Differential and Integral Equations, Volume 2, Number 3, July 1989, pp. 300-309.

OSCILLATION OF DISCRETE ANALOGUES OF DELAY EQUATIONS*

L.H. ERBE AND B.G. ZHANG†

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

(Submitted by: Klaus Schmitt)

Abstract. Oscillation and nonoscillation criteria are established for first order linear and nonlinear difference equations with delay. Some of the conditions are shown to be sharp. Applications are also given to certain nonhomogeneous equations.

1. Introduction. In a number of recent papers [1, 6-9, 11, 12], the oscillatory and asymptotic behaviour of solutions of certain linear and nonlinear difference equations has been extensively investigated. It turns out that many (but not all, see [7]) of the substantial criteria for differential equations have discrete analogues. Further, criteria have also been obtained for the oscillatory and nonoscillatory behaviour of differential delay equations which may also be considered as analogues of criteria in the ordinary differential equations case [10]. It is the purpose of this paper to combine these two approaches in order to obtain oscillation criteria for discrete analogues of first order delay differential equations.

We consider in section 2 first order difference equations of the form

$$y_{n+1} - y_n + p_n y_{n-m} = 0 \tag{1.1}$$

and

$$y_{n+1} - y_n + p_n f(y_{n-m}) = 0 \tag{1.2}$$

where $n = 1, 2, \dots$, and m is a positive integer. Equation (1.2) has also been considered in the numerical analysis of certain functional differential equations ([2]). Equations (1.1) and (1.2) are discrete analogues of

$$y'(t) + p(t)y(t-m) = 0 \tag{1.3}$$

and

$$y'(t) + p(t)f(y(t-m)) = 0 \tag{1.4}$$

Received April 28, 1988.

*Research supported by NSERC - Canada.

†On leave from Shandong College of Oceanology, P.R.C.

AMS Subject Classifications: 39A10.

respectively. In section 3 we study the oscillation of discrete versions of delay logistic equation of the form

$$y_{n+1} - y_n + p_n(1 + y_n)y_{n-m} = 0. \tag{1.5}$$

In section 4 we study the oscillation of difference equations of the form

$$y_{n+1} - y_n + \sum_{i=1}^{k} p_{in} y_{n-m_i} = 0 \tag{1.6}$$

which is a discrete analogue of

$$y'(t) + \sum_{i=1}^{k} p_i(t) y(t - m_i) = 0. \tag{1.7}$$

In section 5 we consider the nonhomogeneous equations of the form

$$y_{n+1} - y_n + p_n y_{n-m} = f_n. \tag{1.8}$$

As usual, a nontrivial solution is said to be oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $y_n y_{n+1} \leq 0$. Otherwise it is nonoscillatory.

2. Difference equations with a single delay. In this section we establish sufficient conditions for oscillation and nonoscillation of the first order equations (1.1) and (1.2).

Theorem 2.1. *Assume that*

$$\begin{aligned} &\liminf_{n\to\infty} p_n \equiv c > 0 \qquad \text{and} \\ &\limsup_{n\to\infty} p_n > 1 - c. \end{aligned} \tag{2.1}$$

Then

(i)
$$y_{n+1} - y_n + p_n y_{n-m} \leq 0 \tag{2.2}$$

has no eventually positive solution;

