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# ON SPATIAL ENERGY DECAY FOR QUASILINEAR BOUNDARY VALUE PROBLEMS IN CONE-LIKE AND EXTERIOR DOMAINS

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**Abstract.** A number of boundary value problems for second order quasilinear partial differential equations in divergence form, in cone-like domains and exterior domains, in two and three dimensions, are considered. For the cone-like domains, homogeneous data of either the Dirichlet, Neumann, or mixed type are prescribed on the lateral sides. New results are obtained concerning the spatial decay of the energy (Dirichlet norm of the solution) at infinity and its maximum rate of growth near the finite end of the domain.

**1. Introduction.** During the last 25 years, much research has been done on spatial decay behavior of solutions of elliptic boundary value problems in semi-infinite domains. A comprehensive review of the work up to the early eighties is given by Horgan and Knowles [1]. Among the subsequent papers on the subject are [2]-[7] and also some of the references therein.

The equations considered here, as in [5]-[7], are single second order equations in divergence form:

$$\frac{\partial}{\partial x_1}\left[\rho(\mathbf{x},u,\nabla u)\frac{\partial u}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[\rho(\mathbf{x},u,\nabla u)\frac{\partial u}{\partial x_2}\right]+\frac{\partial}{\partial x_3}\left[\rho(\mathbf{x},u,\nabla u)\frac{\partial u}{\partial x_3}\right]=0\ , \tag{1.1}$$

for a wide range of positive functions $\rho$. The conditions on $\rho$ are general enough so that (1.1) need not necessarily be elliptic as long as a solution exists which satisfies the given conditions of the problem. The boundary data are of either the Dirichlet, Neumann, or mixed type.

Many important equations of mathematical physics are included in (1.1). For example, when $\rho = 1$, one obtains Laplace's equation, when $\rho = \rho(\mathbf{x}, u)$, (1.1) corresponds to the nonlinear steady-state diffusion equation, and when $\rho = [1 + |\nabla u|^2]^{-\frac{1}{2}}$, the minimal surface equation is obtained.

Throughout this paper, standard index notation will be used, and the usual summation convention will be followed. Thus, we write

$$u_{,j} = \frac{\partial u}{\partial x_j}\ ;$$

$$[\rho u_{,\beta}]_{,\beta} = [\rho u_{,1}]_{,1} + [\rho u_{,2}]_{,2}\ ;$$

$$[\rho u_{,j}]_{,j} = [\rho u_{,1}]_{,1} + [\rho u_{,2}]_{,2} + [\rho u_{,3}]_{,3}\ ;$$

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i.e., Latin indices range over the values 1, 2, 3 while Greek indices range over the values 1 and 2. Using this notation, we may rewrite (1.1) in the form

$$[\rho(\mathbf{x}, u, \nabla u)u_{,j}]_{,j} = 0 \; . \tag{1.2}$$

When the underlying domain is a cylinder of the form

$$R = [(x_1, x_2, x_3)|(x_1, x_2) \in S, x_3 > 0] \; , \tag{1.3}$$

where $S$ is a domain in the $x_1 - x_2$ plane, and the data on the lateral portion of the boundary are homogeneous, a frequently obtained result is exponential decay of the energy integral (Dirichlet integral) of the form

$$E(z) \leq E(0)e^{-kz}, \quad z \geq 0 \; , \tag{1.4}$$

where

$$E(z) = \iiint_{R_z} \rho u_{,j}\, u_{,j} \; d\tau \; , \tag{1.5}$$

with

$$R_z = [(x_1, x_2, x_3)|(x_1, x_2, x_3) \in R, \;\; x_3 > z > 0] \, , \tag{1.6}$$

and $k$ depends only upon the properties of the cross-section $S$ and on the function $\rho$.

In order for (1.4) to be meaningful, as pointed out in [7], one must limit the class of considered solutions to those for which $E(0)$ is finite. Except for [7], either $E(0)$ has been a priori assumed to be finite, or boundary data at the face $z = 0$ were assumed to be sufficiently smooth (for example, continuously differentiable) so that a bound for $E(0)$ could be derived from this data, as in [2]. In fact, it is possible to prescribe continuous boundary data on the face for which a solution to the boundary value problem exists but such that the energy integral is infinite, *even* when the equation is linear. Specific examples are discussed below, in Sections 2, 3, and 4.

In [7] it is shown that, for a semi-infinite cylinder with only the assumption of $u$ being bounded at infinity (but without any assumption on $\nabla u$),

$$E(a) \leq b_1 a^{-r} + b_2 a^{-1} \, , \qquad a > 0 \, , \tag{1.7}$$

where $r \geq 1$ depends upon $\rho$, and $b_1$ and $b_2$ depend upon $\rho$, upon the uniform bound for $u$ and upon the area of the cross-section $S$.

When $\rho$ satisfies additional conditions under which one may obtain a result of the form (1.4), then (1.4) and (1.7) may be combined to give

$$E(z) \leq (b_1 a^{-r} + b_2 a^{-1})e^{-k(z-a)} \, , \tag{1.8}$$

for all $z > a > 0$, thereby obtaining a decay estimate for $E(z)$ which is valid even in cases where $E(0) = +\infty$, provided only that a solution to the considered problem exists.

Equation (1.7) by itself constitutes a decay bound as $a \to +\infty$, although it is a weak bound in that (1.8) implies that the decay near infinity is actually exponential rather than algebraic. However, (1.7) also provides a growth estimate for $E(a)$ as $a \to 0^+$ for all functions satisfying the partial differential equation and the boundary conditions, even those producing infinite energy, and this estimate seems to be close to optimal, as is shown by the examples discussed in Sections 2, 3, and 4.

Most of the results obtained to date have been for cylindrical domains or domains which can be enclosed in a cylinder. In [8], Breuer and Roseman discuss decay results in non-cylindrical (non-striplike) two-dimensional domains. As far as the authors can determine, the only discussion of decay results in cone-like domains is found in the work of Berdichevskii [9] which implies that the analogue of (1.4) for a cone is

$$E(z) \leq E(0)(1 + \frac{z}{c})^{-c\gamma}, \qquad z > 0, \tag{1.9}$$

where the apex of the cone is at $(0, 0, -c)$, and $\gamma$ is a positive exponent related to the conical angle. Once again, (1.9) ceases to be meaningful when $E(0) = +\infty$.

In [7], spatial decay results were obtained for semi-infinite cylindrical domains. The purpose of the work here is to obtain corresponding results for unbounded non-cylindrical domains, such as cone-like and exterior domains. A number of such results are presented. Whereas, in cylinder-like domains, it was only required a priori in [7] that there exist a positive constant $N$ such that $|u| < N$ throughout the cylinder, it is necessary here, in three-dimensional domains, to require either that $u$ decay to zero in the neighborhood of infinity at a specified (rather mild) rate, or that the energy integral over an appropriate unbounded subdomain be finite, but not both. In two dimensions, it is only necessary to assume that $|u| < N$ throughout the domain. None of the theorems here require any a priori bound on any derivative of $u$.

