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AN ESTIMATE FOR THE MINIMA OF THE FUNCTIONALS OF THE CALCULUS OF VARIATIONS

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1. We consider a functional of the Calculus of Variations of the following form

$$J(w) = \int_G [f(x, Dw(x)) - g(x, w(x))]\, dx \tag{1.1}$$

where

i) G is a bounded open subset of R^n,

ii) $f : (x, z) \in G \times R^n \to R$ is a Carathéodory function,

iii) $g : (x, s) \in G \times R \to R$ is measurable in x and differentiable in s.

Let $A : [0, +\infty) \to R$ be a convex function such that $\lim_{r\to 0}(A(r)/r) = 0$ and consider a function $u \geq 0$ which minimizes (1.1) in the Orlicz space $W_0^{1,A}(G)$ (this definition is given in Section 2). Here, we prove an a priori estimate for u. Therefore, we are not concerned with the existence problem of minima, but assuming the existence of a minimum of (1.1), we seek a priori bounds for it.

Before we state our result more precisely, we recall that for each function $\phi \in L^1(G)$ its Schwarz symmetrized, denoted by ϕ^*, is defined in the ball G^* centered at the origin and with the same measure as G. In Section 2, we give some definitions and preliminaries. In Section 3, we prove the following theorem.

Theorem. *Let i), ii) and iii) hold. Moreover, assume that*

iv) there exists a function $A(r)$ with the above properties such that

$$\liminf_{\epsilon\to 0^-} \frac{f(x,(1+\epsilon)z) - f(x,z)}{\epsilon} \geq A(|z|) \quad \forall x, z.$$

v) The partial derivative $g_s(x, s)$ of $g(x, s)$ with respect to s satisfies

$$g_s(x, s) \leq g_s(x, 0) \quad \forall s \geq 0, \quad \text{for a.e. } x \in G$$

with $g_s(x, 0) \in L^1(G)$. Then, if $u \geq 0$ is a minimum in $W_0^{1,A}(G)$ for the functional (1.1),

$$u^*(x) \leq v(x) \quad \text{for a.e. } x \in G^*,$$

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where $v(x) \in W_0^{1,A}(G^*)$ is the minimum of the functional

$$J^*(w) = \int_{G^*} \Big[\int_0^{|Dw|} \frac{A(s)}{s}\, ds - (g_s)^*(x,0) w(x) \Big]\, dx. \tag{1.2}$$

This result extends previous results proved by G. Talenti in [9] and [10]. In fact, if we assume, in addition, that the function $f(x,z)$ is differentiable with respect to the variable z, the minimum u has to satisfy the Euler equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_{u_{x_i}}(x, Du(x)) = g_u(x,u) \tag{1.3}$$

and the analagous estimate for weak solution of (1.3) was first proved in [10]. We emphasize that here, we do not assume any differentiability or convexity condition on the function f. As far as we know, some extensions of Talenti's results [9] and [10] can be found also in [2], [4] and [7], but they are concerned with weak solutions of equations and not with minima of functionals.

2. Let R^n be the n-dimensional Euclidian space and let $\phi \in L^1(R^n)$. The distribution function of ϕ is defined by

$$\mu(t) = \text{meas}\,\{x \in R^n : |\phi(x)| > t\}. \tag{2.1}$$

The Schwarz symmetrization (or spherically decreasing rearrangement) ϕ^* of ϕ is defined by

$$\phi^*(x) = \inf\{t \geq 0 : \mu(t) < S_n |x|^n\},$$

where S_n is the measure of the n-dimensional unit ball. By definition, $\phi^*(x)$ is the unique, positive, radial and decreasing function with the same distribution function as ϕ. If $\phi \in L^1(G)$ and G satisfies i), the Schwarz symmetrization of ϕ is defined on G^*, which is the ball centered at the origin with the same measure as G, by the position

$$\phi^* = (\tilde{\phi})^*/G^*$$

where

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in G \\ 0 & \text{if } x \in R^n - G; \end{cases}$$

i.e., it is the restriction to G^* of the Schwarz symmetrization of $\tilde{\phi}$.