(ii)
$$y_{n+1} - y_n + p_n y_{n-m} \geq 0 \tag{2.3}$$

has no eventually negative solution

(iii) *Every solution of (1.1) oscillates.*

Proof: It is sufficient to establish (i). Parts (ii) and (iii) will then follow. Assume then that $y = \{y_n\}$ is an eventually positive solution such that $y_n > 0$ for $n \geq N_1$. Let $\varepsilon > 0$, $0 < \varepsilon < c$, and choose N_2 such that $p_n \geq c - \varepsilon > 0$ for $n \geq N_2$. Letting $N = \max\{N_1 + m, N_2\}$ we have $y_n \geq p_n y_{n-m} \geq (c - \varepsilon)y_{n-1}$ for $n \geq N$, since y_n is nonincreasing for $n \geq N$. On the other hand, we have $0 \geq y_{n+1} - y_n + p_n y_{n-m} \geq y_{n+1} + y_n(p_n - 1)$ for $n \geq N$ so that $y_n(p_n - 1 + c - \varepsilon) \leq 0$ for $n \geq N$. That is, $p_n \leq 1 - c + \varepsilon$ for $n \geq N$, and hence $\limsup_{n\to\infty} p_n \leq 1 - c + \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $\limsup_{n\to\infty} p_n \leq 1 - c$, which contradicts (2.1). This proves (i); (ii) follows from (i) by letting $z_n = -y_n$ for an eventually negative solution $\{y_n\}$ and (iii) follows from (i) and (ii).

Theorem 2.2. *Assume that*

$$\liminf_{n\to\infty} p_n \equiv c > \frac{m^m}{(m+1)^{m+1}}. \tag{2.5}$$

Then the conclusions of Theorem 2.1 hold.

Proof: To prove (i), suppose the contrary and let $\{y_n\}$ be a solution of (2.2) with $y_n > 0$ for $n \geq N_1$. Setting $r_n = y_n/y_{n+1}$ and dividing inequality (2.2) by y_n and rearranging we have

$$r_n^{-1} \leq 1 - p_n r_{n-m} \cdots r_{n-1}, \quad n \geq N_1 + m. \tag{2.6}$$

From (2.5), $p_n > 0$ for $n \geq N_2$ and setting $N = \max\{N_2, N_1 + m\}$ it follows that y_n is nonincreasing for $n \geq N$ and so $r_n \geq 1$. Also f_n is bounded above - otherwise (2.5) and (2.6) imply that $r_n < 0$ for arbitrarily large n. If we set $\liminf_{n\to\infty} r_n = \ell$, then from (2.6) we get

$$\limsup_{n\to\infty} \frac{1}{r_n} = \frac{1}{\ell} \leq 1 - \liminf_{n\to\infty} \{p_n r_{n-m} \cdots r_{n-1}\}. \tag{2.6}$$

Since $\liminf_{n\to\infty}\{p_n r_{n-m} \cdots r_{n-1}\} \geq c\ell^m$ we have $\frac{1}{\ell} \leq 1 - c\ell^m$ or

$$c \leq \frac{\ell - 1}{\ell^{m+1}} \equiv h(\ell). \tag{2.7}$$

Now it is easy to see that $\max_{\ell\geq 1} h(\ell) = m^m/(m+1)^{m+1}$ and hence we obtain $c \leq m^m/(m+1)^{m+1}$, contradicting (2.5). This completes the proof.

Remark: If $m = 0$ in (2.5), then the right hand side is 1 and (1.1) becomes

$$y_{n+1} - y_n + p_n y_n = 0 \tag{2.8}$$

which is a discrete analog of

$$y'(t) + p(t)y(t) = 0. \tag{2.9}$$

Clearly, (2.9) is nonoscillatory for any continuous $p(t)$ whereas (2.8) is oscillatory if $\liminf_{n\to\infty} p_n > 1$.

The next result shows that condition (2.5) is sharp. In fact if $p_n = m^m/(m+1)^{m+1}$ for all $n \geq 0$, then (2.1) becomes

$$y_{n+1} - y_n + \frac{m^m}{(m+1)^{m+1}} y_{n-m} = 0 \tag{2.10}$$

and with $y_N = d > 0$, and $y_{k+1} = m/(m+1) \cdot y_k$, $k = N, N+1, \ldots$, it follows that $\{y_n\}$ is defined and is a nonoscillatory solution of (2.10).

Theorem 2.3. *Assume $p_n \geq 0$ and*

$$\sup p_n < \frac{m^m}{(m+1)^{m+1}}. \tag{2.11}$$

Then (1.1) has a nonoscillatory solution.