The two-dimensional results are obtained by using conformal mapping principles in conjunction with the techniques used in [7]. The three-dimensional results are obtained by an appropriate variation of the techniques in [7] to take into account the non-cylindrical geometry of the underlying domains, and by suitably restricting the behavior of $u$ at infinity, as described above.

## 2. Preliminaries.

Throughout the following, we shall be working with an equation of the form

$$[\rho(\mathbf{x}, u, \nabla u)u_{,k}]_{,k} = 0, \tag{2.1a}$$

or

$$[\rho(\mathbf{x}, u, \nabla u)u_{,\beta}]_{,\beta} = 0, \tag{2.1b}$$

in a given three-dimensional or two-dimensional domain, where $\rho$ is a differentiable function which satisfies, throughout the domain, an inequality of the form

$$0 < \rho \leq M + K\{\rho|\nabla u|^2\}^{1/p}, \tag{2.2}$$

for some $p > 1$, $M \geq 0$, and $K \geq 0$. In particular, this includes the functions $\rho = (1 + |\nabla u|^2)^{-1/2}$ ($M = 1$, $K = 0$), for which (2.1) represents the equation of a minimal surface, and $\rho = 1 + |\nabla u|^2$ ($M = 1$, $K = 1$, $p = 2$ may be used) and, of course, any bounded function $\rho$ for which $K$ may be taken to be zero.

All the domains considered, as well as their boundaries, are assumed to be sufficiently regular for the application of the divergence theorem. The symbol $\frac{\partial u}{\partial n}$ will denote an outward normal derivative of a function $u$ at a point on the boundary.

In order that this paper be as self-contained as possible, we shall now state the main theorem in [7] and outline the main steps of its proof.

**Theorem.** [7]: *Let $R$ be the cylinder defined in (1.2), $\partial R$ its boundary, and $\Gamma$ the subset of $\partial R$ where $x_3 > 0$. Suppose that $u(x_1, x_2, x_3) \in C^2(R) \cap C^1(R \cup \Gamma)$ and satisfies the conditions*

(i) *$u$ is a solution of (2.1a) in $R$ with a positive function $\rho$, for which there exist $p > 1$, $M \geq 0$, and $K \geq 0$ such that (2.2) is valid in $R$,*
(ii) *$u\frac{\partial u}{\partial n} = 0$ on $\Gamma$, and*
(iii) *there exists $N > 0$ such that $|u| < N$ throughout $R$.*

*Then, there exist positive constants $b_1$ and $b_2$, which depend only upon $N$, $M$, $K$, $p$ and the area of the cross-section $S$, such that*

$$E(a) \leq b_1 a^{-\left(\frac{p+1}{p-1}\right)} + b_2 a^{-1} \quad \text{for all } a > 0\,, \tag{2.3}$$

*where $E(a)$ is defined by (1.5), (1.6). In addition, if $K = 0$, then (2.3) holds with $b_1 = 0$.*

**Sketch of proof:** [7] First, two specific auxiliary cut-off functions of $x_3$, $\psi_a(x_3)$ and $H_a(x_3)$, are constructed, each of which also depends upon a positive parameter $a$, as shown in Figure 1.

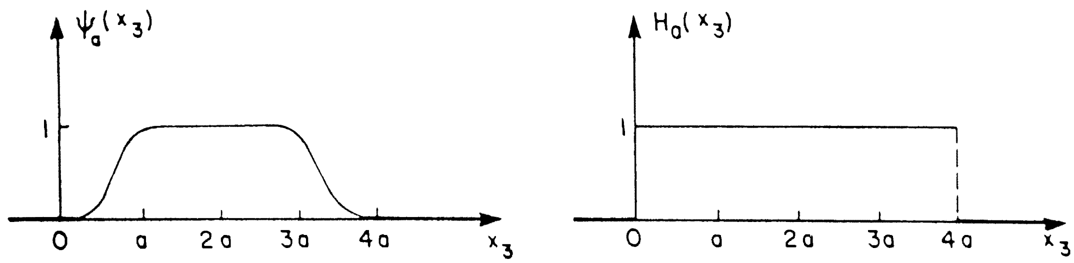


**Figure 1.**

The following relations hold:

$$0 \leq \psi_a(x_3) \leq H_a(x_3)\,, \tag{2.4}$$

$$|\psi_a'(x_3)| \leq \frac{D}{a} H_a(x_3)\,, \tag{2.5}$$

where $D$ is a constant, and

$$|\psi_a'(x_3)|^2 \leq \frac{C(p)}{a^2}[\psi_a(x_3)]^{2/p} H_a(x_3)\,, \tag{2.6}$$

for all $p > 1$, where $C(p)$ depends only upon $p$. (One specific $\psi_a(x_3)$ is explicitly constructed in [7]).

By use of the divergence theorem, condition (ii) of the theorem, equation (2.2), and the construction of $\psi_a$, it is shown in [7] that

$$\frac{1}{4}\iiint\limits_R \psi_a^2 \rho u_{,k} u_{,k}\, d\tau \leq M \iiint\limits_R \psi_a'^2 u^2\, d\tau + K \iiint\limits_R \psi_a'^2 \rho^{1/p}(u_{,k} u_{,k})^{1/p} u^2\, d\tau\,, \tag{2.7}$$

from which it follows that

$$\frac{1}{4q}\iiint\limits_R \psi_a^2 \rho u_{,k}\, u_{,k}\, d\tau \le M \iiint\limits_R \psi_a'^2 u^2\, d\tau + \frac{[4KC(p)]^q}{4qa^{2q}} \iiint\limits_R H_a u^{2q}\, d\tau\,, \tag{2.8}$$

where $q$ is defined by the relation $\frac{1}{p}+\frac{1}{q}=1$.

Next, from (2.5), condition (iii) of the theorem, and the construction of $\psi_a$ and $H_a$, it is seen that

$$\iiint\limits_a^{3a} \rho u_{,k}\, u_{,k}\, d\tau \le \frac{4qMD^2N^2}{a^2}\iiint\limits_0^{4a} d\tau + \frac{(4KC)^q N^{2q}}{a^{2q}}\iiint\limits_0^{4a} d\tau\,, \tag{2.9}$$

where $\iiint\limits_\alpha^\beta F\, d\tau$ represents the integral of $F$ over the subdomain of $R$ in which $\alpha < x_3 < \beta$. This notation will also be used in similar situations throughout this paper.

