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SMALLEST LYAPUNOV FUNCTIONS OF DIFFERENTIAL INCLUSIONS

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Abstract. We provide a first answer to the question: given a differential inclusion, does there exist a *smallest* nonnegative extended lower semicontinuous (i.e., take their values in $\mathbb{R}_+ \cup \{+\infty\}$) Lyapunov function *larger than a given* lower semicontinuous function? Since the lower semicontinuous functions involved in the statement of this problem are not necessarily differentiable, we have to weaken the usual definition of a derivative and replace it by the one of epicontingent derivative. This allows us to characterize lower semicontinuous Lyapunov functions of a differential inclusion. With this definition at hand, we shall answer this question. These results find natural applications in differential games.

The tool for achieving this objective is the existence of largest closed viability (and/or invariance) domains of a differential inclusion contained in a given closed subset. Hence, we shall provide in the appendix the proof of their existence as well as the division of the boundary of a closed subset into areas from where some or all solutions to the differential inclusion remain or leave this closed subset.

1. Lyapunov functions. Let $F : X = \mathbb{R}^n \rightsquigarrow X$ be a set-valued map with which we associate the differential inclusion

$$\text{for almost all } \ t \geq 0, \ x'(t) \in F(x(t)) \tag{1}$$

We also consider a time-dependent function $w(\cdot)$ defined as solutions to a differential equation

$$w'(t) = -\phi(w(t)), \tag{2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a given continuous function with linear growth.

We consider a nonnegative function $V : X \mapsto \mathbb{R}_+ \cup \{+\infty\}$ (called *extended function*), proper in the sense that its domain

$$\mathrm{Dom}\,(V) := \{x \in X \mid V(x) < +\infty\} \tag{3}$$

is not empty. We assume that $\mathrm{Dom}\,(V) \subset \mathrm{Dom}\,(F)$. This function V is said to enjoy the ϕ-Lyapunov property, i.e.,

$$\forall\, t \geq 0, \ \ V(x(t)) \leq w(t), \ \ w(0) = V(x(0)) \tag{4}$$

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along at least one solution $x(\cdot)$ to the differential inclusion (1) and one solution $w(\cdot)$ to (2) or the ϕ-universal Lyapunov property, for which property (4) is satisfied along *all* solutions to (1) and (2).

We shall need the concept of *contingent cone* $T_K(x)$ to a subset K at $x \in K$. Let $d_K(x) := \inf_{y\in K} \|x-y\|$ denote the distance from x to K. Then, $T_K(x)$ is the subset of $v \in X$ such that $\liminf_{h\to 0+} d_K(x+hv)/h = 0$. It is always a closed cone, convex whenever the set-valued map $x' \rightsquigarrow T_K(x')$ is lower semicontinuous at x (in this case, the subset is said to be *sleek at* x). Convex and smooth subsets are sleek.

We recall that the epigraph $\mathcal{E}p(V)$ of V is the subset of pairs (x, w) such that $V(x) \leq w$ and that the contingent epiderivative $D_\uparrow V(x)(v)$ of V at x in the direction v is defined by

$$D_\uparrow V(x)(v) := \liminf_{h\to 0+, u\to v} \frac{V(x+hu)-V(x)}{h}.$$

The epigraph of the function $v \to D_\uparrow V(x)(v)$ is the contingent cone to the epigraph of V at $(x, V(x))$ (see [1], [3, Chapter VII] for further information).

We say that V is contingently epidifferentiable if, for all $x \in \operatorname{Dom}(V)$,

$$D_\uparrow V(x)(v) > -\infty \text{ for all } v \in X \ \& \ D_\uparrow V(x)(v) < \infty \text{ for some } v \in X.$$

Definition 1.1. *We shall say that a nonnegative contingently epidifferentiable extended function V is a Lyapunov function of F associated with a function $\phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}$ if and only if V is a solution to the contingent Hamilton-Jacobi inequalities*

$$\forall x \in \ Dom(V), \quad \inf_{v\in F(x)} D_\uparrow V(x)(v) + \phi(V(x)) \leq 0 \tag{5}$$

and a universal Lyapunov function of F associated with a function ϕ if and only if V is a solution to the upper contingent Hamilton-Jacobi inequalities

$$\forall x \in \ Dom(V), \quad \sup_{v\in F(x)} D_\uparrow V(x)(v) + \phi(V(x)). \leq 0 \tag{6}$$

We refer to [6, 7, 8] and the references of these papers for a thorough study of contingent Hamilton-Jacobi equations arising from optimal control and comparison with viscosity solutions.