We refer to [3], [5], and [8] for classical results on symmetrization such as the following inequality

$$\int_{R^n} |\phi(x) \cdot \psi(x)|\, dx \leq \int_{R^n} \phi^*(x) \psi^*(x)\, dx \tag{2.2}$$

which holds for each ϕ, ψ real-valued measurable functions. Now, let $A : [0,\infty) \to R$ be a convex function such that $\lim_{r\to 0}(A(r)/r) = 0$. We consider the Orlicz convex class

$$\Big\{ \phi \text{ measurable } : \int_G A(|\phi(x)|)\, dx < +\infty \Big\}. \tag{2.3}$$

It is well known (see, for example, [1]) that the Orlicz class (2.3) is linear if and only if $A(r)$ satisfies the so-called Δ_2-condition$^{(\circ)}$. The linear hull of (2.3) is the Orlicz space

$$L^A(G) = \Big\{\phi \text{ measurable } : \exists\lambda \text{ such that } \int_G A(|\frac{\phi(x)}{\lambda}|)\,dx < +\infty\Big\}.$$

$L^A(G)$ is a Banach space under the norm

$$\|\phi\|_{L^A(G)} = \inf\Big\{\lambda > 0 : \int_G A(\frac{|\phi(x)|}{\lambda})\,dx \le 1\Big\}.$$

The Orlicz space $W^{1,A}(G)$ is defined as the subset of $L^A(G)$ whose elements have distributional first derivatives in $L^A(G)$ too. By $W_0^{1,A}(G)$, we denote the closure of $C_0^1(G)$ with respect to the $W^{1,A}$-norm. For details on Orlicz spaces, we refer to [1].

In the following, we need some notations. We denote by "$\{u > t\}$" the set $\{x \in G : u(x) > t\}$ and for any monotone function $\phi(t)$, we denote by $d\phi/dt$ the Radon-Nykodim derivative of ϕ with respect to the Lebesgue measure; i.e.,

$$\frac{d\phi}{dt} = \lim_{h\to 0^+} \frac{\phi(t+h)-\phi(t)}{h}\,.$$

3. In order to prove the theorem, we give some preliminary lemmas.

Lemma 3.1. *Let the nonnegative function $u(x)$ minimize the functional (1.1) in the space $W_0^{1,A}(G)$. Under the assumptions of the theorem, for a.e., $t > 0$, the following inequality holds:*

$$-\frac{d}{dt}\int_{\{u>t\}} A(|Du(x)|)\,dx \le \int_{\{u^*>t\}} (g_u)^*(x,0)\,dx\,. \tag{3.1}$$

Proof: For $s > 0$, $t > 0$ and $h > 0$, set

$$F_{t,h}(s) = \begin{cases} 0 & \text{if } s < t \\ s-t & \text{if } t \le s \le t+h \\ h & \text{if } s > t+h\,. \end{cases}$$

For $u \in W_0^{1,A}(G)$ one can easily see that $F_{t,h}(u) \in W_0^{1,A}(G)$. Now, since u minimizes (1.1), for ϵ small, we have

$$J(u) \le J\big(u + \epsilon F_{t,h}(u)\big);$$

that is, by using the definition of $F_{t,h}$,

$$J(u) \le \int_{\substack{\{u>t+h\}\\ \{u<t\}}} f(x,Du)\,dx + \int_{\{t<u<t+h\}} f(x,(1+\epsilon)Du)\,dx$$
$$- \int_{\{u<t\}} g(x,u)\,dx - \int_{\{t<u<t+h\}} g(x,u+\epsilon(u-t))\,dx - \int_{\{u>t+h\}} g(x,u+\epsilon h)\,dx\,.$$

($\circ$) We say that $A(r)$ satisfies the Δ_2-condition if there exists a constant $K > 0$ such that $A(2r) \le KA(r)$, $\forall r > 0$.

Therefore, easy calculations for $\epsilon < 0$, give

$$\int_{\{t<u<t+h\}} \frac{f(x,(1+\epsilon)Du) - f(x,Du)}{\epsilon}\,dx \le$$
$$\int_{\{u>t+h\}} \frac{g(x,u+\epsilon h) - g(x,u)}{\epsilon}\,dx + \int_{\{t<u<t+h\}} \frac{g(x,u+\epsilon(u-t)) - g(x,u)}{\epsilon}\,dx\,.$$

In the previous inequality, we estimate from below the left hand side by using Fatou's lemma and assumptions iv) and we estimate from above the right hand side by using v) and the dominated convergence theorem, so we get:

$$\begin{aligned}\frac{1}{h}\int_{\{t<u<t+h\}} A(|Du|)\,dx &\le \frac{1}{h}\int_{\{t<u<t+h\}} \liminf_{\epsilon\to 0^-} \frac{f(x,(1+\epsilon)Du) - f(x,Du)}{\epsilon}\,dx \\ &\le \int_{\{u>t\}} g_u(x,u)\,dx + \frac{1}{h}\int_{\{t<u<t+h\}} g_u(x,u)(u-t)\,dx\,.\end{aligned}$$

To the limit for $h \to 0^+$, for a.e. $t > 0$, we have

$$-\frac{d}{dt}\int_{\{u>t\}} A(|Du|)\,dx \le \int_{\{u>t\}} g_u(x,u)\,dx. \tag{3.2}$$

Now, given any function $v(x)$, denote by $\chi_{\{v>t\}}$ the characteristic function of the set $\{x : v(x) > t\}$. From v) and (2.2), we have

$$\begin{aligned}\int_{\{u>t\}} g_u(x,u)\,dx &\le \int_{\{u>t\}} g_u(x,0)\,dx = \int_{R^n} g_u(x,0)\chi_{\{u>t\}}(x)\,dx \\ &\le \int_{R^n} (g_u)^*(x,0)\chi_{\{u^*>t\}}(x)\,dx = \int_{\{u^*>t\}} (g_u)^*(x,0)\,dx.\end{aligned} \tag{3.3}$$

Inequality (3.1) follows from (3.2) and (3.3).