Since $\iiint\limits_0^{4a} d\tau = 4||S||a$, (2.9) implies

$$\iiint\limits_a^{3a} \rho u_{,j}\, u_{,j}\, d\tau \le \frac{16qMD^2N^2||S||}{a} + \frac{4(4KC)^q N^{2q}||S||}{a^{2q-1}}\,, \tag{2.10}$$

or

$$\iiint\limits_a^{3a} \rho u_{,j}\, u_{,j}\, d\tau \le \frac{B_1}{a} + \frac{B_2}{a^{2q-1}}\,, \tag{2.11}$$

for all $a>0$, with $B_1$ and $B_2$ independent of $a$.

Because (2.11) holds for all $a>0$, we have that

$$\iiint\limits_{3^k a}^{3^{k+1}a} \rho u_{,j}\, u_{,j}\, d\tau \le \frac{B_1}{3^k a} + \frac{B_2}{3^{(2q-1)k} a^{2q-1}}\,, \tag{2.12}$$

for all $a>0$ and any non-negative integer $k$.

Next, separate consideration is given in [7] to the cases where $a \ge 1$ and $a \le 1$, an appropriate summation is made over $k$, and the analysis leads to (2.3), with $b_1 = 0$ if $K=0$.

We note that in passing from (2.9) to (2.10), the geometry of the cylinder was exploited; this step breaks down for a cone-like domain for which the subvolume of $R$ between 0 and $4a$ is of the order of $a^3$ for $a>1$.

Furthermore, the proof will go through without changes for an $n$-dimensional cylinder, $n \ge 2$.

We shall now look at a specific example of a problem of the type discussed in the above theorem with infinite $E(0)$. Let $R = [(x_1,x_2,x_3)\,|\,0<x_1<1\,,0<x_2<1\,,x_3>0]$, $\rho=1$, and let the boundary data be

$$u(0,x_2,x_3) = u(1,x_2,x_3) = 0\,, \tag{2.13a}$$

$$u_{,2}(x_1,0,x_3) = u_{,2}(x_1,1,x_3) = 0\,, \tag{2.13b}$$

$$u(x_1, x_2, 0) = F(x_1) = \sum_{k=1}^{\infty} \frac{\sin(k^m \pi x_1)}{k^2}, \tag{2.13c}$$

where $m$ is a positive integer $\geq 4$. (The function $F(x_1)$ was suggested by an exercise in Garabedian [10, p. 286]).

In this example, (2.1a) reduces to the linear Laplace equation whose solution is readily obtained in the form

$$u(x_1, x_2, x_3) = \sum_{k=1}^{\infty} \frac{\sin(k^m \pi x_1)}{k^2} e^{-k^m \pi x_3}, \tag{2.14}$$

and then, by direct calculation,

$$E(a) \equiv \iiint_{R_a} u_{,j}\, u_{,j}\, d\tau = \frac{\pi}{2} \sum_{k=1}^{\infty} k^{m-4} e^{-2k^m \pi a}, \tag{2.15}$$

for $a > 0$. By standard asymptotic methods, we obtain

$$E(a) \sim \Gamma(1 - 3/m) \frac{(2\pi)^{3/m}}{4m} a^{-(1-3/m)} \quad \text{as} \quad a \to 0^+, \tag{2.16}$$

for $m \geq 4$. By comparing (2.16) with (2.3) with $b_1 = 0$ for the case of very large $m$, we see that the estimate given by the theorem for the growth rate of $E(a)$ as $a \to 0^+$ seems to be close to the best possible.

**3. Results for two-dimensional problems.** We begin by considering the case of an unbounded domain which can be contained in a sector of the plane as illustrated in Figure 2.

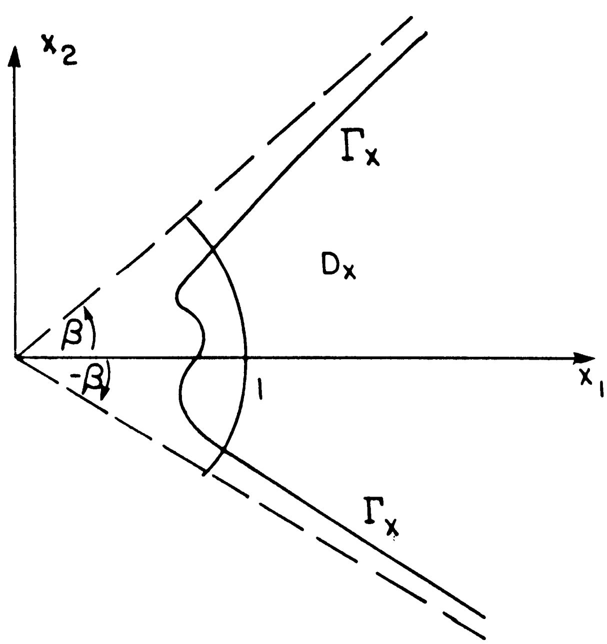


**Figure 2.**

**Theorem 1.** *Let $D_x$ be an unbounded simply connected domain in the $x_1 - x_2$ plane which is a subdomain of the sector $[(x_1, x_2)| - \beta <$ arc $\tan(\frac{x_2}{x_1}) < \beta]$ , $0 < \beta < \pi$, and define $\Gamma_x$ to be that subset of $\partial D_x$ on which $x_1^2 + x_2^2 > 1$. Suppose that $u \in C^2(D_x) \cap C^1(D_x \cup \Gamma_x)$ and*

(i) *$u$ satisfies (2.1b) and (2.2) in $D_x$,*

(ii) *$u\frac{\partial u}{\partial n} = 0$ on $\Gamma_x$, and*

(iii) *there exists $N > 0$ such that $|u| < N$ throughout $D_x$.*

*Let $D_x(a)$ be the subdomain of $D_x$ in which $x_\alpha x_\alpha > a^2 > 1$, and define*

$$\mathcal{E}(a) = \int\int_{D_x(a)}\int \rho u_{,\alpha}\, u_{,\alpha}\; dx_1\, dx_2\,. \tag{3.1}$$

*Then, there exist positive constants $b_1$ and $b_2$, which depend only upon $N$, $M$, $K$, and $p$, such that*

$$\mathcal{E}(a) \le b_1(\log a)^{-(\frac{p+1}{p-1})} + b_2(\log a)^{-1}\,, \tag{3.2}$$

*for all $a > 1$. Moreover, if $K = 0$ in (2.2), then $b_1 = 0$.*

**Remark:** Equation (3.2) implies *logarithmic* decay as $a \to \infty$ and *algebraic* decay as $a \to 1^+$. We also note that $b_1$ and $b_2$ are independent of $\beta$.