Theorem 1.1. *Let V be a nonnegative contingently epidifferentiable lower semicontinuous extended function and $F : X \rightsquigarrow X$ be a nontrivial set-valued map (this means that its graph is not empty).*

— *If F is upper semicontinuous with compact convex images and linear growth, then V is a Lyapunov function of F associated with $\phi(\cdot)$ if and only if for any initial state $x_0 \in \ Dom(V)$, there exist solutions $x(\cdot)$ to (1) and $w(\cdot)$ to (2) satisfying property (4).*

— *If F is Lipschitzean on the interior of its domain with compact values, then an extended function V such that $Dom(V) \subset \ Int \ Dom(F)$ is a universal Lyapunov function associated with ϕ if and only if for any initial state $x_0 \in \ Dom(V)$, all solutions $x(\cdot)$ to (1) and $w(\cdot)$ to (2) do satisfy property (4).*

Proof: We set $G(x, w) := F(x) \times \{-\varphi(w)\}$. Evidently, the system (1), (2) has a solution satisfying (4) if and only if the system of differential inclusions

$$(x'(t), w'(t)) \in G(x(t), w(t)) \tag{7}$$

has a solution starting at $(x_0, V(x_0))$ viable in $\mathcal{K} := \mathcal{E}p(V)$. We first observe that $\mathcal{K}$ is a viability domain for G (i.e., $G(z) \cap T_K(z) \neq \emptyset$ for all $z \in K$) if and only if V is a Lyapunov function for F with respect to φ. If $\mathcal{K}$ is a viability domain of G, by taking $z = (x, V(x))$, we infer that $(v, -\varphi(V(x))) \in T_{\mathcal{K}}(x, V(x)) = \mathcal{E}p(D_\uparrow V(x))$ for some $v \in F(x)$, hence (5).

Conversely, since $F(x)$ is compact and $v \mapsto D_\uparrow V(x)(v)$ is lower semicontinuous, (5) implies that there exists $v \in F(x)$ such that the pair $(v, -\phi(V(x)))$ belongs to $T_{\mathcal{K}}(x, V(x))$. Hence, $z_n := (x+h_n v_n, V(x)+h_n s_n) \in \mathcal{K}$ with $h_n \to 0+$, $v_n \to v$ and $s_n \to -\varphi(V(x))$. If $w > V(x)$, this implies that for large n

$$(x + h_n v_n, w - h_n\varphi(w)) = z_n + (0, w - V(x) - h_n(s_n + \varphi(w))) \in \mathcal{K} + \{0\} \times \mathbb{R}_+ = \mathcal{K}$$

so that $(v, -\varphi(w)) \in T_{\mathcal{K}}(x, w)$.

In the same way, one can check that the closed subset $\mathcal{E}pV$ is an invariance domain (i.e., $G(z) \subset T_{\mathcal{K}}(z)$ for all $z \in \mathcal{K}$) of the set-valued map G if and only if V is a universal Lyapunov function, a consequence of the Invariance Theorem (see [2, Theorem 4.6.2]. ∎

Example. (W-monotone set-valued maps). Let $W : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a nonnegative extended function. We say that a set-valued map F is W-monotone (with respect to ϕ) if

$$\forall\, x, y,\ \forall\, u \in F(x),\ v \in F(y),\ D_\uparrow W(x-y)(v-u) + \phi(W(x-y)) \leq 0. \tag{8}$$

We obtain, for instance, the following consequence:

Corollary 1.1. *Let W be an nonnegative contingently epidifferentiable extended lower semicontinuous function and $F : X \rightsquigarrow X$ be a nontrivial upper semicontinuous set-valued map with compact convex images and linear growth such that $-F$ is W-monotone with respect to some ϕ. Let $\bar{x}$ be an equilibrium of F (i.e., $0 \in F(\bar{x})$). Then, for any initial state x_0, there exist solutions $x(\cdot)$ and $w(\cdot)$ satisfying*

$$\forall\, t \geq 0,\ W(x(t) - \bar{x}) \leq w(t)$$

In particular, for $W(z) := \frac{1}{2}\|z\|^2$, we find the usual concept of monotonicity (with respect to ϕ):

$$\forall\, x, y,\ \forall\, u \in F(x),\ v \in F(y),\ \langle u - v\,,\, x - y\rangle \geq \phi(\frac{1}{2}\|x-y\|^2).$$

