.2).

Existence. Let us first introduce some notations and recall some results of convex analysis.

For any normal convex integrand $g(x,r)$ on $\Omega \times \mathbb{R}$ with value in $\mathbb{R} \cup \{+\infty\}$ satisfying H2 ii) (with $g(x,r)$ in place of $\psi(x,r)$), the functional I_g defined on $L^2(\Omega)$ by

$$I_g(u) = \begin{cases} \int_\Omega g(x,u(x))\,dx & \text{if } g(x,u(x)) \in L^1(\Omega), \\ +\infty & \text{if not,} \end{cases}$$

is convex and *l.s.c* on $L^2(\Omega)$.

For any λ positive, let $g_\lambda(x,\cdot)$ be the Yosida regularization of $g(x,\cdot)$ (with respect to r) which is the C^1 function defined on $\mathbb{R}$ by

$$g_\lambda(x,r) = \inf_{z\in\mathbb{R}} \big\{\frac{1}{2\lambda}|r-z|^2 + g(x,z)\big\}.$$

Under the assumptions on $g(x,r)$, it is known that $g_\lambda(x,u)$ is a Carathéodory function on $\Omega \times \mathbb{R}$, with Lipschitz derivative with respect to r ([1]). Moreover, it is obvious that $g_\lambda(x,r)$ satisfies H2 ii) with $(\lambda/2)a^2(x) + b(x)$ in place of $b(x)$.

If we then assume that there exists u_0 in $L^2(\Omega)$ such that $g(x,u_0(x))$ lies in $L^1(\Omega)$, it can be proved that

$$(I_g)_\lambda(u) = I_{g_\lambda}(u) \quad \text{for every } u \in L^2(\Omega),$$

where the Yosida regularization of I_g defined on $L^2(\Omega)$ by

$$(I_g)_\lambda(u) = \inf_{v\in L^2(\Omega)} \big\{\frac{1}{2\lambda} \parallel u - v \parallel_{L^2(\Omega)} + I_g(v)\big\}$$

is C^1 convex, with Lipschitz derivative on $L^2(\Omega)$.

Concerning the subdifferentials of I_g and I_{g_λ}, we have the localization property for every u in $L^2(\Omega)$,

$$\partial I_g(u) = \big\{v \in L^2(\Omega);\ v(x) \in \partial g(x,u(x)) \text{ a.e. in } \Omega\big\}$$
$$\partial I_{g_\lambda}(u) = \partial(I_g)_\lambda(u) = \partial g_\lambda(x,u(x)) \text{ a.e. in } \Omega.$$

Finally, we have the following

Convergence lemma.

$$I_g(u) = \lim_{\lambda\to 0^+} \nearrow I_{g_\lambda}(u) \quad \text{for all } u \text{ in } L^2(\Omega),$$

hence I_{g_λ} converges to I_g in the sense of Mosco ([14]); consequently, if u_λ is a sequence of $L^2((0,T)\times\Omega)$ such that

- $u_\lambda \rightharpoonup u$ weakly in $L^2((0,T)\times\Omega)$,
- $\partial I_{g_\lambda}(u_\lambda) \rightharpoonup v$ weakly in $L^2((0,T)\times\Omega)$,
- $\overline{\lim} \int_{(0,T)\times\Omega} \partial I_{g_\lambda}(u_\lambda(t,x))u_\lambda(t,x)\,dx\,dt \le \int_{(0,T)\times\Omega} v(t,x)u(t,x)\,dt\,dx,$

as λ goes to zero, then

$$v(t,x) \in \partial g(x,u(t,x)) \text{ a.e. in } (0,T)\times\Omega.$$

The proof of Theorem 3.1 is given in two steps. In step 1, we prove the existence of a solution under the assumption that θ_0 lies in $H^2(\Omega) \cap H_0^1(\Omega)$ and that χ_0 and $\partial\psi(\cdot, \chi_0(\cdot))$ belong to $L^2(\Omega)$. In step 2, the initial conditions are only assumed to verify the assumption H3. They are then approximated by two sequences θ_0^n and χ_0^n satisfying the hypothesis of step 1, and we pass to the limit as n goes to infinity.

Step 1. Assume that θ_0 belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and that χ_0 lies in $L^2(\Omega)$ and is such that $\partial\psi(\cdot, \chi_0(\cdot))$ belongs to $L^2(\Omega)$.

We consider the following regularization of problem (3.7)-(3.8)

$$c(x)\frac{d\theta_\lambda}{dt} - \operatorname{div}(k(x)\nabla\theta_\lambda) + (id + \partial\phi(x, \cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x, \chi_\lambda)) \ni 0 \text{ in } \Omega \times (0, T), \quad (3.10)$$

$$\frac{d\chi_\lambda}{dt} - (id + \partial\phi(x, \cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x, \chi_\lambda)) \ni 0 \text{ in } \Omega \times (0, T), \quad (3.11)$$

$$\theta_\lambda(t = 0) = \theta_0 \text{ in } \Omega, \quad (3.12)$$

$$\chi_\lambda(t = 0) = \chi_\lambda \text{ in } \Omega, \quad (3.13)$$

$$\theta_\lambda = 0, \text{ on } \partial\Omega. \quad (3.14)$$

Using assumption H1 on $c(x)$, the Hilbert space

$$H = L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)),$$

may be endowed with the scalar product

$$(U, V)_H = \int_\Omega c(x)u_1(x)v_1(x)\,dx + \int_\Omega u_2(x)v_2(x)\,dx.$$

Let A be the linear operator defined on H by

$$\begin{pmatrix} -(1/c(x))\operatorname{div}(k(x)\nabla u_1 \\ 0 \end{pmatrix}, \quad (3.15)$$

with domain

$$D(A) = \{U \in H; u_1 \in H_0^1(\Omega), \operatorname{div}(k(x)\nabla u_1) \in L^2(\Omega)\}.$$

Using the notations and the results recalled at the beginning of this section, one sees that the operator $u \mapsto \partial\psi_\lambda(\cdot, u(\cdot))$ is the derivative of the functional $(I_\psi)_\lambda$ and is Lipschitz continuous from $L^2(\Omega)$ into $L^2(\Omega)$.