Remark 1. Let $v(x)$ be the minimum of (1.2) in $W_0^{1,A}(G^*)$ (the existence has been investigated in [10]). Since (1.2) is differentiable, $v(x)$ has to satisfy the Euler equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big[\frac{A(|Dv|)}{|Dv|^2} v_{x_i}(x)\Big] = (g_u)^*(x,0).$$

By standard calculations, such as multiplication by the test function $F_{t,h}(v(x))$ and integration by parts, we get:

$$\frac{1}{h}\int_{\{t<v<t+h\}} A(|Dv(x)|)\,dx = \frac{1}{h}\Big[\int_{\{v>t+h\}} (g_u)^*(x,0)h\,dx + \int_{\{t<v<t+h\}} (g_u)^*(x,0)(v-t)\,dx\Big]\,.$$

Now, for $h \to 0^+$, we have

$$-\frac{d}{dt}\int_{\{v>t\}} A(|Dv|)\,dx = \int_{\{v>t\}} (g_u)^*(x,0)\,dx, \quad \text{for a.e. } t > 0. \tag{3.4}$$

Remark 2. From our assumptions on the function $A(r)$, it follows that $A(r)$ and $A(r)/r$ are positive and increasing.

As in [10], we assume, for convenience, that $A(r)$ is strictly increasing and twice continuously differentiable; therefore, the function $B(r) = A(r)/r$ is also strictly increasing.

We recall the following lemma which has been proved in [10].

Lemma 3.2. *Let $A : r \in [0,+\infty) \to R$ be a positive increasing convex function such that $\lim_{r\to 0^+}(A(r)/r) = 0$. Let G satisfy i). For $u \in W_0^{1,A}(G)$, let $\mu(t)$ be the distribution function of u. Then, for a.e. $t > 0$, the following inequality holds:*

$$nS_n^{1/n}\mu(t)^{1-1/n} \leq -\mu'(t)B^{-1}\Big[\frac{-\frac{d}{dt}\int_{\{u>t\}} A(|Du|)\,dx}{nS_n^{1/n}\mu(t)^{1-1/n}}\Big]. \tag{3.5}$$

The inequality (3.5) becomes an equality if u is spherically symmetric; in fact, we can prove the following lemma.

Lemma 3.3. *Under the assumptions of Lemma 3.2, if $v(x) \in W_0^{1,A}(G^*)$ is spherically symmetric and $\nu(t)$ is its distribution function, for a.e. $t > 0$, the following equality holds:*

$$nS_n^{1/n}\nu(t)^{1-1/n} = -\nu'(t)B^{-1}\Big[\frac{-\frac{d}{dt}\int_{\{v>t\}} A(|Dv|)\,dx}{nS_n^{1/n}\nu(t)^{1-1/n}}\Big]. \tag{3.6}$$

Proof: Let us observe that if $v(x)$ is spherically symmetric, denoting by $b(0,r)$, the ball centered at the origin with radius r, we have

$$\int_{\{v>t\}} |Dv(x)|\,dx = \int_{b(0,v^{-1}(t))} |Dv(|x|)|\,dx = nS_n \int_0^{(\nu(t)/S_n)^{1/n}} |Dv(r)|r^{n-1}\,dr$$

and, setting $r = (s/S_n)^{1/n}$, we have

$$\int_{\{v>t\}} |Dv(x)|\,dx = \int_0^{\nu(t)} |Dv((s/S_n)^{1/n})|\,ds\,. \tag{3.7}$$

In a similar way,

$$\int_{\{v>t\}} A(|Dv(x)|)\,dx = \int_0^{\nu(t)} A(|Dv((s/S_n)^{1/n}|)\,ds. \tag{3.8}$$

Now, a derivation with respect to the variable t in (3.8) gives

$$-\frac{d}{dt}\int_{\{v>t\}} A(|Dv(x)|)\,dx = -\nu'(t)A(|Dv((\nu(t)/S_n)^{1/n})|)$$

and, recalling that $B(r) = A(r)/r$, the right hand side is equal to

$$B[|Dv((\nu(t)/S_n)^{1/n})|]|Dv((\nu(t)/S_n)^{1/n})|(-\nu'(t)).$$

Therefore,

$$B^{-1}\Big[\frac{-\frac{d}{dt}\int_{\{v>t\}} A(|Dv(x)|)\,dx}{(-\nu'(t))|Dv((\nu(t)/S_n)^{1/n})|}\Big] = |Dv((\nu(t)/S_n)^{1/n})|\,. \tag{3.9}$$