We can reformulate the viability and invariance theorems by saying that the indicator

$$\psi_K(x) := \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}$$

of a closed subset is a (universal) Lyapunov function:

Corollary 1.2. *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth.*

A closed subset K enjoys the viability property if and only if its indicator Ψ_K is a solution to the contingent equation

$$\inf_{v \in F(x)} D_\uparrow \Psi_K(x)(v) = 0.$$

— *If F is Lipschitzean on the interior of its domain with compact values, then K is invariant by F if and only if its indicator Ψ_K is a solution to the contingent equation*

$$\sup_{v \in F(x)} D_\uparrow \Psi_K(x)(v) = 0.$$

Let us also introduce attractors.

Definition 1.2. *We shall say that a closed subset K is an "attractor" of order $\alpha \geq 0$ if and only if for any $x_0 \in$ Dom(F), there exists at least one solution $x(\cdot)$ to the differential inclusion (1) such that*

$$\forall t \geq 0,\ d_K(x(t)) \leq d_K(x_0)e^{-\alpha t}. \tag{9}$$

It is said to be a "universal attractor" of order $\alpha \geq 0$ if and only if for any $x_0 \in$ Dom(F), all solutions $x(\cdot)$ to the differential inclusion (1) satisfy the above property.

We can recognize attractors by checking whether the distance function to K is a Lyapunov function:

Corollary 1.3. *Assume that F is a nontrivial upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. Then a closed subset $K \subset$ Dom(F) is an attractor if and only if the function $d_K(\cdot)$ is a solution to the contingent inequalities:*

$$\forall x \in \text{Dom}(F),\ \inf_{v \in F(x)} D_\uparrow d_K(x)(v) + \alpha d_K(x) \leq 0.$$

If F is Lipschitzean with compact images, then K is a universal attractor if and only if

$$\forall x \in \text{Dom}(F),\ \sup_{v \in F(x)} D_\uparrow d_K(x)(v) + \alpha d_K(x) \leq 0.$$

For $\alpha = 0$, a sufficient condition for K to be an attractor of order 0 is then to satisfy

$$\forall x \in \text{Dom}(F), \exists y \in \pi_K(x) \,|\, F(x) \cap T_K(y) \neq \emptyset$$

where $\pi_K(x)$ denotes the subset of $y \in K$ such that $d_K(x) = \|x - y\|$. Indeed, it is easy to check that

$$D_\uparrow d_K(x)(v) \leq d(v, T_K(\pi_K(x))).$$

This is a particular case of the situation where the function V is defined through a nonnegative function $U : X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$ in the following way:

$$V(x) := \inf_{y \in Y} U(x, y).$$

(Take $U(x,y) := \|x - y\| + \Psi_K(y)$). When we assume that the infimum is achieved at a point y_x, formula

$$D_\uparrow V(x)(u) \leq \inf_{v \in Y} D_\uparrow U(x, y_x)(u, v)$$

holds true, because $V(x + hu) - V(x) \leq U(x + hu, y_x + hv) - U(x, y_x)$.

Hence, under the assumptions of Theorem 1.1, we infer that assumption

$$\forall x,\ \inf_{u \in F(x), v \in Y} D_\uparrow U(x, y_x)(u, v) + \phi(U(x, y_x)) \leq 0 \tag{10}$$

implies that there exists a solution $x(\cdot)$ satisfying

$$\forall t \geq 0,\ \inf_{y \in Y} U(x(t), y) \leq w(t).$$

We can derive from this inequality and the calculus of contingent epiderivatives many consequences.

Remark. With an extended nonnegative function V, we can associate affine functions $w \to aw - b$ for which V is a solution to the contingent Hamilton-Jacobi inequalities (5).