**Proof.** Let us define

$$r = (x_\alpha x_\alpha)^{1/2}\,, \tag{3.3a}$$

$$\theta = \text{arc}\,\cos(x_1/r) = \text{arc}\,\sin(x_2/r)\,, \quad -\pi < \theta \le \pi\,, \tag{3.3b}$$

and

$$\xi_1 = \log r\,, \tag{3.4a}$$

$$\xi_2 = \theta\,. \tag{3.4b}$$

The transformation from $x_1 - x_2$ to $\xi_1 - \xi_2$ coordinates is a one to one conformal mapping which maps the sector containing $D_x$ onto an infinite strip and $D_x$ onto $D_\xi$, a subdomain of the strip, as shown in Figure 3.

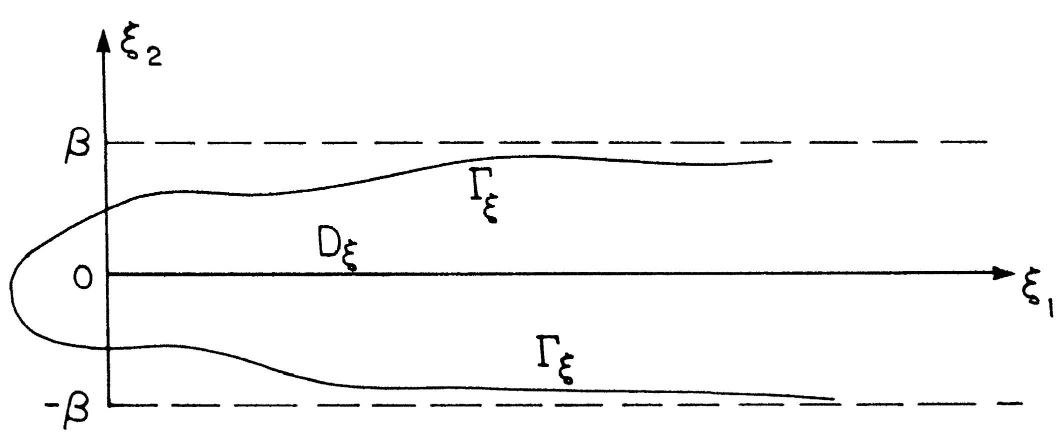


**Figure 3.**

The subdomain $D_x(a)$ is mapped onto $D_\xi(\log a)$, which is the subdomain of $D_\xi$ in which $\xi > \log a$; $\Gamma_x$ is mapped onto $\Gamma_\xi$, which is the portion of $\partial R_\xi$ on which $\xi_1 > 0$. The homogeneous boundary condition $u\frac{\partial u}{\partial n} = 0$ on $\Gamma_x$ is preserved, so that $u\frac{\partial u}{\partial n} = 0$ on $\Gamma_\xi$. In addition, the form of equation (2.1b) is also unchanged in that

$$\frac{\partial}{\partial \xi_\alpha}(\rho \frac{\partial u}{\partial \xi_\alpha}) = 0\,, \qquad (\xi_1, \xi_2) \in D_\xi\,. \tag{3.5}$$

Furthermore, from (2.2),

$$0 < \rho \le M + K\{\rho e^{-2\xi_1} \frac{\partial u}{\partial \xi_\alpha} \frac{\partial u}{\partial \xi_\alpha}\}^{1/p}\,, \tag{3.6}$$

and since $e^{-2\xi_1} < 1$ in $D_\xi$,

$$0 < \rho \le M + K\{\rho \frac{\partial u}{\partial \xi_\alpha} \frac{\partial u}{\partial \xi_\alpha}\}^{1/p} \qquad \text{in} \quad D_\xi\,. \tag{3.7}$$

Thus, we see that in $D_\xi$ the function $u$ fulfills all the conditions of the two-dimensional version of the main theorem in [7] discussed above. Now, since

$$\mathcal{E}(a) = \int\limits_{D_x(a)}\!\!\int \rho \frac{\partial u}{\partial x_\alpha} \frac{\partial u}{\partial x_\alpha}\, dx_1\, dx_2 = \int\limits_{D_\xi(\log a)}\!\!\int \rho \frac{\partial u}{\partial \xi_\alpha} \frac{\partial u}{\partial \xi_\alpha}\, d\xi_1\, d\xi_2 = E(\log a)\,, \tag{3.8}$$

it follows from (2.3) that

$$\mathcal{E}(a) = E(\log a) \le b_1 (\log a)^{-(\frac{p+1}{p-1})} + b_2 (\log a)^{-1}\,, \tag{3.9}$$

for all $a > 1$, which is the desired result.

For a general semi-infinite strip, the constants $b_1$ and $b_2$ depend linearly upon the thickness of the strip. Here, the thickness is $2\beta$, and since $\beta < \pi$, they can be chosen to be independent of $\beta$. ▮

Having obtained a result for a two-dimensional cone-like domain, we now turn our attention to an exterior problem; that is, a problem in a domain which is the exterior of a simple closed curve, as illustrated in Figure 4.

**Theorem 2.** *Let $C$ be a simple closed curve which is contained in the closed unit disk in the $x_1 - x_2$ plane, such that the origin belongs to Int $(C)$. Assume that $u \in C^2(D_x)$, where $D_x = $ Ext $(C)$, satisfies (2.1b) and (2.2) in $D_x$, and there exists $N > 0$ such that $|u| < N$ in $D_x$. Then, defining $D_x(a)$ and $\mathcal{E}(a)$ as in Theorem 1, we have that*

$$\mathcal{E}(a) \le b_1 (\log a)^{-\frac{p+1}{p-1}} + b_2 (\log a)^{-1}\,, \tag{3.10}$$

*for $a > 1$, where $b_1$ and $b_2$ depend only upon $N$, $M$, $K$, and $p$. In addition, $b_1 = 0$ if $K = 0$.*

**Remark.** As in Theorem 1, (3.10) implies *algebraic* growth as $a \to 1^+$.

**Proof:** We define the coordinates $\xi_1$ and $\xi_2$ as in Theorem 1, and let $D_x^* \subset D_x$ be the domain obtained by deleting all points on the negative real line from $D_x$ (see Figure 4). The transformation from $x_1 - x_2$ coordinates to $\xi_1 - \xi_2$ coordinates is one to one and

