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HEREDITARY CONTROL SYSTEMS GOVERNED BY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper examines nonlinear, hereditary control systems, with dynamics described by an integrodifferential equation. It is proved that the set of trajectories is compact in an appropriate space of continuous functions and under some additional hypotheses on the velocity field, that it is also connected. Analogous properties are obtained for the attainable sets, and it is also shown that the attainability multifunctions is Hausdorff continuous in the time variable. Then a relaxation and a "bang-bang" type theorem is proved. Finally some optimal control problems are solved.

1. Introduction. The purpose of this paper is to examine hereditary control systems (i.e., control systems with memory), governed by an integrodifferential equation. So the state velocity does not only depend on the past history of the state, but also on the past values of the control function. Our work extends earlier works of Angell [1], Cesari [5], Hermes-LaSalle [8] and Oguztoreli [13].

The hereditary control system under consideration is the following:

$$\begin{cases} \dot{x}(t) = g(t, x_t) + \displaystyle\int_0^t K(t,s) f(s, x(s), u(s))\, ds, \quad \text{a.e.. on } \; T = [0, b] \\ x(v) = \phi(v) \;\; \text{for} \;\; v \in T_0 = [-r, 0], \quad u(t) \in U(t, x(t)) \;\; \text{a.e.} \end{cases} \tag{*}$$

Here, $x : T_b = [-r, b] \to X$ is the state function and $x_t(\cdot) = x|_{[t-r,t]}(\cdot)$, i.e., represents the past history of the state up to time t.

In the next section we will recall some basic defintions from the theory of multifunctions that we will need in the sequel and we will state the main hypotheses concerning the data of our system (*). In section 3, we study in detail the structure of the set of trajectories of (*) and of the corresponding attainable set. In section 4, we prove a relaxation result which, roughly speaking, says that convexification of the velocity field does not alter the reachability properties of the system. Also, we prove a "bang-bang" type theorem. Finally, in section 5, we solve various optimization problems involving system (*).

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2. Preliminaries. Let (Ω, Σ) be a measurable space and X Polish space (i.e., a complete, separable, metrizable space). We will be using the following notations (convexity makes sense when X is linear):

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\}$$

and

$$P_{k(c)}(X) = \{A \subseteq X : \text{ nonempty, compact (convex)}\}.$$

A multifunction $F : \Omega \to P_f(X)$ is measurable if it satisfies one of the following equivalent conditions:

(i) $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$ for each open subset U of X

(ii) $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable, for each $x \in X$

(iii) there exists a sequence $\{f_n\}_{n\geq 1}$ of measurable functions $f_n : \Omega \to X$ such that, for all $\omega \in \Omega$, $F(\omega) = \operatorname{cl}\{f_n(\omega)\}_{n\geq 1}$ (Castaing's representation).

If there is a σ-finite measure $\mu(\cdot)$ with respect to which Σ is complete, then (i), (ii) and (iii) above are equivalent to:

(iv) $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel σ-field of X (graph measurability).

Let X be a finite dimensional Banach space. By S_F^1, we will denote the set of integrable selectors of $F(\cdot)$, i.e., $S_F^1 = \{f \in L^1(X) : f(\omega) \in F(\omega)\ \mu\text{- a.e}\}$. Clearly, this set may be empty. Using Aumann's selection theorem, it is not difficult to see that $S_F^1 \neq \emptyset$ if and only if $\inf\{\|z\| : z \in F(\omega)\} \in L_+^1$. We say that a multifunction $F : \Omega \to P_f(X)$ is integrably bounded if and only if it is measurable and $\sup\{\|z\| : z \in F(\omega)\} = |F(\omega)| \in L_+^1$. It is clear that in this case $S_F^1 \neq \emptyset$. Using this set S_F^1, we can define a set valued integral for $F(\cdot)$ by setting

$$\int_\Omega F = \left\{ \int_\Omega f \, d\mu : f \in S_F^1 \right\}.$$

Next, let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \backslash \{\emptyset\}$ is said to upper semi-continuous (u.s.c.), if for all $V \subseteq Z$ open, the set $G^+(V) = \{y \in Y : G(y) \subseteq V\}$ is open in Y. Also, on $P_f(X)$, we can define a metric $h(\cdot,\cdot)$, by setting $h(A, B) = \max\{\sup(d(a, B), a \in A), \sup(d(b, A), b \in B)\}$. We know that for any complete metric space $X, (P_f(X), h)$ is also complete. This metric is known as the Hausdorff metric and a multifunction $F : Y \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from the topological space Y into the metric space $(P_f(X), h)$.