For that purpose, we consider the convex function b defined by

$$b(a) := \sup_{x \in \text{Dom}(F)} \left(\inf_{v \in F(x)} D_\uparrow V(x)(v) + aV(x) \right).$$

Then it is clear that V is a solution to the contingent Hamilton-Jacobi inequalities

$$\forall x \in \text{Dom}(F), \ \inf_{v \in F(x)} D_\uparrow V(x)(v) + aV(x) - b(a) \leq 0.$$

Therefore, we deduce that there exists a solution to the differential inclusion satisfying

$$\forall t \geq 0, \ V(x(t)) \leq (V(x_0) - \frac{b(a)}{a})e^{-at} + \frac{b(a)}{a}.$$

A reasonable choice of a is the largest of the minimizers of $a \in]0, \infty[\to \max(0, b(a)/a)$, for which $V(x(t))$ decreases as fast as possible to the smallest level set $V^{-1}(] - \infty, \frac{b}{a}])$ of V.

Remark. By using the necessary condition of the Viability Theorem, we obtain the following result. We denote by $\mathcal{H}yp(V) := \{(x, w) \mid w \leq V(x)\}$ the *hypograph* of an extended function V and by $D_\downarrow V(x)(v)$ the contingent hypoderivative of V, whose hypograph is the contingent cone to the hypograph of V at $(x, V(x))$, and defined by

$$D_\downarrow V(x)(v) := \limsup_{h \to 0+, u \to v} \frac{V(x + hu) - V(x)}{h}.$$

Theorem 1.2. *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth and a continuous function $\phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}$.*

Let V be an nonnegative contingently epidifferentiable extended function. If, for some $x_0 \in$ Dom(F), we have

$$\sup_{v \in F(x_0)} D_\downarrow V(x_0)(v) + \phi(V(x_0)) < 0,$$

then for any solution $x(\cdot)$ to the (1) starting at x_0 and any solution $w(\cdot)$ to (2) starting at $V(x_0)$, there exists $T > 0$ such that

$$\forall t \in]0, T], \ V(x(t)) < w(t).$$

Proof: Indeed, if there exists a solution $x(\cdot)$ to (1) starting at x_0 and a solution $w(\cdot)$ to (2) starting at $V(x_0)$ satisfying

$$\forall T > 0, \exists t \in]0, T] \mid V(x(t)) \geq w(t),$$

then it is easy to deduce that there exists $v \in F(x_0)$ such that

$$-\phi(V(x_0)) \leq D_\downarrow V(x_0)(v) \leq \sup_{v \in F(x_0)} D_\downarrow V(x_0)(v)$$

which contradicts our assumption.

2. Smallest Lyapunov functions. The functions ϕ and $U : X \to \mathbb{R}_+ \cup \{+\infty\}$ being given, we shall construct the smallest lower semicontinuous Lyapunov function of a set-valued map F associated to ϕ larger than or equal to U, i.e., the smallest nonnegative lower semicontinuous solution U_ϕ to the contingent Hamilton-Jacobi inequalities (5) larger than or equal to U.

Theorem 2.1. *Let us consider a nontrivial set-valued map $F : X \rightsquigarrow X$, a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with linear growth and a proper nonnegative extended function U such that $Dom\,(U) \subset\ Dom\,(F)$.*

— Let us assume that F is upper semicontinuous with compact convex images and linear growth. Then there exists a smallest nonnegative lower semicontinuous solution U_ϕ : $Dom\,(F) \mapsto \mathbb{R} \cup \{+\infty\}$ to (5) larger than or equal to U (which can be the constant $+\infty$), which enjoys the property:

$\forall x \in\ Dom\,(U_\phi)$, there exist solutions to (1) and (2) satisfying

$$\forall t \geq 0,\ U(x(t)) \leq U_\phi(x(t)) \leq w(t).$$

— If F is Lipschitzean on the interior of its domain with compact values and ϕ is Lipschitzean, then there exists a smallest nonnegative lower semicontinuous solution $\widetilde{U}_\phi$: $Dom\,(F) \mapsto \mathbb{R} \cup \{+\infty\}$ to (6) larger than or equal to U (which can be the constant $+\infty$), which enjoys the property:

$\forall x \in\ Dom\,(U_\phi)$, all solutions of (1) and (2) satisfy

$$\forall t \geq 0,\ U(x(t)) \leq U_\phi(x(t)) \leq w(t).$$

Proof: By Theorem 3.1 of the appendix, we know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(U)$ (the viability kernel of the epigraph of U) of the set-valued map $(x, w) \rightsquigarrow G(x, w) := F(x) \times \{-\phi(w)\}$. If it is empty, it is the epigraph of the constant function equal to $+\infty$.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function U_ϕ defined by

$$U_\phi(x) := \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, the epigraph of any solution $V \geq U$ to the contingent inequalities (5) being a closed viability domain of the set-valued map G, is contained in the epigraph of U_ϕ, so that U_ϕ is the smallest of the lower semicontinuous solutions of (5) larger than U. Since

$$\mathcal{E}p(U_\varphi) = \ \text{Graph}(U_\varphi) + \{0\} \times \mathbb{R}_+ \subset \mathcal{K} + \{0\} \times \mathbb{R}_+$$

it is therefore enough to show that $\mathcal{K} + \{0\} \times \mathbb{R}_+ \subset \mathcal{K}$. In fact, we prove

if $\mathcal{M} \subset Dom(F) \times \mathbb{R}_+$ is a closed viability domain of G,, then so is the subset

$$\mathcal{M}_0 := \mathcal{M} + \{0\} \times \mathbb{R}_+\,.$$

Obviously, $\mathcal{M}_0$ is closed. To see that $G(x, w) \cap T_{\mathcal{M}_0}(x, w) \neq \emptyset$, let

$$U_{\mathcal{M}}(x) := \inf_{(x,\lambda)\in\mathcal{M}} \lambda,\ d := -\phi(U_M(x)).$$

By assumption, there exists $v \in F(x)$ such that (v, d) belongs to the contingent cone to $\mathcal{M}$ at the point $(x, U_{\mathcal{M}}(x)) \in \mathcal{M}$. Hence, there exist sequences $h_n > 0$ converging to 0, v_n converging to v and d_n converging to d such that

$$\forall n \geq 0,\ (x + h_n v_n, U_{\mathcal{M}}(x) + h_n d_n) \in \mathcal{M}\,.$$

This proves the claim when $w = U_{\mathcal{M}}(x)$ and the case $w > U_{\mathcal{M}}(x)$ follows like in the proof of Theorem 1.1.

When F and ϕ are Lipschitzean, Theorem 3.2 of the Appendix implies that there exists a largest closed invariance domain $\widetilde{\mathcal{K}}$ contained in the epigraph of U. We prove that it is the epigraph of the smallest lower semicontinuous solution

$$\widetilde{U}_\phi = \inf_{(x,\lambda)\in\widetilde{\mathcal{K}}} \lambda$$

of (6) we are looking for. This can be checked in an analogous way by showing that

if $\mathcal{M} \subset Dom(F) \times \mathbb{R}_+$ is a closed invariance domain of the set-valued map G, then so is the subset $\mathcal{M} + \{0\} \times \mathbb{R}_+$.

Corollary 2.1. *We posit the assumptions of Theorem 2.1.*

a) Let us assume that F is upper semicontinuous with compact convex images and linear growth.

— The indicator $\Psi_{\text{Viab}(K)}$ of the viability kernel Viab(K) of a closed subset K (i.e., the largest closed viability domain of F contained in K) is the smallest nonnegative lower semicontinuous solution of

$$\forall x \in Dom(V), \inf_{v\in F(x)} D_\uparrow V(x)(v) \le 0 \tag{11}$$

larger than or equal to Ψ_K.

— For all $a \ge 0$, there exists a smallest lower semicontinuous function $d_{\mathcal{M}_a} : X \to \mathbb{R} \cup \{+\infty\}$ larger than or equal to $d_{\mathcal{M}}$ such that

$\forall x_0 \in Dom(d_{\mathcal{M}_a})$, there exists a solution $x(\cdot)$ of (1) such that

$$d_{\mathcal{M}}(x(t)) \le d_{\mathcal{M}_a}(x_0)e^{-at}.$$

b) Assume that F is Lipschitzean on the interior of its domain with compact values.

— The indicator $\Psi_{\text{Inv}(K)}$ of the invariant kernel Inv(K) of a closed subset K (i.e., the largest closed invariance domain of F contained in K) is the smallest nonnegative lower semicontinuous solution of

$$\forall x \in Dom(V), \sup_{v\in F(x)} D_\uparrow V(x)(v) \le 0 \tag{12}$$

larger than or equal to Ψ_K.