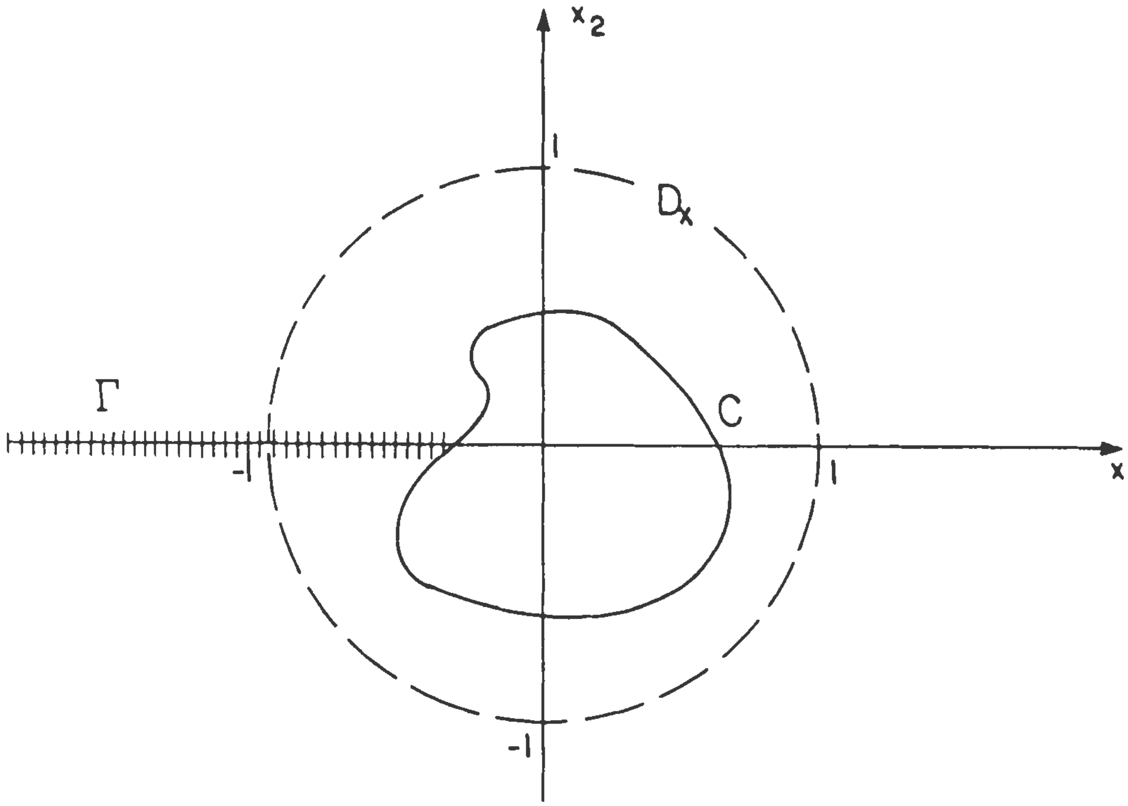


**Figure 4.**

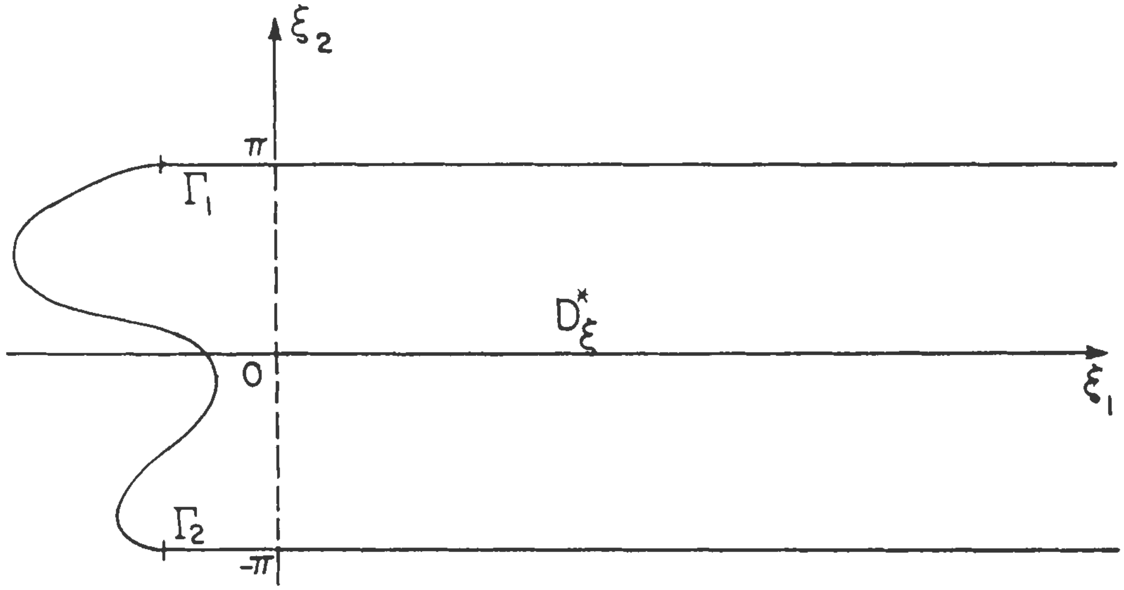


**Figure 5.**

conformal on $D_x^*$, and maps $D_x^*$ onto the domain $D_\xi^*$, which is of the type indicated in Figure 5.

*Denoting*

$$D_\xi^*(a) = [(\xi_1, \xi_2) | (\xi_1, \xi_2) \in D_\xi^* \text{ and } \xi_1 > a > 0], \tag{3.11}$$

we see that the subdomain of $D_x^*$ which is outside the unit circle is mapped onto the semi-infinite strip $D_\xi^*(0)$. As one approaches the cut $\Gamma$ in Figure 4 from above, $\Gamma_1$ is obtained as the image of $\Gamma$, while $\Gamma_2$ is obtained as one approaches from below. The portion of the boundary curve which connects $\Gamma_1$ and $\Gamma_2$ in the $\xi_1 - \xi_2$ plane is the image of $C$ in the $x_1 - x_2$ plane.

Once again, the form of the differential equation is not changed, so that

$$\frac{\partial}{\partial \xi_k}(\rho \frac{\partial u}{\partial \xi_k}) = 0 \quad \text{in} \quad D_\xi^* . \tag{3.12}$$

This time, the boundary conditions are the periodicity conditions which follow from the nature of the transformation; that is,

$$u(\xi_1, -\pi) = u(\xi_1, \pi) , \tag{3.13a}$$

$$\frac{\partial u}{\partial \xi_2}(\xi_1, -\pi) = \frac{\partial u}{\partial \xi_2}(\xi_1, \pi) . \tag{3.13b}$$

Equations (3.6) and (3.7) also hold in this situation.

Accordingly, all the conditions for the theorem in [7] are seen to be fulfilled in $D_\xi^*$, except that instead of the boundary condition $u\frac{\partial u}{\partial n} = 0$ on $\Gamma_1$ and $\Gamma_2$, we have the periodicity conditions (3.13). However, examination of the proof of the Roseman-Zimering theorem [7] shows that the proof goes through, without any changes, for the conditions (3.13). This is because of the fact that the only time the boundary condition $u\frac{\partial u}{\partial n} = 0$ comes into play is when the divergence theorem is used, and the significance of this boundary condition is that it causes the surface integral terms to vanish. The same effect is obtained with the conditions (3.13).