Proof: We show that

$$r_n^{-1} = 1 - p_n r_{n-m} \cdots r_{n-1} \tag{2.12}$$

has a positive solution. To see this, define

$$s_{N-m} = \cdots = s_{N-1} = q = \frac{m+1}{m} > 1 \tag{2.13}$$

and

$$s_N = (1 - p_N s_{N-m} \cdots s_{N-1})^{-1} > 1. \tag{2.14}$$

From (2.13) and (2.14) it follows that $s_N < q$ so we define

$$s_{N+1} = (1 - p_{N+1} s_{N+1-m} \cdots s_N)^{-1} < q \tag{2.15}$$

and by induction $1 < s_{N+k} < q$ for all $k = 1, 2, \ldots$ so that the sequence $\{s_n\}$, $n \geq N$, is a solution of (2.12). Next, defining $y_N = 1$, $y_{N+1} = y_N/s_N$ and so on, it follows that $\{y_n\}$ satisfies (1.1). ∎

Our next result gives sufficient conditions for oscillation of the nonlinear equation (1.2) which extend the criterion of Theorem 2.2. In this and in several of the other results to follow, the conclusion could be stated in terms of the nonexistence of positive or negative solutions to inequalities, as in Theorems 2.1, 2.2, but for simplicity we state the conclusion in terms of oscillation of the equation.

Theorem 2.4. *Assume that f is continuous on $\mathbb{R} = (-\infty, +\infty)$ and satisfies*

(i) $xf(x) > 0$, $x \neq 0$ *and*

(ii) $\liminf_{x \to 0} \frac{f(x)}{x} = M$, $0 < M < +\infty$.

(iii) $cM > \frac{m^m}{(m+1)^{m+1}}$, $m \geq 1$

where $c = \liminf_{n\to\infty} p_n > 0$. Then every solution of (1.2) oscillates.

Proof: We suppose not and assume that $\{y_n\}$ is a positive nonoscillatory solution. We first claim that $\lim_{n\to\infty} y_n = 0$. Clearly, from (1.2) it follows that $\{y_n\}$ is decreasing and hence $\lim_{n\to\infty} y_n = \alpha \geq 0$ exists. But then taking the limit in (1.2) we have $f(\alpha) = 0$ and so $\alpha = 0$. Now with $r_n = y_n/y_{n+1} \geq 1$, (1.2) becomes

$$\frac{1}{r_n} = 1 - p_n r_{n-1} \cdots r_{n-m} \frac{f(y_{n-m})}{y_{n-m}}. \tag{2.16}$$

The remainder of the argument is similar to the proof of Theorem 2.2. Setting $\ell = \liminf_{n\to\infty} r_n$ and observing that r_n must be bounded above we have from (2.16)

$$\frac{1}{\ell} \leq 1 - cM\ell^m \tag{2.17}$$

which gives

$$cM \leq \frac{\ell - 1}{\ell^{m+1}} \leq \frac{m^m}{(m+1)^{m+1}}, \tag{2.18}$$

contradicting condition (iii) of the hypothesis. This proves the theorem.

The next result relaxes the strict positivity assumptions on the p_n in the previous theorems.

Theorem 2.5. *Assume that $p_n \geq 0$ and the assumptions (i) and (ii) of Theorem 2.4 hold. Assume further that f is nondecreasing and*

$$\limsup_{n\to\infty} \sum_{k=n-m}^{n} p_k > \frac{1}{M}. \tag{2.19}$$

Then every solution of (1.2) oscillates.

Proof: Let $\{y_n\}$ be a positive solution of (1.2). Then, as in the proof of Theorem 1.4, $y_n \to 0$ as $n \to \infty$. Summing (1.2) from N to $N+m$, we have

$$y_{N+m+1} - y_N + \sum_{k=N}^{N+m} p_k f(y_{k-m}) = 0. \tag{2.20}$$

Hence, we obtain

$$y_{N+m+1} - y_N + f(y_N) \sum_{k=N}^{N+m} p_k \leq 0. \tag{2.21}$$

Rearranging (2.21) we have

$$y_{N+m+1} - y_N \Big[1 - \frac{f(y_N)}{y_N} \sum_{k=N}^{N+m} p_k\Big] \leq 0 \tag{2.22}$$

and hence

$$\sum_{k=N}^{N+m} p_k \leq \frac{y_N}{f(y_N)}. \tag{2.23}$$

Therefore, taking limit suprema as $N \to \infty$ of both sides of (2.23) gives a contradiction to (2.19). This completes the proof.