— For all $a \ge 0$, there exists a smallest lower semicontinuous function $\widetilde{d_{\mathcal{M}_a}} : X \to \mathbb{R} \cup \{+\infty\}$ larger than or equal to $d_{\mathcal{M}}$ such that

$\forall x_0 \in Dom(d_{M_a})$, any solution $x(\cdot)$ of (1) satisfies

$$d_{\mathcal{M}}(x(t)) \le \widetilde{d_{\mathcal{M}_a}}(x_0)e^{-at}.$$

We can regard the subsets $\text{Dom}(d_{\mathcal{M}_a})$ and $\text{Dom}(\widetilde{d_{\mathcal{M}_a}})$ as the basins of exponential attraction and of universal exponential attraction of $\mathcal{M}$.

Proof: Let us check that the smallest lower semicontinuous solution U_0 larger than or equal to $U \equiv 0$ is equal to the indicator of $K_0 := \text{Viab}(K)$. Since it is clear that ψ_{K_0} is a solution to the above contingent inequalities (11), we have

$$U_0(x) \le \Psi_{K_0}(x) \text{ on } K_0.$$

Let x_0 belong to the domain of U_0. Then there exists a solution $x(\cdot)$ to the system of differential inclusions (7) starting at $(x_0, U_0(x_0))$ satisfying $U_0(x(t)) \le U_0(x_0)$ since $w(t) \equiv U_0(x_0)$. Therefore, x_0 belongs to K_0 and thus, $U_0(x_0) \le \Psi_{K_0}(x_0) = 0$.

The proof of the second statement follows similarly.

Theorem 2.2. *We posit the assumptions of Theorem 2.1. Assume furthermore that ϕ vanishes at* 0. *Then if U vanishes on an equilibrium $\bar{x}$ of F, so does the function U_ϕ associated with ϕ.*

Let L be the set-valued map associating to any solution $x(\cdot)$ of the differential inclusion (1) its limit set, and let $\mathcal{S}$ be the solution map. If ϕ is asymptotically stable, then for any $x_0 \in$ Dom (U_ϕ), there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that $L(x(\cdot)) \subset U^{-1}(0) \cap F^{-1}(0)$.

Proof: If $\overline{x}$ is an equilibrium of F such that $U(\bar{x}) = 0$, then $(\bar{x}, 0)$ is an equilibrium of G restricted to the epigraph of U (because $\phi(0) = 0$), so that the singleton $(\bar{x}, 0)$, as a viability domain, is contained in the epigraph of U_ϕ. Hence, $0 \leq U(\bar{x}) \leq U_\phi(\bar{x}) \leq 0$.

If ϕ is asymptotically stable, then the solutions $w(\cdot)$ to the differential equation $w'(t) = -\phi(w(t))$ do converge to 0 when $t \to +\infty$. Let x_0 belong to the domain of U_ϕ and $x(\cdot)$ be a solution satisfying

$$U(x(t)) \leq U_\phi(x(t)) \leq w(t)\,.$$

Hence, any cluster point ξ of $L(x(\cdot))$ belongs to $U_\phi^{-1}(0)$ because the epigraph of U_ϕ is closed. Hence, $0 \leq U(\xi) \leq U_\phi(\xi) \leq 0$.

Remark. Since the epigraph of U_ϕ is the viability kernel of the epigraph of U, we deduce that for any initial situation (x_0, w_0) such that $w_0 < U_\phi(x_0)$, for any solution $(x(\cdot), w(\cdot))$ to the system (7) starting at (x_0, w_0), then there exists $T > 0$ such that $w(T) < U(x(T))$. This happens whenever the initial state x_0 does not belong to the domain of U_ϕ.

3. Appendix: the anatomy of a closed subset. Let us consider the differential inclusion (1).

Definition 3.1. [Viability and invariance properties]. *Let K be a subset of Dom (F). We shall say that K enjoys the local viability property (for F) if, for any initial state $x_0 \in K$, there exist $T > 0$ and a solution $x(\cdot)$ on $[0, T]$ of (1) starting at x_0 which is viable in K (i.e., $x(t) \in K$ for all $t \in [0, T]$). It enjoys the global viability property (or, simply, the viability property) if we can always take $T = \infty$.*

The subset K is said to be invariant by F if, for any initial state x_0 of K, all solutions of (1) are viable.