Therefore, as in Theorem 1, we obtain

$$\begin{aligned} \mathcal{E}(a) = E(\log a) &= \int\limits_{D_\xi^*(\log a)} \int \rho \frac{\partial u}{\partial \xi_k} \frac{\partial u}{\partial \xi_k} \, d\xi_1 \, d\xi_2 \\ &\leq b_1 (\log a)^{-(\frac{p+1}{p-1})} + b_2 (\log a)^{-1} , \quad a > 1 , \end{aligned} \tag{3.14}$$

and, as before, $b_1 = 0$ if $K = 0$. ∎

An example of a two-dimensional exterior Dirichlet problem, for which there exists a classical solution $u = u(r, \theta)$ with $\mathcal{E}(1)$ being infinite, arises when $C$ is the unit circle, $\rho = 1$, and the given data are

$$\lim_{r \to 1^+} u(r, \theta) = \sum_{k=1}^{\infty} \frac{\sin(k^m \theta)}{k^2} , \tag{3.15}$$

where $r$ and $\theta$ are the usual polar coordinates in the plane, and $m \geq 4$ is a fixed integer. (cf. [10, p. 286]).

**4. Three-dimensional results** As previously remarked in Section 2, the proof of the Roseman-Zimering theorem [7] made use of the special geometry of a cylinder and in fact was also extended [7] to some semi-infinite domains which are not containable in a cylinder. However, the proof there breaks down when the domains are truely cone-like. In the following, in Theorems 4 and 5, we are able to treat cone-like domains and even exterior domains by replacing the boundedness condition at infinity by other (relatively mild) growth conditions.

**Theorem 3.** *Let $T$ be an unbounded subdomain of the three-dimensional half-space $x_3 > 0$, and let $\partial T$ be the boundary of $T$, $\overline{T}$ the closure of $T$, and $\Gamma$ the subset of $\partial T$ on which $x_3 > 0$. Assume that the intersection of $\overline{T}$ with any plane $x_3 = X_3 \geq 0$ is a two-dimensional region of finite positive area, and let $\gamma > 0$ be such that the sub volume of $T$ between the planes $x_3 = 0$ and $x_3 = X_3 \leq 4$ is less than $\gamma X_3$.*

*Suppose $u = u(x_1, x_2, x_3) \in C^2(T) \cap C^1(T \cup \Gamma)$ satisfies (2.1a) and (2.2) in $T$, satisfies $u\frac{\partial u}{\partial n} = 0$ on $\Gamma$, and, for some $N > 0$, $|u(x_1, x_2, x_3)| < N$ for $(x_1, x_2, x_3) \in T$ with $0 < x_3 < 4$.*

*Define $E(a)$ to be the energy integral over $T_a$, the subdomain of $T$ in which $x_3 > a$, and assume that $E(1) < \infty$ (and hence $E(a) < \infty$ for all $a > 0$).*

*Then,*

$$E(a) \leq \begin{cases} \tilde{b}_1 a^{-\left(\frac{p+1}{p-1}\right)}, & 0 < a < 1, \quad K \neq 0, \\ \tilde{b}_2 a^{-1}, & 0 < a < 1, \quad K = 0, \end{cases} \tag{4.1}$$

*where $\tilde{b}_1$ and $\tilde{b}_2$ depend upon $N$, $M$, $K$, $p$, $\gamma$ and $E(1)$.*

**Remark.** Theorem 3 says in effect that if one is willing to assume finite energy at infinity, the Roseman-Zimering [7] result will go through at the left end for very general semi-infinite domains.

**Proof:** We follow the proof of the theorem in [7], as described in Section 2 above, up to (2.9). Then we use the fact that here $\iiint_0^{4a} d\tau < 4a\gamma$ for $0 < a < 1$, to obtain

$$\iiint_a^{3a} \rho u_{,k}\, u_{,k}\; d\tau \leq \frac{16qMD^2N^2\gamma}{a} + \frac{4(4KC)^q N^{2q}\gamma}{a^{2q-1}}, \tag{4.2}$$

or

$$\iiint_a^{3a} \rho u_{,k}\, u_{,k}\; d\tau \leq \frac{\tilde{B}_1}{a} + \frac{\tilde{B}_2}{a^{2q-1}}, \tag{4.3}$$

for $0 < a < 1$. Equation (4.3) then implies that

$$\iiint_a^{3a} \rho u_{,k}\, u_{,k}\; d\tau \leq \begin{cases} \dfrac{\tilde{B}_2}{a^{2q-1}}, & K \neq 0, \\ \dfrac{\tilde{B}_1}{a}, & K = 0, \end{cases} \tag{4.4}$$

for $0 < a < 1$. Now,

$$E(a) = \iiint_a^{1} \rho u_{,k}\, u_{,k}\; d\tau + E(1), \tag{4.5}$$

and it follows that (cf. [7]),

$$E(a) \leq E(1) + \begin{cases} \dfrac{\tilde{B}_3}{a^{2q-1}} = \tilde{B}_3 a^{-\left(\frac{p+1}{p-1}\right)}, & K \neq 0, \\ \dfrac{\tilde{B}_4}{a}, & K = 0, \end{cases} \tag{4.6}$$

where $\tilde{B}_3 = \frac{3}{2}\tilde{B}_2$, $\tilde{B}_4 = \frac{3}{2}\tilde{B}_1$, which leads to (4.1) with $\tilde{b}_1 = \tilde{B}_3 + E(1)$, $\tilde{b}_2 = \tilde{B}_4 + E(1)$.

**Theorem 4.** *Let $T$ be a domain of the type described in Theorem 3 and also such that it is contained within a right circular cone whose vertex is at $(0,0,-1)$ in $x_1-x_2-x_3$ space, its axis coincides with the $x_3$-axis, and with the angle between one of its generators and the $x_3$-axis being $\beta < \frac{1}{2}\pi$.*

*Suppose $u = u(x_1,x_2,x_3) \in C^2(T) \cap C^1(T \cup \Gamma)$ satisfies (2.1a) and (2.2) in $T$, satisfies $u\frac{\partial u}{\partial n} = 0$ on $\Gamma$ and, for some $A > 0$, $1 > \alpha > \frac{1}{2}$, $\alpha \neq \frac{3}{2q}$,*

$$|u(x_1,x_2,x_3)| \leq A(x_3+1)^{-\alpha}, \qquad x_3 > 0. \tag{4.7}$$

*Then,*

$$E(a) \leq \frac{b_1 \tan^2\beta}{a^{(p+1)/(p-1)}} + \frac{b_2 \tan^2\beta}{a} + \frac{b_3\tan^2\beta}{a^{2\alpha-1}}, \tag{4.8}$$

*for $a > 0$, where the coefficients $b_1$, $b_2$, and $b_3$ depend upon $A$, $D$, $\alpha$, $p$, $M$, and $K$, and (4.8) holds with $b_1 = 0$ if $K = 0$.*

**Remark.** We note that if (4.7) holds for some $\alpha_1 > \frac{1}{2}$, it also holds for any $\alpha_2$, $\frac{1}{2} < \alpha_2 < \alpha_1$. Therefore the requirement that $\alpha < 1$ in (4.7) is a convenient normalization rather than a restriction, as is the condition that $\alpha \neq \frac{3}{2q}$.