Remark 2.1: The previous results also apply to the equation

$$y_{n+1} - y_n + p_n y_{\tau(n)} = 0 \tag{2.24}$$

where $\tau(n) = n - \sigma(n)$ and $0 \leq \sigma(n) < n$. If $\sigma(n)$ is bounded and integer, say $\sigma(n) \leq m$ (integer), then the above results apply to (2.24). For example, (2.24) is oscillatory if (2.5) holds. We also remark that the results obtained in the theorems above also apply to discrete analogues of advanced-type equations. For example, every solution of

$$y_{n+1} - y_n = p_n y_{n+m}, \quad m \geq 1 \tag{2.25}$$

is oscillatory if $\liminf_{n\to\infty} p_n > (m-1)^{m-1}/m^m$. We omit the details.

Remark 2.2: Gyori and Ladas [6] studied the difference equation with constant coefficients of the form

$$y_{n+1} - y_n + \sum_{j=0}^{m} P_j y_{n-j} = 0 \tag{2.26}$$

and showed that it oscillates if and only if the characteristic equation

$$\lambda - 1 + \sum_{j=0}^{m} P_j \lambda^{-j} = 0 \tag{2.27}$$

has no positive roots. It follows, as a special case that

$$y_{n+1} - y_n + py_{n-m} = 0$$

oscillates if and only if

$$p > \frac{m^m}{(m+1)^{m+1}}. \tag{2.28}$$

Theorems 2.2 and 2.3 are consistent with (2.28).

3. Oscillation of discrete versions of delay logistic equations. We consider the discrete version of the delay logistic equation of the form

$$\Delta y_n + p_n(1+y_n)y_{n-m} = 0, \quad n = 1, 2, \cdots. \tag{3.1}$$

From an ecological viewpoint, we must have $p_n \geq 0$ ([14]) for $n \geq 1$ and $1 + y_n > 0$ for $h \geq -m$.

Theorem 3.1. *Assume that*

$$\liminf_{n\to\infty} p_n = c > \frac{m^m}{(m+1)^{m+1}}. \tag{3.2}$$

Then every solution of (3.1) oscillates.

Proof: If not, assume that $y_n > 0$ for $n \geq N$ is a solution of (3.1). Then we have

$$\Delta y_n + p_n y_{n-m} \leq 0, \; n = 1, 2, \cdots \tag{3.3}$$

By Theorem 2.2, (3.3) has no positive solution under assumption (3.2).

Assume now that $y_n < 0$ for $n \geq N$ is a solution of (3.1). Then $\Delta y_n \geq 0$ so $\lim y_n = \alpha \leq 0$ exists. It is obvious that $\alpha = 0$. Setting $r_n = y_n/y_{n+1}$ then (3.1) becomes

$$\frac{1}{r_n} - 1 + p_n(1+y_n)r_{n-m}\cdots r_{n-1} = 0$$

or

$$\frac{1}{r_n} = 1 - p_n(1+y_n)r_{n-m}\cdots r_{n-1}. \tag{3.4}$$

Setting $\liminf_{n\to\infty} r_n = \ell$, then $1 \leq \ell < +\infty$. Taking the limit supremum in (3.4) we have

$$\begin{aligned}\frac{1}{\ell} &\leq 1 - \liminf_{n\to\infty}\{p_n(1+y_n)r_{n-m}\cdots r_{n-1}\} \\ &\leq 1 - \ell^m(\liminf_{n\to\infty} p_n).\end{aligned}$$

Hence

$$\liminf_{n\to\infty} p_n \leq \frac{m^m}{(m+1)^{m+1}}$$

which contradicts (3.2). The proof is completed.