Similarly, the operator $u \mapsto (id + \partial\phi(\cdot, \cdot))^{-1}(u(\cdot))$ is Lipschitz continuous from H into H (remark that H2 ii) is trivially satisfied for $\phi(x, \cdot)$, since by H2 i) $\phi(x, \cdot) \geq 0$).

In view of assumption H1, the operator B_λ defined on H by

$$B_\lambda(U) = \begin{pmatrix} (1/c(x))\big((id + \partial\phi(x, \cdot))^{-1}(u_1(x) - \partial\psi_\lambda(x, u_2(x)))\big) \\ -\big((id + \partial\phi(x, \cdot))^{-1}(u_1(x) - \partial\psi_\lambda(x, u_2(x))\big) \end{pmatrix} \quad (3.16)$$

is Lipschitz continuous from H onto H, as the composition of two Lipschitz operators.

The problem (3.10)-(3.14) can be written as the abstract evolution problem in H for $U_\lambda = (\theta_\lambda, \chi_\lambda)$,

$$\frac{dU_\lambda}{dt} + AU_\lambda + B_\lambda U_\lambda = 0 \text{ for all } t \text{ in } (0, T), \quad (3.17)$$

$$U_\lambda(0) = (\theta_0, \chi_0). \tag{3.18}$$

Concerning (3.17)-(3.18), the following lemma is a direct application of [5].

Lemma 3.1. *Problem (3.17)-(3.18) has a unique strong solution $U_\lambda = (\theta_\lambda, \chi_\lambda)$ with the regularity*

$$\theta_\lambda \in C^0([0,T];\ D(A)) \cap C^1([0,T], L^2(\Omega)) \tag{3.19}$$

$$\chi_\lambda \in C^1([0,T];\ L^2(\Omega)). \tag{3.20}$$

The proof of Lemma 3.1 is a simple consequence of

- the maximal monotonicity of the linear operator A on H endowed with the scalar product $(\cdot,\cdot)_H$,
- the Lipschitz property of B_λ on H,
- the fact that (θ_0, χ_0) lies in $D(A)$.

We pass to the limit as λ goes to zero using the following a priori estimates.

Lemma 3.2. *The solution $(\theta_\lambda, \chi_\lambda)$ of (3.17)-(3.18) satisfies the following estimates:*

$$\theta_\lambda \textit{ is bounded in } W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \tag{3.21}$$

$$\chi_\lambda \textit{ is bounded in } W^{1,2}(0,T;L^2(\Omega)), \tag{3.22}$$

$$\partial\psi_\lambda(x,\chi_\lambda) \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.23}$$

independently of λ.

Proof: System (3.17) is equivalent to

$$c(x)\frac{d\theta_\lambda}{dt} + \frac{d\chi_\lambda}{dt} - \operatorname{div}(k(x)\nabla\theta_\lambda) = 0 \tag{3.24}$$

$$\frac{d\chi_\lambda}{dt} + \partial\phi(x, \frac{d\chi_\lambda}{dt}) + \partial\psi_\lambda(x,\chi_\lambda) \ni \theta_\lambda, \tag{3.25}$$

and due to the regularity (3.19)-(3.20) and the Lipschitz property of $\partial\psi_\lambda(x,\cdot)$, both hold in $L^2(\Omega)$, for all t in $(0,T)$.

Multiplying (3.24) by θ_λ and (3.25) by $d\chi_\lambda/dt$, adding the resulting equations and integrating over $\Omega \times (0,t)$, yields

$$\begin{aligned}\frac{1}{2}\|\sqrt{c}\theta_\lambda(t)\|^2_{L^2(\Omega)} + \int_0^t \|\sqrt{k}\nabla\theta_\lambda(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\frac{d\chi_\lambda}{dt}(s)\|^2_{L^2(\Omega)}\,ds \\ + \int_0^t\int_\Omega \partial\phi(x,\frac{d\chi_\lambda}{dt})\frac{d\chi_\lambda}{dt}\,dx\,ds + \int_\Omega \psi_\lambda(x,\chi_\lambda(t))\,dx \\ = \int_\Omega \psi_\lambda(x,\chi_0)\,dx + \frac{1}{2}\|\sqrt{c}\theta_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Assumption H2 i) implies that

$$\int_\Omega \partial\phi(x,\frac{d\chi_\lambda}{dt})\frac{d\chi_\lambda}{dt}\,dx \geq 0 \ \text{ for all } t \in [0,T].$$

Also by H3, $\psi(x,\chi_0(x))$ lies in $L^1(\Omega)$, so that we conclude

$$\lim_{\lambda\to 0+}\int_\Omega \psi_\lambda(x,\chi_0(x))\,dx = \int_\Omega \psi(x,\chi_0(x))\,dx.$$