Let us emphasize that the concept of invariance *depends upon the behavior of F on Dom$(F)\backslash K$.*

A subset $K \subset$ Dom (F) is called a *viability domain* of F if and only if $F(x) \cap T_K(x) \neq \emptyset$ for all $x \in K$ and an *invariance domain* of F if and only if $F(x) \subset T_K(x)$ on K.

Let K be a closed subset of the domain of F. We shall prove the existence of the largest closed viability and invariance domains (which may be empty) contained in K. They will be called the *viability kernel of K* and *invariance kernel* of K, and denoted by Viab (K) and Inv (K), respectively.

We begin by proving that such a viability kernel does exist and characterize it.

Theorem 3.1. *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth. Let $K \subset$ Dom (F) be closed. Then the viability kernel of K exists (possibly empty) and is the subset of initial points such that at least one solution starting from them remains in K.*

Proof: Let $\mathcal{K} := \mathcal{C}(0, T; K)$ and $K_0 := \{x \in K \,|\, \mathcal{S}(x) \cap \mathcal{K} \neq \emptyset\}$. A standard argument shows that that K_0 is closed (see for instance the Convergence Theorem of ([2, Theorem 2.2.1.]). If $K_0 \neq \emptyset$, we also have $F(x) \cap T_{K_0}(x) \neq \emptyset$ on K_0, since this condition is necessary for the existence, thanks to the Viability Theorem (see [9], [2, Proposition 4.2.1]). For the same reason, any closed viability domain of F restricted to K is contained in K_0, hence $K_0 = \text{Viab}(K)$. ∎

The viability kernels may inherit properties of both F and K. For instance, if the graph of F and the subset K are convex, so is the viability kernel of K. If F is a closed convex process (i.e., its graph is a closed convex cone) and if K is a closed convex cone, the viability kernel is a closed convex cone.

We prove now the existence of an invariance kernel:

Theorem 3.2. *Let us assume that F is Lipschitzean on the interior of its domain and has compact values. For any closed subset $K \subset \text{Int}(\text{Dom}(F))$, there exists an invariance kernel of K (possibly empty). It is the subset of initial points such that all solutions starting from them remain in K.*

Proof: We set $K_1 := \{x \in K \mid \mathcal{S}(x) \subset \mathcal{K}\}$, with $\mathcal{K} = C(0,T;K)$. The argument is similar to the previous proof, but this time we use Filippov's Theorem (see [2, Corollary 2.4.1, p.121]) saying that for all $T > 0$, the solution map $\mathcal{S}$ is Lipschitzean from the interior of the domain of F to $\mathcal{C}(0,T;X)$ or even, to $W^{1,1}(0,T;X)$. In particular, it is lower semicontinuous, and thus, lower semicontinuous from the interior of the domain of F to $\mathcal{C}(0,T;X)$ supplied with the topology of pointwise convergence. ∎

It is clear that

$$\text{Inv}(K_1 \cap K_2) = \text{Inv}(K_1) \cap \text{Inv}(K_2) \tag{13}$$

and more generally, that the invariance kernel of any intersection of closed subsets K_i $(i \in I)$ is the intersection of the invariance kernels of the K_i.

Let us also introduce another "tangent cone" which is useful in optimization under constraints (see for instance [5]).

Definition 3.2. *The Dubovitskii-Milyutin tangent cone $D_K(x)$ to K is defined by:*

$$\begin{cases} v \in D_K(x) \quad \text{if and only if} \\ \exists\, \varepsilon > 0,\ \exists\, \alpha > 0 \quad \text{such that} \quad x +]0,\alpha](v + \varepsilon B) \subset K. \end{cases}$$

Let K be closed and $\widehat{K} := \overline{X \backslash K}$. Then the Dubovitskii-Milyutin tangent cone is related to contingent cone by the relation $X \backslash T_K(x) = D_{\widehat{K}}(x)$.

This observation and the proof of the Viability Theorem imply the following useful result:

Proposition 3.1. *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images. Let $K \subset \text{Dom}(F)$ be closed with nonempty interior and $x_0 \in \partial K$. Then each of the following conditions implies the next one:*

i) $F(x_0) \subset D_K(x_0)$

ii) *for any solution starting from x_0, $\exists\, T > 0 \mid \forall t \in]0,T]$, $x(t) \in \text{Int}(K)$*

iii) *for any solution starting from x_0, there exists $T > 0$ such that $x(T) \in \text{Int}(K)$*

iv) *$\exists$ a sequence $x_n \in \partial K$ converging to x_0 such that $F(x_n) \subset D_K(x_n)$.*

All these statements are equivalent if we assume that the set-valued map R defined by

$$x \in \partial K \rightsquigarrow R(x) := F(x) \cap T_{\widehat{K}}(x)$$

is lower semicontinuous on ∂K at x_0.