Theorem 3.2. *Assume that $p_n \geq 0$ and*

$$\sup p_n < \frac{m^m}{(m+1)^{m+1}} \tag{3.5}$$

Then (3.1) has a nonoscillatory solution.

Proof: We shall construct a negative solution of (3.1) under the assumption (3.5).

Define

$$S_{N-m} = \cdots = S_{N-1} = q = \frac{m+1}{m} > 1$$

and $y_{N-1} = -1/(N-1)$, where N is a positive integer with $N > 2$. Then $y_N = y_{N-1}/S_{N-1} < 0$

$$S_N = (1 - P_N(1+y_N)S_{N-m}\cdots S_{N-1})^{-1} < q$$

$$Y_{N+1} = \frac{y_N}{S_N}\,.$$

By induction $S_{N+1}, \ldots S_n, \ldots$ and $y_{N+1}, \ldots, y_n \ldots$ are defined by the preceding argument and $\{y_n\}$ is a negative solution of (3.1). The proof is completed.

4. Equations with several delays. In this section we wish to extend the previous results to

$$y_{n+1} - y_n + \sum_{i=1}^{k} p_{in} y_{n-m_i} = 0. \tag{1.6}$$

Theorem 4.1. *Assume that $p_{ij} \geq 0$ and*

$$\sum_{i=1}^{k} (\liminf_{n\to\infty} p_{in}) \frac{(m_i+1)^{m_i+1}}{(m_i)^{m_i}} > 1\,. \tag{4.1}$$

Then every solution of (1.6) oscillates.

Proof: Let $\{y_n\}$ be a solution of (1.6) with $y_n > 0$. Setting $r_n = y_n/y_{n+1}$ equation (1.6) becomes

$$\frac{1}{r_n} = 1 - \sum_{i=1}^{k} p_{in} r_{n-m_i} \cdots r_{n-1}\,. \tag{4.2}$$

Letting $\ell = \liminf_{n\to\infty} r_n$ it follows that $1 \leq \ell < +\infty$.

Taking the supremum limit on both sides of (4.2) we have

$$\frac{1}{\ell} - 1 \leq -\sum_{i=1}^{k} \liminf_{n\to\infty} (p_{in} r_{n-m_i} \cdots r_{n-1})$$

$$\leq -\sum_{i=1}^{k} (\liminf_{n\to\infty} p_{in}) \ell^{m_i}.$$

Hence, $\ell > 1$ must hold and therefore

$$1 \geq \sum_{i=1}^{k} (\liminf_{n\to\infty} p_{in}) \frac{\ell^{m_i+1}}{\ell - 1}$$

$$\geq \sum_{i=1}^{k} (\liminf_{n\to\infty} p_{in}) \frac{(m_i+1)^{m_i+1}}{(m_i)^{m_i}} \tag{4.3}$$

which contradict (4.1). The proof is completed

Corollary 4.2. *Assume that* $p_{ij} \geq 0$ *and*

$$k\Big(\prod_{i=1}^{k}(\liminf_{n\to\infty} p_{in})\Big)^{1/k} > \frac{(\bar{m})^{\bar{m}}}{(\bar{m}+1)^{\bar{m}+1}} \tag{4.4}$$

where $\bar{m} = \frac{1}{k}\big(\sum_{i=1}^{k} m_i\big)$. *Then every solution of (1.6) oscillates.*

Proof: In fact, from (4.3) and the arithmetic-geometric mean inequality, we have

$$1 \geq k\frac{\ell^{\bar{m}+1}}{\ell-1}\Big(\prod_{i=1}^{k}\liminf_{n\to\infty} p_{in}\Big)^{1/k} \geq k\frac{(\bar{m}+1)^{\bar{m}+1}}{(\bar{m})^{\bar{m}}}\Big(\prod_{i=1}^{k}\liminf_{n\to\infty} p_{in}\Big)^{1/k}$$

which contradicts (4.4).