Then inequality (3.26) together with assumption H2 ii) yields

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{c}\theta_\lambda(t)\|^2_{L^2(\Omega)} + \int_0^t \|\sqrt{k}\nabla\theta_\lambda(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \Big\|\frac{d\chi_\lambda}{dt}(s)\Big\|^2_{L^2(\Omega)}\,ds\\
&\le \int_\Omega \psi(x,\chi_0(x))\,dx + \frac{1}{2}\|\sqrt{c}\theta_0\|^2_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\chi_\lambda(t)\|_{L^2(\Omega)} + \|b\|_{L^1(\Omega)}\\
&\le \int_\Omega \psi(x,\chi_0(x))\,dx + \frac{1}{2}\|\sqrt{c}\theta_0\|^2_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\int_0^t \Big\|\frac{d\chi_\lambda}{dt}(s)\Big\|_{L^2(\Omega)}\,ds\\
&\qquad + \|a\|_{L^2(\Omega)}\|\chi_0\|_{L^2(\Omega)} + \|b\|_{L^1(\Omega)}.
\end{aligned}$$

In view of assumption H1, the last inequality implies that

$$\theta_\lambda \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)) \tag{3.27}$$

$$\chi_\lambda \text{ is bounded in } W^{1,2}(0,T;L^2(\Omega)), \tag{3.28}$$

the bounds depending only upon $\|\psi(x,\chi_0(x)\|_{L^1(\Omega)}$, $\|\theta_0\|_{L^2(\Omega)}$, $\|\chi_0\|_{L^2(\Omega)}$, $\|a\|_{L^2(\Omega)}$, $\|b\|_{L^1(\Omega)}$ and T.

The equivalence of (3.10) and (3.24), the boundedness of $d\chi_\lambda/dt$ in $L^2((0,T)\times\Omega))$ and the fact that θ_0 belongs to $H^1_0(\Omega)$ (in this step) show that

$$\frac{d\theta_\lambda}{dt} \text{ is bounded in } L^2((0,T)\times\Omega), \tag{3.29}$$

the bound depending on the bounds which appear in (3.27)-(3.28) but also on $\|\theta_0\|_{H^1_0(\Omega)}$.

To conclude the proof of Lemma 3.2, we have to derive the estimate (3.23). Using the results recalled at the beginning of this section and the regularity (3.20) of χ_λ, the quantity $\partial\psi_\lambda(x,\chi_\lambda)$ belongs to $W^{1,2}(0,T;L^2(\Omega))$ and the chain rule holds ([11],[13])

$$\frac{d}{dt}(\partial\psi_\lambda(x,\chi_\lambda)) = \partial^2\psi_\lambda(x,\chi_\lambda)\cdot\frac{d\chi_\lambda}{dt} \quad \text{a.e. in } (0,T)\times\Omega.$$

Multiplying (3.25) by $(d/dt)(\partial\psi_\lambda(x,\chi_\lambda))$, integrating over Ω then over $(0,t)$, we obtain the following equality:

$$\begin{aligned}
&\int_0^t\int_\Omega \partial^2\psi_\lambda(x,\chi_\lambda)\Big|\frac{d\chi_\lambda}{dt}\Big|^2\,dx\,ds +\\
&\int_0^t\int_\Omega \partial\phi(x,\frac{d\chi_\lambda}{dt})\frac{d\chi_\lambda}{dt}\partial^2\psi_\lambda(x,\chi_\lambda)\,dx\,ds + \frac{1}{2}\big\|\partial\psi_\lambda(x,\chi_\lambda)\big\|^2_{L^2(\Omega)}(t)\\
&= \int_0^t\int_\Omega \theta_\lambda\frac{d}{dt}(\partial\psi_\lambda(x,\chi_\lambda))\,dx\,ds + \frac{1}{2}\big\|\partial\psi_\lambda(x,\chi_0)\big\|^2_{L^2(\Omega)} \quad \text{for all } t\in[0,T].
\end{aligned}\tag{3.30}$$

The convexity of $\psi_\lambda(x,\cdot)$ and property H2 i) of $\phi(x,\cdot)$ leads to

$$\int_0^t\int_\Omega \partial^2\psi_\lambda(x,\chi_\lambda)\Big|\frac{d\chi_\lambda}{dt}\Big|^2\,dx\,ds \ge 0, \tag{3.31}$$

and

$$\int_0^t \int_\Omega \partial\phi(x, \frac{d\chi_\lambda}{dt}) \frac{d\chi_\lambda}{dt} \partial^2\psi_\lambda(x, \chi_\lambda)\, dx\, ds \geq 0, \tag{3.32}$$

for all $t \in [0, T]$. Integrating the right-hand side of (3.30) by part, inequalities (3.31) and (3.32) yield

$$\frac{1}{2}\|\partial\psi_\lambda(\cdot, \chi_\lambda(\cdot))\|^2_{L^2(\Omega)}(t) \leq -\int_0^t \int_\Omega \frac{d\theta_\lambda}{dt} \partial\psi_\lambda(x, \chi_\lambda)\, dx\, ds + \int_\Omega \theta_\lambda(t)\partial\psi_\lambda(x, \chi_\lambda(t))\, dx$$
$$- \int_\Omega \theta_0 \partial\psi_\lambda(x, \chi_0)\, dx + \frac{1}{2}\|\partial\psi_\lambda(\cdot, \chi_0(\cdot))\|^2_{L^2(\Omega)}. \tag{3.33}$$

Since, by hypothesis in this step, $\partial\psi(\cdot, \chi_0(\cdot))$ belongs to $L^2(\Omega)$, we have

$$\|\partial\psi_\lambda(x, \chi_0)\|_{L^2(\Omega)} \leq \|\partial\psi(x, \chi_0)\|_{L^2(\Omega)},$$

so inequality (3.33), together with the estimates (3.23) and (3.29) and Gronwall's lemma, allow us to conclude that

$$\partial\psi_\lambda(x, \chi_\lambda) \text{ is bound in } L^\infty(0, T; L^2(\Omega)), \tag{3.34}$$

independently of λ, the bound depending on the bound on $d\theta_\lambda/dt$ in $L^2((0, T) \times \Omega)$ and $\partial\psi(x, \chi_0)$ in $L^2(\Omega)$. This ends the proof of Lemma 3.2. ∎

Passage to the limit as λ goes to 0.