Proof: The statement of this proposition can be reformulated in this way; each condition implies the next one:

i) $\exists\, r > 0$ such that for all $x \in \partial K \cap (x_0 + rB)$, we have $F(x) \cap T_{\widehat{K}}(x) \neq \emptyset$

ii) $\exists\, T > 0$ and a viable solution starting at x_0 on $[0, T]$

iii) $\exists$ a solution starting at x_0 such that $\forall T > 0$, $\exists t \in]0, T] \mid x(t) \in \widehat{K}$

iv) $F(x_0) \cap T_{\widehat{K}}(x_0) \neq \emptyset$.

The first implication follows from the proof of the sufficient condition of the Viability Theorem applied to $\widehat{K}$, the second implication is obvious and the third one ensues from the proof of the necessary condition of the Viability Theorem still applied to $\widehat{K}$.

If, in addition,

$$x \in \partial K \rightsquigarrow R(x) := F(x) \cap T_{\widehat{K}}(x) \text{ is lower semicontinuous at } x_0 \in \partial K,$$

then when $v_0 \in R(x_0)$, there exists a neighborhood $\partial K \cap (x_0 + rB)$ such that $(v_0 + B) \cap R(x) \neq \emptyset$ on this neighborhood. Hence, the pointwise viability property implies the local one, and thus, the existence of at least one local viable solution starting from 0. ∎

As a consequence, we obtain the

Theorem 3.3. (Strict invariance theorem). Let F and K be as in Proposition 3.1. If $F(x) \subset D_K(x)$ on ∂K and $x_0 \in \partial K$, then any solution of (1) starting from x_0 remains in the interior of K on some interval $]0, T]$.

We then can divide the boundary of ∂K into five areas:

$$\begin{cases} \overline{K_e} & := \{\, x \in \partial K \mid F(x) \cap T_K(x) \neq \emptyset \,\} \\ \overline{K_s} & := \{\, x \in \partial K \mid F(x) \cap T_{\widehat{K}}(x) \neq \emptyset \,\} \\ K_e & := \partial K \backslash \overline{K_s} = \{\, x \in \partial K \mid F(x) \subset D_K(x) \,\} \subset \overline{K_e} \\ K_s & := \partial K \backslash \overline{K_e} = \{\, x \in \partial K \mid F(x) \subset D_{\widehat{K}}(x) \,\} \subset \overline{K_s} \\ K_b & := \{\, x \in \partial K \mid F(x) \cap T_{\partial K}(x) \neq \emptyset \,\} \subset \overline{K_e} \cap \overline{K_s}\,. \end{cases}$$

Proposition 3.2. *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and a closed subset K of its domain with a nonempty interior.*

— *Whenever $x \in K_e$, all solutions starting at x must enter the interior of K on some open time interval $]0, T[$, and whenever $x \in K_s$, all solutions starting at x must leave the subset K on some $]0, T[$.*

— *If $\partial K \cap (x + rB) \subset \overline{K_e}$ for some $r > 0$, then at least one solution starting at x is viable in K on some $[0, T]$ and the analogous statement holds true for $\overline{K_s}$.*

— *If $\partial K \cap (x + rB) \subset K_b$ for some $r > 0$, then at least one solution starting at x remains in the boundary ∂K on some $[0, T]$.*

In summary, the boundary of K can be partitioned into K_e, K_s, $\overline{K_e} \cap \overline{K_s}$ and K_b. From K_e, all solutions must enter K, from K_s, all solutions must leave K, from K_b, a solution can remain in the boundary if $F(\cdot) \cap T_{\partial K}(\cdot)$ is lower semicontinuous, from $(\overline{K_e} \cap \overline{K_s}) \backslash K_b$, a solution can remain in K or in $\widehat{K}$ if either $F(\cdot)\cap T_K(\cdot)$ or $F(\cdot)\cap T_{\widehat{K}}(\cdot)$ is lower semicontinuous on the boundary of K.

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