Theorem 4.3. *Assume that* $p_{ij} \geq 0$ *and*

$$\liminf_{n\to\infty}\big(\sum_{i=1}^{k} p_{in}\big) > \frac{(\hat{m})^{\hat{m}}}{(\hat{m}+1)^{\hat{m}+1}} \tag{4.5}$$

where $\hat{m} = \min(m_1, \cdots, m_k) \geq 1$. *Then every solution of (1.6) oscillates*

Proof: Let $\{y_n\}$ be a solution of (1.6) with $y_n > 0$. Then y_n is nonincreasing. Therefore

$$y_{n+1} - y_n + \big(\sum_{i=1}^{k} p_{in}\big)y_{n-\hat{m}} \leq 0$$

By Theorem 2.2 we get the conclusion of Theorem 4.3.

The next result is the analogue of Theorem 2.1. The proof is omitted since it is similar to that of Theorem 2.1.

Theorem 4.4. *Assume that*

$$\liminf_{n\to\infty}\sum_{i=1}^{k} p_{in} = c > 0$$

and

$$\limsup_{n\to\infty}\sum_{i=1}^{k} p_{in} > 1 - c.$$

Then every solution of (1.6) is oscillatory.

5. Nonhomogeneous difference equations.

We shall consider in this section the nonhomogeneous equation

$$y_{n+1} - y_n + p_n y_{n-m} = f_n. \tag{5.1}$$

where $p_n \geq 0$ for all $n \geq 1$.

Theorem 5.1. *Assume that $f_n = g_{n+1} - g_n$ and for each $n \geq 1$ there exists $n' > n$ such that $g_n g_{n'} < 0$. Let g_n^+ and g_n^- be defined by $g_n^+ = \max\{0, g_n\}$, $g_n^- = \max\{0, -g_n\}$ and assume that*

$$\sum_{n=1}^{\infty} p_n g_{n-m}^+ = +\infty \quad \text{and} \qquad \sum_{n=1}^{\infty} p_n g_{n-m}^- = +\infty. \tag{5.2}$$

Then every solution of (5.1) is oscillatory.

Proof: If not, then (5.1) has a solution $\{y_n\}$ which is either eventually positive or eventually negative. Assume first that $y_n > 0$ for all $n \geq N$ and rewrite (4.1) in the form

$$(y_{n+1} - g_{n+1}) - (y_n - g_n) + p_n y_{n-m} = 0. \tag{5.3}$$

Therefore, $\{(y_n - g_n)\}$ is a monotone decreasing sequence for $n \geq N+m$. Clearly, $y_n - g_n \geq 0$ must also hold for $n \geq N + m$. For if $y_n - g_n < 0$ for all large n then since g_n changes sign for arbitrarily large values of n, we would contradict $y_n > 0$ for $n \geq N$. Setting $\lim_{n\to\infty}(y_n - g_n) \equiv \alpha \geq 0$, (5.3) implies that $\sum_{n=N+m}^{\infty} p_n y_{n-m} < \infty$. Since $y_n \geq g_n$ for $n \geq N + m$, we have that $\sum_{n=N+m}^{\infty} p_n g_{n-m}^+ < \infty$, a contradiction to (5.2). The proof for the case that $y_n < 0$ for all large n is similar.

Theorem 5.2. *Assume that $f_n = g_{n+1} - g_n$ and there exist two constants $h_1 < h_2$ and sequences $\{i_k\}$, $\{j_k\}$ with*

$$g_{i_k} = h_1,\ g_{j_k} = h_2,\ k = 1, 2 \ldots \quad \text{and} \qquad h_1 \leq g_n \leq h_2 \quad \text{for all} \quad n.$$

Assume further that

$$\liminf_{n\to\infty} p_n = c > \frac{m^m}{(m+1)^{m+1}}. \tag{5.4}$$

Then every solution of (5.1) oscillates.