Through the application of Aubin's lemma (see [12], e.g.), estimates (3.21)-(3.22) imply the existence of two subsequences of θ_λ and χ_λ, still denoted θ_λ and χ_λ, such that

$$\theta_\lambda \to \theta \text{ in } C^0([0, T]; L^2(\Omega)) \tag{3.35}$$

and

$$\chi_\lambda \rightharpoonup \chi \text{ weakly in } L^2((0, T) \times \Omega), \tag{3.36}$$

for some θ in $W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ and χ in $W^{1,2}(0, T; L^2(\Omega))$.

Passing to the limit in (3.24) is easy and gives

$$c(x)\frac{d\theta}{dt} - \operatorname{div}(k(x)\nabla\theta) + \frac{d\chi}{dt} = 0 \text{ in } (0, T) \times \Omega. \tag{3.37}$$

Due to (3.23), there exists a subsequence of $\partial\psi_\lambda(x, \chi_\lambda)$, still denoted $\partial\psi(x, \chi_\lambda)$, such that

$$\partial\psi_\lambda(x, \chi_\lambda) \rightharpoonup Y \text{ weak-* in } L^\infty(0, T; L^2(\Omega)) \tag{3.38}$$

for some Y identified via the following lemma.

Lemma 3.3. *Equation (3.11) and the convergences (3.35)-(3.36) and (3.38) imply that*

$$\chi_\lambda \to \chi \text{ in } C^0([0, T]; L^2(\Omega)), \tag{3.39}$$

and

$$Y \in \partial\psi(\cdot, \chi) \text{ a.e. in } (0, T) \times \Omega. \tag{3.40}$$

Proof: Let λ and μ be two strictly positive reals. Subtraction of the two equations corresponding to λ and μ, multiplication of the resulting equation by $\chi_\lambda - \chi_\mu$ and integration over $(0,t) \times \Omega$ yields

$$\frac{1}{2}\|\chi_\lambda - \chi_\mu\|^2_{L^2(\Omega)}(t) = \int_0^t \int_\Omega \big[(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x,\chi_\lambda)) \\ - (id + \partial\phi(x,\cdot))^{-1}(\theta_\mu - \partial\psi_\mu(x,\chi_\mu))\big](\chi_\lambda - \chi_\mu)\,dx\,ds \tag{3.41}$$

for all t in $[0,T]$. The right-hand side A of (3.41) is evaluated as follows:

$$\begin{aligned} A = &\int_0^t \int_\Omega \big[(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x,\chi_\lambda)) \\ &- (id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\mu(x,\chi_\mu))\big](\chi_\lambda - \chi_\mu)\,dx\,ds \\ &+ \int_0^t \int_\Omega \big[(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\mu(x,\chi_\mu)) \\ &- (id + \partial\phi(x,\cdot))^{-1}(\theta_\mu - \partial\psi_\mu(x,\chi_\mu))\big](\chi_\lambda - \chi_\mu)\,dx\,ds. \end{aligned}$$

Using the Lipschitz property of the function $r \mapsto (id + \partial\phi(x,\cdot))^{-1}(r)$ for almost every x in Ω, and the easy identity

$$r_1 - r_2 = \lambda\partial\psi_\lambda(x,r_1) - \mu\partial\psi_\mu(x,r_2) + (id + \lambda\partial\psi(x,\cdot))^{-1}(r_1) - (id + \mu\partial\psi(x,\cdot))^{-1}(r_2),$$

we obtain

$$A \le \int_0^t \|\theta_\lambda - \theta_\mu\|_{L^2(\Omega)}(s)\|\chi_\lambda - \chi_\mu\|_{L^2(\Omega)}(s)\,ds + A_1 + A_2$$

where

$$\begin{aligned} A_1 = \int_0^t \int_\Omega &\big[(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x,\chi_\lambda)) - (id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\mu(x,\chi_\mu))\big] \\ &\big[(id + \lambda\partial\psi(x,\cdot))^{-1}(\chi_\lambda) - (id + \mu\partial\psi(x,\cdot))^{-1}(\chi_\mu)\big]\,dx\,ds, \end{aligned}$$

and

$$\begin{aligned} A_2 = \int_0^t \int_\Omega &\big[(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x,\chi_\lambda)) - (id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\mu(x,\chi_\mu))\big] \\ &\big[\lambda\partial\psi_\lambda(x,\chi_\lambda) - \mu\partial\psi_\mu(x,\chi_\mu)\big]\,dx\,ds. \end{aligned}$$

As $r \mapsto (id + \partial\phi(x,\cdot))^{-1}(r)$ is Lipschitz, A_1 may be rewritten as

$$\begin{aligned} A_1 = \int_0^t \int_\Omega &\frac{(id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\lambda(x,\chi_\lambda)) - (id + \partial\phi(x,\cdot))^{-1}(\theta_\lambda - \partial\psi_\mu(x,\chi_\mu))}{\partial\psi_\lambda(x,\chi_\lambda) - \partial\psi_\mu(x,\chi_\mu)} \\ &(\partial\psi_\lambda(x,\chi_\lambda) - \partial\psi_\mu(x,\chi_\mu))\big[(id + \lambda\partial\psi(x,\cdot))^{-1}(\chi_\lambda) - (id + \mu\partial\psi(x,\cdot))^{-1}(\chi_\mu)\big]\,dx\,ds. \end{aligned}$$