On the other hand, a derivation with respect to the variable t in (3.7) and easy calculations (see, for example, step 3 in [6]) give

$$nS_n^{1/n}\nu(t)^{1-1/n} = -\frac{d}{dt}\int_{\{v>t\}} |Dv(x)|\,dx = (-\nu'(t))|Dv((\nu(t)/S_n)^{1/n})|\,. \tag{3.10}$$

From (3.9) and (3.10), we obtain (3.6).

Now we are able to prove the theorem.

Proof. Let $u(x)$ and $v(x)$ minimize, respectively, (1.1) in $W_0^{1,A}(G)$ and (1.2) in $W_0^{1,A}(G^*)$ and denote by $\mu(t)$ and $\nu(t)$, respectively, the distribution functions of $u(x)$ and $v(x)$. Following [6], the idea of the proof is to obtain a differential inequality between μ and ν from which we deduce that $\mu(t) \leq \nu(t)$ for a.e. $t > 0$ and, obviously, that $u^*(x) \leq v(x)$ since μ is also the distribution function of u^* and u^* and v are both radial and decreasing. By applying (3.5) and (3.6), respectively, to $u(x)$ and $v(x)$, for a.e. $t > 0$, we have:

$$\mu'(t)\mu(t)^{-1+1/n}B^{-1}\Big[\frac{-\frac{d}{dt}\int_{\{u>t\}}A(|Du(x)|)\,dx}{nS_n^{1/n}\mu(t)^{1-1/n}}\Big]$$
$$\leq \nu'(t)\nu(t)^{-1+1/n}B^{-1}\Big[\frac{-\frac{d}{dt}\int_{\{v>t\}}A(|Dv(x)|)\,dx}{nS_n^{1/n}\nu(t)^{1-1/n}}\Big].$$

Therefore, by using (3.1) and (3.4), since B^{-1} is strictly increasing and μ' is negative, we have

$$\mu'(t)\mu(t)^{-1+1/n}B^{-1}\Big[\frac{\int_{\{u^*>t\}}(g_u)^*(x,0)\,dx}{nS_n^{1/n}\mu(t)^{1-1/n}}\Big]$$
$$\leq \nu'(t)\nu(t)^{-1+1/n}B^{-1}\Big[\frac{\int_{\{v>t\}}(g_u)^*(x,0)\,dx}{nS_n^{1/n}\nu(t)^{1-1/n}}\Big].$$

A change of variables gives

$$\mu'(t)\mu(t)^{-1+1/n}B^{-1}\Big[\frac{\int_0^{\mu(t)}(g_u)^*((s/S_n)^{1/n},0)\,ds}{nS_n^{1/n}\mu(t)^{1-1/n}}$$
$$\leq \nu'(t)\nu(t)^{-1+1/n}B^{-1}\Big[\frac{\int_0^{\nu(t)}(g_u)^*((s/S_n)^{1/n},0)\,ds}{nS_n^{1/n}\nu(t)^{1-1/n}}.$$

Now, set $\phi(s) = (g_u)^*((s/S_n)^{1/n},0)$, we consider the function

$$F(\lambda) = \lambda^{-1+1/n}B^{-1}\Big[\frac{\int_0^{\lambda}\phi(s)\,ds}{nS_n^{1/n}\lambda^{1-1/n}}\Big]$$

and, as in [6], we observe that $F(\lambda)$ is a continuous function on each interval $[\epsilon, \text{meas}\, G]$, $\epsilon > 0$, and $F(\lambda) > 0$ for $\lambda > 0$. The last inequality may be written in the following way:

$$\mu'(t)F(\mu(t)) \leq \nu'(t)F(\nu(t)).$$

Let $\tilde{F}$ be a function such that $(d\tilde{F}/dt) = F$; $\tilde{F} \in C^1((0, \text{meas}\, G])$. By integration, we have

$$\tilde{F}(\mu(t)) - \tilde{F}(\mu(0)) \leq \tilde{F}(\nu(t)) - \tilde{F}(\nu(0))$$

and, since $\mu(0) = \nu(0) = \text{meas}\, G$,

$$\tilde{F}(\mu(t)) \leq \tilde{F}(\nu(t)) \quad \text{for a.e. } t > 0,$$

which, since $\tilde{F}$ is increasing, implies the desired inequality

$$\mu(t) \leq \nu(t) \quad \text{for a.e. } t > 0.$$

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