**Proof:** We follow the steps in the proof of the Roseman-Zimering theorem [7], as described in Section 2, up to an equation analogous to (2.8), which is

$$\frac{1}{4q}\iiint_T \psi_a^2 \rho u_{,k} u_{,k}\, d\tau \leq M \iiint_T \psi_a'^2 u^2\, d\tau + \frac{[4KC(p)]^q}{4qa^{2q}} \iiint_T H_a u^{2q}\, d\tau. \tag{4.9}$$

Now, using (4.7), we obtain

$$\begin{aligned} \iiint_T \psi_a'^2 u^2\, d\tau &\leq \frac{D^2}{a^2}\iiint_T H_a u^2\, d\tau \leq \frac{D^2}{a^2}\iiint_0^{4a} u^2\, d\tau \\ &\leq \frac{\pi\tan^2\beta\, D^2A^2}{a^2}\int_0^{4a}(x_3+1)^{-2\alpha}(x_3+1)^2\, dx_3 \\ &= \frac{\pi\tan^2\beta\, D^2A^2}{a^2}\left[\frac{(4a+1)^{3-2\alpha}-1}{3-2\alpha}\right], \end{aligned} \tag{4.10}$$

which implies that there exist constants $B_1$ and $B_2$ depending upon $A$, $D$, and $\alpha$ such that

$$\iiint_T \psi_a'^2 u^2\, d\tau \leq \begin{cases} B_1 \tan^2\beta\, a^{1-2\alpha}, & a > 1, \\ B_2\tan^2\beta\, a^{-1}, & 0 < a < 1. \end{cases} \tag{4.11}$$

Similarly,

$$\begin{aligned} \iiint_T H_a u^{2q}\, d\tau &\leq \pi A^{2q}\tan^2\beta \int_0^{4a}(x_3+1)^{2-2q\alpha}\, dx_3 \\ &= \pi A^{2q}\tan^2\beta\left[\frac{(4a+1)^{3-2q\alpha}-1}{3-2q\alpha}\right], \end{aligned} \tag{4.12}$$

and thus there exist $B_3$, $B_4$, and $B_5$ depending upon $A$, $p$, and $\alpha$ such that

$$\iiint\limits_T H_a u^{2q}\, d\tau \le \begin{cases} B_3 \tan^2\beta\, a^{3-2q\alpha}\,, & a>1,\ \alpha<\frac{3}{2q}\,, \\ B_4 \tan^2\beta\,, & a>1,\ \alpha>\frac{3}{2q}\,, \\ B_5 \tan^2\beta a\,, & 0<a<1\,. \end{cases} \tag{4.13}$$

Combination of (4.13), (4.11) and (4.9), together with the properties of $\psi_a$, yield

$$\iiint\limits_a^{3a} \rho u_{,k}\, u_{,k}\, d\tau \le \begin{cases} B_6 \tan^2\beta\, a^{-1} + B_7 \tan^2\beta\, a^{-(2q-1)} & 0<a<1\,, \\ B_8 \tan^2\beta\, a^{-(2\alpha-1)} & \\ +B_9 \tan^2\beta a^{-[2q(\alpha+1)-3]} + B_{10}\tan^2\beta a^{-2q}\,, & a>1, \end{cases} \tag{4.14}$$

where $B_i$ depends upon $A$, $\alpha$, $p$, $D$, $M$, and $K$.

By comparison of orders of magnitude, a simpler expression can be obtained as follows:

$$\iiint\limits_a^{3a} \rho u_{,k}\, u_{,k}\, d\tau \le \tan^2\beta[B_{11}a^{-1} + B_{12}a^{-(\frac{p+1}{p-1})} + B_{13}a^{-(2\alpha-1)}]\,, \tag{4.15}$$

for all $a>0$. By an analysis similar to that in [7] as described in Section 2, following (2.12), one obtains (4.8). ∎

Our final theorem is for a three-dimensional exterior problem. In the following, $\tilde{E}(a)$ is defined as the energy integral over the domain $[(x_1,x_2,x_3)|r=\sqrt{x_k x_k}>1+a]$.

**Theorem 5.** *Let $\Sigma$ be a piecewise smooth closed surface contained in the closed unit ball and let $G = Ext\,(\Sigma)$.*

*Suppose $u=u(x_1,x_2,x_3)$ satisfies (2.1a) and (2.2) in $G$ and also the inequality*

$$|u| \le Ar^{-\alpha} \quad \text{for} \quad r>1\,, \tag{4.16}$$

*for some $A>0$, $\alpha>\frac{1}{2}$. Then,*

$$\tilde{E}(a) \le \frac{b_1}{a^{(p+1)/(p-1)}} + \frac{b_2}{a} + \frac{b_3}{a^{2\alpha-1}}\,, \tag{4.17}$$

*for all $a>0$, where the coefficients $b_1$, $b_2$, and $b_3$ depend upon $A$, $D$, $\alpha$, $p$, $M$, and $K$, and (4.17) holds with $b_1=0$ if $K=0$.*

**Proof:** First, we define the following cut-off functions of $r=\sqrt{x_k x_k}$ : $\zeta_a(r)=\psi_a(r-1)$ and $h_a(r)=H_a(r-1)$. They satisfy relations similar to (2.4), (2.5), and (2.6); that is,

$$0 \le \zeta_a(r) \le h_a(r) \le 1\,, \tag{4.18}$$

$$\frac{\partial\zeta_a(r)}{\partial x_k}\frac{\partial\zeta_a(r)}{\partial x_k} \le \frac{D}{a^2}h_a(r)\,, \tag{4.19}$$

$$\frac{\partial\zeta_a(r)}{\partial x_k}\frac{\partial\zeta_a(r)}{\partial x_k} \le \frac{C(p)}{a^2}[\zeta_a(r)]^{2/p}h_a(r)\,, \tag{4.20}$$

for all $p>1$, for some constant $D$ and for some $C(p)$ which depends only upon $p$.