Using the usual relation

$$\partial\psi_\lambda(x,r) \in \partial\psi(x,(id + \lambda\partial\psi(x,\cdot))^{-1}(r)),$$

we obtain

$$(\partial\psi_\lambda(x,\chi_\lambda) - \partial\psi_\mu(x,\chi_\mu))\left[(id + \lambda\partial\psi(x,\cdot))^{-1}(\chi_\lambda) - (id + \mu\partial\psi(x,\cdot))^{-1}(\chi_\mu)\right] \geq 0$$

almost everywhere in $(0,T)\times\Omega$. From the monotonicity of the function $r \mapsto (id + \partial\phi(x,\cdot))^{-1}(r)$, we then deduce that

$$A_1 \leq 0. \tag{3.42}$$

Concerning A_2, the Lipschitz property of the function $r \mapsto (id + \partial\phi(x,\cdot))^{-1}(r)$ leads to

$$A_1 \leq C(\lambda+\mu)\int_0^t (\|\partial\psi_\lambda(x,\chi_\lambda)\|_{L^2(\Omega}(s) + \|\partial\psi_\mu(x,\chi_\mu)\|_{L^2(\Omega)}(s))\,ds, \tag{3.43}$$

where C is a constant.

At last, using (3.38), (3.41), (3.42) and (3.43), we obtain

$$\frac{1}{2}\|\chi_\lambda - \chi_\mu\|^2_{L^2(\Omega)}(t) \leq \int_0^t \|\theta_\lambda - \theta_\mu\|_{L^2(\Omega)}(s)\|\chi_\lambda - \chi_\mu\|_{L^2(\Omega)}(s)\,ds + Ct(\lambda+\mu).$$

From Gronwall's lemma, this last inequality yields

$$\|\chi_\lambda - \chi_\mu\|_{L^2(\Omega)}(t) \leq \sqrt{2CT(\lambda+\mu)} + \int_0^t \|\theta_\lambda - \theta_\mu\|_{L^2(\Omega)}(s)\,ds. \tag{3.44}$$

Since, by (3.35), θ_λ converge in $C^0([0,T];L^2(\Omega))$, (3.44) shows that χ_λ is a Cauchy sequence in $C^0([0,T];L^2(\Omega))$, which proves (3.39).

To identify Y, we remark that the convergence (3.38) and (3.39) imply that

$$\lim_{\lambda\to 0+}\int_0^T\int_\Omega \partial\psi_\lambda(x,\chi_\lambda(s))\chi_\lambda(s)\,dx\,ds = \int_0^T\int_\Omega Y\cdot\chi\,dx\,ds. \tag{3.45}$$

Invoking the convergence lemma, (3.38)-(3.39) and equality (3.45) allow us to conclude that

$$Y \in \partial\psi(x,\chi) \quad \text{a.e. in } \Omega\times(0,T).$$

This ends the proof of lemma 3.3. ∎

To complete the proof of step 1, we have to pass to the limit in (3.25) as λ goes to zero.

Using (3.21), (3.22), (3.23) and (3.36), equation (3.25) leads to the existence of a subsequence of $\partial\phi(x, d\chi_\lambda/dt)$, still denoted by $\partial\phi(x, d\chi_\lambda/dt)$, such that

$$\partial\phi(x,\frac{d\chi_\lambda}{dt}) \rightharpoonup Z \quad \text{weakly in } L^2((0,T)\times\Omega) \tag{3.46}$$

where Z is an element of $L^2((0,T)\times\Omega)$ which, in view of (3.40), satisfies

$$\frac{d\chi}{dt} + Z + \partial\psi(x,\chi) \ni \theta \quad \text{a.e. in } (0,T)\times\Omega. \tag{3.47}$$

To identify Z, we multiply (3.25) by $d\chi_\lambda/dt$ and integrate over $(0,T)\times\Omega$ to get

$$\begin{aligned}&\int_0^T \|\frac{d\chi_\lambda}{dt}\|^2_{L^2(\Omega)}(t)\,dt + \int_0^T\int_\Omega \partial\phi(x,\frac{d\chi_\lambda}{dt})\frac{d\chi_\lambda}{dt}\,dx\,dt + \int_\Omega \psi_\lambda(x,\chi_\lambda(T))\,dx\\ &= \int_\Omega \psi_\lambda(x,\chi_0)\,dx + \int_0^T\int_\Omega \theta_\lambda\frac{d\chi_\lambda}{dt}\,dx\,dt.\end{aligned}\tag{3.48}$$