Proof: Suppose the contrary and let $\{y_n\}$ be an eventually positive solution. Setting $x_n = y_n - g_n$, we claim that $x_n + h_1 > 0$ for all large n. Since x_n is nonincreasing, if $x_N + h_1 < 0$ for some large N, then for $i_k \geq N$, we have $0 < y_{i_k} = x_{i_k} + g_{i_k} = x_{i_k} + h_1 \leq U$, a contradiction. Hence, $x_n + h_1 > 0$ for $n \geq N_1$. If we put $z_n = x_n + h_1$, then z_n satisfies

$$\Delta z_n + p_n z_{n-m} \leq 0 \tag{5.5}$$

which contradicts Theorem 2.2. A similar argument applies in the case that y_n is an eventually negative solution, with now $x_n + h_2 < 0$ for all large n, and $z_n = x_n + h_2$. This proves the theorem.

Remark: We observe also that condition (5.4) may also be replaced in Theorem 5.2 by condition (2.1).

Example: Given f_n in Theorem 5.1, then the sequence g_n is determined by specifying g_1. For example, if $f_n = (-1)^{n+1} a_n$ where $0 < a_n \leq a_{n+1}$ for all n and if $g_1 = -a_1/2$, then the sequence g_n alternates in sign and one can easily establish that $g_{2n} \geq a_1/2$, $g_{2n+1} \leq -a_2/2$

for all n. Therefore, (5.2) will hold if $\sum_{n=1}^{\infty} p_n = +\infty$ and so equation (5.1) is oscillatory in this case.

As a further example, if $f_n = (-1)^n(2n+1)/(n(n+1))$, then $f_n = g_{n+1} - g_n$ where $g_n = (-1)^{n+1}/n$. Hence, (5.2) holds if, for example, m is *even* and

$$\sum^{\infty} \frac{p_{2n+1}}{2n+1-m} = \sum^{\infty} \frac{p_{2n}}{2n-m} = +\infty.$$

REFERENCES

[1] S.Z. Chen and L.H. Erbe, *Riccati techniques and discrete oscillations*, J. Math. Anal. Appl., (to appear).

[2] C.W. Cryer, *Numerical methods for functional differential equations*, On: Delay and Functional Differential Equations and their Applications, (Ed. K. Schmitt, pp., 17-102), Academic Press 1972.

[3] L.H. Erbe, *Oscillation criteria for second order nonlinear delay equations*, Canad. Math. Bull., 16 (1973), 49-56.

[4] T. Fort, "Finite Differences and Difference Equations in the Real Domain," Oxford Univ. Press, London, 1948.

[5] G.B. Gustafson, *Bounded oscillations of linear and nonlinear delay differential equations of even order*, J. Math. Anal. Appl., 46 (1974), 175-189.

[6] I. Gyori and G. Ladas, *Linearized oscillations of differential equation with piecewise constant arguments*, Differential and Integral Equations, (to appear).

[7] J.W. Hooker and W.T. Patula, *Riccati type transformations for second order linear differential equations*, J. Math. Anal. Appl., 82 (1981), 451-462.

[8] J.W. Hooker and W.T. Patula, *A second order nonlinear difference equation: Oscillation and asymptotic behaviour,* J. Math. Anal. Appl., 91 (1983), 9-29.

[9] J.W. Hooker, Man Kam Kwong and W.T. Patuta, *Oscillatory second order linear difference equations and Riccati equations,* SIAM J. Math. Anal., 18 (1987), 54-63.

[10] A.B. Mingarelli, "Volterra-Stieljes Integral Equations and Generalized Ordinary Differential Expressions", Lecture Notes in Mathematics, 989, Springer-Verlag, N.Y., 1983.

[11] G.S. Ladde, V. Lakshmikantham and B.G. Zhang, "Oscillation Theory of Differential Equations with Deviating Arguments," Marcel Dekker Inc. 1987.

[12] W.T. Patula, *Growth and oscillation properties of second order linear difference equations,* SIAM J. Math. Anal., 10 (1979), 55-61.

[13] W.T. Patula, *Growth, oscillation and comparison theorems for second order linear difference equations,* SIAM J. Math. Anal., 10 (1979), 1272-1279.

[14] B.G. Zhang and K. Gopalsamy, *Oscillation and nonoscillation in a nonautonomous delay logistic equation,* Quart. Appl. Math. (to appear).