We multiply (2.1a) by $\zeta_a^2 u$ and integrate over $G$. Again, in a manner analogous to that of the proof in Theorem 4, we obtain

$$\frac{1}{4q}\iiint_G \zeta_a^2 \rho u_{,k}\, u_{,k}\, d\tau \leq M \iiint_G \zeta_{a,k}\, \zeta_{a,k}\, u^2\, d\tau + \frac{[4KC(p)]^q}{4qa^{2q}} \iiint_G h_a u^{2q}\, d\tau\,, \tag{4.21}$$

and then we use (4.16), which leads to (4.17). ▮

We shall now present an example in which $\lim_{a\to 0^+} \tilde{E}(a) = +\infty$. The equation is Laplace's equation ($\rho = 1$) and $\Sigma$ is the surface of the unit ball. In the following $r, \theta, \phi$ represent the usual spherical coordinates with $\theta$ the azimuthal angle ($0 \leq \theta < 2\pi$) and $\phi$ the co-latitude ($0 \leq \phi \leq \pi$), while $R,\ \theta,\ z$ represent the usual cylindrical coordinates. First we consider the function $w = W_1(R,\theta) = W_2(r,\theta,\phi)$ defined in Int $(\Sigma)$ as

$$w = \sum_{k=1}^{\infty} \frac{1}{k^2} R^{k^m} \sin(k^m\theta) = \sum_{k=1}^{\infty} \frac{1}{k^2} r^{k^m} (\sin\phi)^{k^m} \sin(k^m\theta)\,, \tag{4.22}$$

where $m$ is a fixed positive integer $\geq 7$. The function $w$ is harmonic in Int $(\Sigma)$ and continuous up to the boundary. Next we define $u$ in $G = \text{Ext}(\Sigma)$ as

$$u = \frac{1}{r} W_2(\frac{1}{r}, \theta, \phi) = \frac{1}{r}\sum_{k=1}^{\infty} \frac{1}{k^2} r^{-k^m} (\sin\phi)^{k^m} \sin(k^m\theta)\,. \tag{4.23}$$

The function $u$ is harmonic in $G$ and continuous up to the boundary, where $r = 1$. We shall demonstrate that

$$\lim_{a\to 0^+} \tilde{E}(a) = \lim_{a\to 0^+} \int\!\!\int\!\!\int_{r>1+a} |\nabla u|^2\, d\tau = +\infty\,. \tag{4.24}$$

Now,

$$\begin{aligned}\tilde{E}(a) &= \int\!\!\int\!\!\int_{r>1+a} |\nabla u|^2\, d\tau \\ &= \int_0^{2\pi}\int_0^{\pi}\int_{1+a}^{\infty} \left[(\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \phi})^2 + (\frac{1}{r^2\sin^2\phi})(\frac{\partial u}{\partial \theta})^2\right] r^2 \sin\phi\, dr\, d\phi\, d\theta\,.\end{aligned} \tag{4.25}$$

From (4.22) and (4.23), it follows that

$$\tilde{E}(a) = \iiint_{r<b} |\nabla w|^2\, d\tau + \iiint_{r<b} [\frac{2w}{r}\frac{\partial w}{\partial r} + \frac{w^2}{r^2}]\, d\tau\,, \tag{4.26}$$

where $b = (1+a)^{-1}$.

The second term on the right hand side of (4.26) can be integrated. Indeed,

$$\begin{aligned}\iiint_{r<b} \left[\frac{2w}{r}\frac{\partial w}{\partial r} + \frac{w^2}{r^2}\right] d\tau &= \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \left[2rw\frac{\partial w}{\partial r} + w^2\right] dr \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \frac{\partial}{\partial r}(rw^2)\, dr \sin\phi\, d\phi\, d\theta \\ &= b\int_0^{2\pi}\int_0^{\pi} W_2^2(b,\theta,\phi) \sin\phi\, d\phi\, d\theta\,.\end{aligned} \tag{4.27}$$

Equations (4.26) and (4.27) imply that

$$\tilde{E}(a) = \iiint\limits_{r<b} |\nabla w|^2 \, d\tau + b \iint\limits_{r=b} w^2 \, d\Omega \,, \tag{4.28}$$

where $d\Omega$ is the differential element of solid angle. From (4.28), we see that

$$\tilde{E}(a) \geq \iiint\limits_{r<b} |\nabla w|^2 \, d\tau \geq \iiint\limits_{r<b} (\frac{\partial w}{\partial R})^2 \, d\tau \,, \tag{4.29}$$

or

$$\tilde{E}(a) \geq \int_{-b}^{b} \int_{0}^{\sqrt{b^2-z^2}} \int_{0}^{2\pi} (\frac{\partial w}{\partial R})^2 \, d\theta R \, dR \, dz \,. \tag{4.30}$$

Now,

$$\frac{\partial w}{\partial R} = \sum_{k=1}^{\infty} k^{m-2} R^{k^m-1} \sin(k^m\theta) \,, \tag{4.31}$$

so that by inserting (4.31) into (4.30) and making use of the orthogonality property of the sine functions on the interval $(0, 2\pi)$, we obtain

$$\iiint\limits_{r<b} (\frac{\partial w}{\partial R})^2 \, d\tau = \pi \sum_{k=1}^{\infty} k^{2(m-2)} \int_{-b}^{b} \int_{0}^{\sqrt{b^2-z^2}} R^{2k^m-1} \, dR \, dz \,, \tag{4.32}$$

and then, after some standard manipulations,

$$\iiint\limits_{r<b} (\frac{\partial w}{\partial R})^2 \, d\tau = \frac{\pi^{3/2} b}{2} \sum_{k=1}^{\infty} \frac{b^{2k^m}}{k^4} \frac{k^m \Gamma(k^m+1)}{\Gamma(k^m+\frac{3}{2})} \,. \tag{4.33}$$

Asymptotic evaluation, with the aid of Stirling's formula, yields

$$\frac{k^m \Gamma(k^m+1)}{\Gamma(k^m+\frac{3}{2})} \sim k^{m/2} \qquad \text{as} \;\; k \to \infty \,, \tag{4.34}$$

and thus,

$$\lim_{b\to 1^-} \iiint\limits_{r<b} (\frac{\partial w}{\partial R})^2 \, d\tau = \lim_{a\to 0^+} \iiint\limits_{r<b} (\frac{\partial w}{\partial R})^2 \, d\tau = +\infty \,, \tag{4.35}$$

when $m \geq 7$. Finally, (4.24) now follows from (4.30).

**Remark.** The techniques used in this section can be used to obtain similar results in higher-dimensional spaces. For the analogues of Theorems 4 and 5, the assumed rate of decay at infinity is expected to depend upon the dimension of the space.

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