As we have seen, $\int_\Omega \psi_\lambda(x,\chi_0)\,dx$ converges to $\int_\Omega \psi(x,\chi_0)\,dx$ which is finite by assumption H3. Due to estimates (3.27)-(3.28) and property H3 i) for $\phi(x,\cdot)$, (3.48) implies that $\int_\Omega \psi_\lambda(x,\chi_\lambda(T))\,dx$ is bounded in λ. The convergence of $\chi_\lambda(T)$ to $\chi(T)$ in $L^2(\Omega)$ (see (3.39)) and the *l.s.c.* of $I_{\psi_{\lambda_0}}$ imply then

$$\begin{aligned}\int_\Omega \psi_{\lambda_0}(x,\chi(T))\,dx &\le \underline{\lim}_{\lambda\to 0^+}\int_\Omega \psi_{\lambda_0}(x,\chi_\lambda(T))\,dx\\ &\le \underline{\lim}_{\lambda\to 0^+}\int_\Omega \psi_\lambda(x,\chi_\lambda(T))\,dx < +\infty,\end{aligned}\tag{3.49}$$

for any λ_0 strictly positive. Appealing to the same argument as for χ_0, we know that $I_{\psi_{\lambda_0}}(\chi(T)) = \int_\Omega \psi_{\lambda_0}(x,\chi(T))\,dx$ converges to $I_\psi(\chi(T))$ as λ_0 goes to zero and (3.49) shows that $I_\psi(\chi(T)) = \int_\Omega \psi(x,\chi(T))\,dx$ is finite and

$$\int_\Omega \psi(x,\chi(T))\,dx \le \underline{\lim}_{\lambda\to 0^+}\int_\Omega \psi_\lambda(x,\chi_\lambda(T))\,dx. \tag{3.50}$$

Due to (3.28), (3.35), (3.39) and (3.50), passing to the $\overline{\lim}$ as λ goes to zero in (3.48) leads to

$$\begin{aligned}&\overline{\lim}\int_0^T\int_\Omega \partial\phi(x,\frac{d\chi_\lambda}{dt})\frac{d\chi_\lambda}{dt}\,dx\,dt \le\\ &-\int_0^T \|\frac{d\chi}{dt}\|^2_{L^2(\Omega)}(t)\,dt - \int_\Omega \psi(x,\chi(T))\,dx + \int_\Omega \psi(x,\chi_0)\,dx + \int_0^T\int_\Omega \theta\frac{d\chi}{dt}\,dx\,dt,\end{aligned}\tag{3.51}$$

where the lower-semi-continuity of the norm in $L^2((0,T)\times\Omega)$ has also been used.

Multiplication of (3.47) by $d\chi/dt$ and integration over $(0,T)\times\Omega$ while taking account of (3.40), gives

$$\int_0^T\int_\Omega Z\frac{d\chi}{dt}\,dx\,dt = -\int_0^T \|\frac{d\chi}{dt}\|^2_{L2(\Omega)}(t)\,dt - \int_0^T\int_\Omega Y\frac{d\chi}{dt}\,dx\,dt + \int_0^T\int_\Omega \theta\frac{d\chi}{dt}\,dx\,dt. \tag{3.52}$$

Appealing now to lemma 2.1, p. 189 of [2], and using (3.39), we get

$$\int_0^T\int_\Omega Y\frac{d\chi}{dt}\,dx\,dt = \int_\Omega \psi(x,\chi(T))\,dx - \int_\Omega \psi(x,\chi_0)\,dx.$$

Equality (3.52) then becomes

$$\begin{aligned}&\int_0^T\int_\Omega Z\frac{d\chi}{dt}\,dx\,dt =\\ &-\int_0^T \|\frac{d\chi}{dt}\|^2_{L^2(\Omega)}(t)\,dt - \int_\Omega \psi(x,\chi(T))\,dx + \int_\Omega \psi(x,\chi_0)\,dx + \int_0^T\int_\Omega \theta\frac{d\chi}{dt}\,dx\,dt.\end{aligned}\tag{3.53}$$

Comparing (3.51) and (3.53) gives

$$\overline{\lim_{\lambda\to 0^+}} \int_0^T \int_\Omega \partial\phi\big(x, \frac{d\chi_\lambda}{dt}\big)\frac{d\chi_\lambda}{dt}\, dx\, dt \le \int_0^T \int_\Omega Z\frac{d\chi}{dt}\, dx\, dt. \tag{3.54}$$

Using the weak convergence of $d\chi_\lambda/dt$ and $\partial\psi(x, d\chi_\lambda/dt)$, respectively, to $d\chi/dt$ and Z, given by (3.28), (3.36) and (3.46), (3.54) together with the convergence lemma implies that

$$Z \in \partial\phi\big(x, \frac{d\chi}{dt}\big) \quad \text{a.e. in } \Omega \times (0, T). \tag{3.55}$$

In view of (3.37), (3.47) and (3.55), we conclude that (θ, χ) is a solution of (3.3)-(3.4). The convergence (3.35) implies that $\theta(t = 0) = \theta_0$. Estimate (3.27)-(3.28), equation (3.37) and assumption H1 show that θ and χ satisfy (3.1)-(3.2). This concludes the proof of step 1.

Step 2. In this step, we only assume that θ_0 and χ_0 satisfy assumption H3.

Due to the inclusion of $\operatorname{dom} I_\phi$ in the closure of dom ∂I_ϕ in $L^2(\Omega)$, we can choose a sequence (θ_0^n, χ_0^n) satisfying the conditions of step 1 and converging to (θ_0, χ_0) in $(L^2(\Omega))^2$ as n goes to infinity. Let (θ^n, χ^n) be the solution of (3.3)-(3.4) associated to (θ_0^n, χ_0^n) through step 1 with the regularity (3.1)-(3.2).

As we have noted in step 1, the estimates (3.27)-(3.28) only depend on $\|\theta_0\|_{L^2(\Omega)}$ and $\|\psi(x, \chi_0)\|_{L^1(\Omega)}$ which implies that

$$\theta^n \quad \text{is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \tag{3.56}$$

and

$$\chi^n \text{ is bounded in } W^{1,2}(0, T; L^2(\Omega)), \tag{3.57}$$

independently of n.

Using assumption H1, equation (3.3), with θ^n and χ^n in place of θ and χ, shows that

$$\frac{d\theta^n}{dt} \quad \text{is bounded in } L^2(0, T; H^{-1}(\Omega)). \tag{3.58}$$

By Aubin's lemma, estimates (3.56)-(3.57) and (3.58) imply the existence of a subsequence (θ^n, χ^n), still denoted by (θ^n, χ^n), so that

$$\theta^n \longrightarrow \theta \quad \text{in } L^2((0, T) \times \Omega) \cap C^0([0, T]; H^{-1}(\Omega)). \tag{3.59}$$

$$\chi^n \rightharpoonup \chi \quad \text{weakly in } W^{1,2}(0, T; L^2(\Omega)), \tag{3.60}$$

as n goes to infinity, for some appropriate θ and χ.

The convergence (3.59) shows that

$$\theta(t = 0) = \theta_0. \tag{3.61}$$

In view of (3.56), (3.59), (3.60) and assumption H1, passing to the limit as n goes to infinity in (3.3) with (θ^n, χ^n) in place of (θ, χ), leads to

$$c(x)\frac{d\theta}{dt} + \frac{d\chi}{dt} - \operatorname{div}(k(x)\nabla\theta) = 0 \quad \text{in } (0, T) \times \Omega. \tag{3.62}$$

To pass to the limit as n goes to infinity in (3.4) (with (θ^n, χ^n) in place of (θ, χ)), we will use the following lemma.

Lemma 3.4. *The sequence* χ^n *satisfies*

$$\chi^n \longrightarrow \chi \text{ in } C^0([0,T];\, L^2(\Omega)). \tag{3.63}$$

Proof: It is very similar to the proof of Lemma 3.3. As already mentioned, (3.4) is equivalent to (3.8). Subtraction of the two equations (3.8) corresponding to (θ^n, χ^n) and (θ^m, χ^m), multiplication by $\chi^n - \chi^m$ and integration over $\Omega \times (0,t)$ leads to

$$\begin{gathered}\|\chi^n - \chi^m\|^2_{L^2(\Omega)}(t) \leq \\ \frac{1}{2}\|\chi_0^n - \chi_0^m\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega [(id + \partial\phi(x,\cdot))^{-1}(\theta^n - \partial\psi(x,\chi^n)) \\ -(id + \partial\phi(x,\cdot))^{-1}(\theta^m - \partial\psi(x,\chi^m))]\,[\chi^n - \chi^m]\,dx\,ds,\end{gathered} \tag{3.64}$$

for all integers n and m. The right-hand side of (3.64) is evaluated as in Lemma 3.3 to give

$$\frac{1}{2}\|\chi^m - \chi^n\|^2_{L^2(\Omega)}(t) \leq \frac{1}{2}\|\chi_0^n - \chi_0^m\|^2_{L^2(\Omega)} + c_1 + c_2, \tag{3.65}$$

where

$$c_1 = \int_0^t \int_\Omega [(id + \partial\phi(x,\cdot))^{-1}(\theta^n - \partial\psi(x,\chi^n)) - (id + \partial\phi(x,\cdot))^{-1}(\theta^n - \partial\psi(x,\chi^m))] \\ [\chi^n - \chi^m]\,dx\,ds.$$

$$c_2 = \int_0^t \int_\Omega [(id + \partial\phi(x,\cdot))^{-1}(\theta^n - \partial\psi(x,\chi^m)) - (id + \partial\phi(x,\cdot))^{-1}(\theta^m - \partial\psi(x,\chi^m))] \\ [\chi^n - \chi^m]\,dx\,ds.$$

For any fixed real z, the graph $r \mapsto -(id + \partial\phi(x,\cdot))^{-1}(z - \partial\psi(x,r))$ is monotone for almost any x in Ω, which implies that c_1 is negative for all t in $(0,T)$.

Using the contraction property of the application $r \mapsto (id + \partial\phi(x,\cdot))^{-1}(r)$, for almost every x in Ω, (3.65) then implies

$$\begin{gathered}\frac{1}{2}\|\chi^m - \chi^n\|^2_{L^2(\Omega)}(t) \leq \\ \frac{1}{2}\|\chi_0^n - \chi_0^m\|^2_{L^2(\Omega)} + \int_0^t \|\theta^n - \theta^m\|_{L^2(\Omega)}(s)\|\chi^n - \chi^m\|_{L^2(\Omega)}(s)\,ds,\end{gathered} \tag{3.66}$$

for all t in $[0,T]$. By Gronwall's lemma, (3.66) gives

$$\|\chi^n - \chi^m\|_{L^2(\Omega)}(t) \leq \|\chi_0^n - \chi_0^m\|_{L^2(\Omega)} + \int_0^t \|\theta^n - \theta^m\|_{L^2(\Omega)}(s)\,ds, \tag{3.67}$$

for all t in $[0,T]$. Since θ^n converges to θ in $L^2((0,T)\times\Omega)$ (c.f. (3.59)) and χ_0^n converges to χ_0 in $L^2(\Omega)$, (3.67) implies that

$$\chi^n \longrightarrow \chi \text{ in } C^0([0,T];L^2(\Omega)). \tag{3.68}$$

This concludes the proof of the lemma. ∎

To pass to the limit in (3.4), we rewrite it as

$$-\frac{d\chi^n}{dt} \in \rho_n(t,x,\chi^n)$$

where ρ_n is given by

$$\rho_n(t,x,z) = -(id + \partial\phi(x,\cdot))^{-1}(\theta^n(t,x) - \partial\psi(x,z)).$$

Note that ρ_n is measurable in (t,x,z) and monotone in z.

Furthermore, from (3.4), we also know that

$$-\frac{d\chi^n}{dt} \in D(\partial\phi(x,\cdot)) \quad \text{a.e. in } \Omega \times (0,T).$$

Let

$$c_-(x) = \inf D(\partial\phi(x,\cdot)), \quad c_+(x) = \sup D(\partial\phi(x,\cdot)),$$
$$d_-(x) = \inf D(\partial\psi(x,\cdot)), \quad d_+(x) = \sup D(\partial\psi(x,\cdot)).$$

We define $\tilde{\rho}_n(t,x,\cdot)$ as the unique maximal monotone extension of $\rho_n(t,x,\cdot)$ with domain in $[d_-(x), d_+(x)]$. Note that $\tilde{\rho}_n$ coincides with ρ_n on $(d_-(x), d_+(x))$. Now, (3.4) reads as

$$-\frac{d\chi^n}{dt} \in \tilde{\rho}_n(t,x,\chi^n),$$

or functionally,

$$-\frac{d\chi^n}{dt} \in \tilde{R}_n(\chi^n),$$

where $\tilde{R}_n$ denotes the canonical extension of $\tilde{\rho}_n$ to $L^2((0,T)\times\Omega)$.

From the definition of ρ_n, the convergence of θ^n in $L^2(\Omega\times(0,T))$ shows that $\tilde{R}_n$ converges to $\tilde{R}$ (associated to $\tilde{\rho}$ corresponding to θ) in the sense of maximal monotone graphs. Now the strong convergence of χ^n in $C^0([0,T];L^2(\Omega))$, hence in $L^2(\Omega\times(0,T))$ and that of $d\chi^n/dt$ weakly to $d\chi/dt$ in $L^2(\Omega\times(0,T))$ implies that

$$-\frac{d\chi}{dt} \in \tilde{R}(\chi) \tag{3.69}$$

which is the limit of equation (3.4).

However, one can prove more because of assumption H2 and the properties of $\tilde{R}$. We claim that $-d\chi/dt \in \rho(t,x,\chi)$ almost everywhere, which can be expressed as $d\chi/dt + \partial\phi(x, d\chi/dt) + \partial\psi(x,\chi) \ni \theta$ almost everywhere.

Indeed, on the set $\{(t,x) \in (0,T)\times\Omega, d_-(x) < \chi < d_+(x)\}$, $\tilde{R}(\chi)$ is explicitly known to be $-(id + \partial\phi(x,\cdot))^{-1}(\theta(t,x) - \partial\psi(x,\cdot))$, so that $-d\chi/dt \in \tilde{R}(\chi)$ can be rewritten as

$$\frac{d\chi}{dt} + \partial\phi(x, \frac{d\chi}{dt}) + \partial\psi(x,\chi) \ni \theta\,.$$

We now consider $\tilde{R}(\chi)$ on the sets

$$G_- = \{(t,x) \in (0,T)\times\Omega;\ \chi = d_-(x)\}$$

provided $d_- \neq -\infty$ (resp. $G_+ = \{(t,x) \in (0,T) \times \Omega;\ \chi = d_+(x)\}$ provided $d_+(x) \neq +\infty$).

We only investigate the case G_- (the case G_+ being similar). From the definitions of $\tilde{\rho}$, $d_-(x)$ and $c_+(x)$, one has

$$\tilde{\rho}(t,x,d_-(x)) = \rho(t,x,d_-(x)) \cup (-\infty, -c_+(x)], \tag{3.70}$$

where $\rho(t,x,d_-(x)) = \emptyset$ if $d_-(x) \notin D(\partial\psi(x,\cdot))$ and $(-\infty,-c_+(x)] = \emptyset$ if $c_+(x) = +\infty$. Since χ lies in $W^{1,2}(0,T;L^2(\Omega)) = L^2(\Omega;W^{1,2}(0,T))$, Stampachia's lemma (cf. [11]) applied to $W^{1,2}(0,T)$ implies

$$\frac{d\chi}{dt} = 0 \quad \text{a.e. in } G_-, \tag{3.71}$$

and equation (3.69) then leads to

$$0 \in \tilde{\rho}(t,x,d_-(x) \quad \text{a.e. in } G_-. \tag{3.72}$$

Let (t,x) be such that (3.72) holds. For such an (t,x) in G_-, relation (3.70) shows that

i) either $0 \in \rho(t,x,d_-(x))$

ii) or $0 \leq -c_+(x)$.

In the case i), (3.71) implies that

$$\frac{d\chi}{dt} + \partial\phi(x, \frac{d\chi}{dt}) + \partial\psi(x,d_-(x)) \ni \theta,$$

which shows that (3.4) is satisfied.

In the case ii), one has $c_+(x) \leq 0$, and using assumption H2 iii) $c_+(x) = 0$ (because $0 \in D(\partial\phi(x,\cdot))$ implies that $c_+(x) \geq 0$). With the help of assumption H2 iii), we can conclude, therefore, that $d_-(x) \in D(\partial\psi(x,\cdot))$. For such a (t,x) in G^-, the set $\theta(t,x) - \partial\psi(x,d_-(x))$ is not empty and contains strictly positive reals. Using again $c_+(x) = 0$ one has

$$(id + \partial\phi(x,\cdot))^{-1}(z) = 0 \quad \text{for all } z > 0,$$

so that finally

$$0 \in \rho(t,x,d_-(x)),$$

which concludes the proof of the claim. Therefore, (3.4) holds almost everywhere in G_-. As already mentioned, the proof is similar on G_+, which concludes the proof of Theorem 3.1.

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