Finally, if X is any Banach space and $\{A_n\}_{n\geq 1} \subseteq 2^X \backslash \{\emptyset\}$ we define

$$w - \overline{\lim} A_n = \{x \in X : x = w - \lim x_k,\ \ x_k \in A_{n_k},\ \ k \geq 1\},$$

where w denotes the weak topology on X. When X is finite dimensional, weak and norm topologies coincide and in this case we have

$$\overline{\lim} A_n = \{x \in X : \underline{\lim}\, d(x, A_n) = 0\}.$$

Now, let us state the assumptions concerning the data of our system (*). Recall that $T = [0, b]$, $T_b = [-r, b]$ and $T_0 = [-r, 0]$.

H: X, Z are finite dimensional Banach spaces. These are the state and the control space, respectively.

H(g): $g : T \times C(T_0, X) \to X$ is a function such that

(1) for every $z \in C(T_0, X)$, $t \to g(t, z)$ is measurable;

(2) $\|g(t, z)\| \le a_1(t) + b_1(t)\|z\|_\infty$ almost everywhere with $a_1(\cdot)$, $b_1(\cdot) \in L^1_+$;

(3) $\|g(t, z_1) - g(t, z_2)\| \le \psi_1(t)\|z_1 - z_2\|_\infty$ μ-almost everywhere, with $\psi_1(\cdot) \in L^1_+$.

H(K): $K : \Delta = \{(t, s) : 0 \le s \le t \le b\} \to L(X) = \{$ space of continuous linear operators from X into itself$\}$, $(t, s) \to K(t, s)$ is measurable and $\|K(t, s)\| \le M$.

H(U): $U : T \times X \to P_{kc}(Z)$ is a multifunction such that

(1) $(t, x) \to U(t, x)$ is measurable;

(2) $x \to U(t, x)$ is u.s.c.;

(3) $|U(t, x)| \le a(t) + b(t)\|x\|$ almost everywhere, where $a(\cdot)$, $b(\cdot) \in L^1_+$.

H(f): $f : T \times X \times Z \to X$ is a function such that

(1) $t \to f(t, x, u)$ is measurable;

(2) $(x, u) \to f(t, x, u)$ is continuous;

(3) $\|f(t, x, u)\| \le a_2(t) + b_2(t)\|x\|$ almost everywhere, where $a_2(\cdot)$, $b_2(\cdot) \in L^1_+$, $u \in u(t, x)$;

(4) $\|f(t, x_1, u) - f(t, x_2, u)\| \le \psi_1(t)\|x_1 - x_2\|$ almost everywhere, where $\psi_1(\cdot) \in L^1_+$, $u \in B \subseteq Z$ bounded.

A function $u : T \to Y$ is an admissible control if there exists a solution (trajectory) $x(u)(\cdot)$ of (*) corresponding to $u(\cdot)$ such that $u(t) \in U(t, x(t))$ almost everywhere on T. A trajectory corresponding to an admissible control is said to be an admissible trajectory. A pair $(x(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding admissible trajectory, is said to be an admissible pair. We will denote the set of admissible pairs by $A(\phi)$. We will make the following controllability type hypothesis concerning $A(\phi)$.

$\text{H}_c :\ A(\phi) \neq \emptyset$.

Furthermore, it is clear from the previous hypotheses that for every admissible control $u(\cdot)$, the corresponding admissible trajectory is unique.

3. Set of trajectories, attainable set. We will start with a result concerning the topological properties of the set $A(\phi)$. This result will allow us to solve the optimal control problems considered in section 5.

In the proof we will need the following lemma, which here we state and prove for a general Banach space X. Recall that if $A \subseteq X$ is nonempty, then the support function of A is the function $\sigma(\cdot, A) : X^* \to \bar{R} = RU\{+\infty\}$, defined by $\sigma(x^*, A) = \sup\{(x^*, x) : x \in A\}$. Here, X^* denotes the topological dual of X and $(\cdot, \cdot)$ the duality brackets for the pair (X, X^*).

Lemma 3.1. *If X is Banach space, $K \in L(X)$, $\{A_n\}_{n\ge1} \subseteq 2^X \backslash \{\emptyset\}$ and $A_n \subseteq G$ where $G \in P_{wkc}(X) = \{$ nonempty, weakly compact, convex subsets of $X\}$ then, $w - \overline{\lim} K(A_n) \subseteq \overline{\text{conv}}\, K(w - \overline{\lim} A_n)$.*

Proof: For every $x^* \in X^*$, we have that $\overline{\lim}\, \sigma(x^*, K(A_n)) = \overline{\lim}\, \sigma(K^*x^*, A_n)$. Using Proposition 3.1 of [15], we get that

$$\overline{\lim}\, \sigma(K^*x^*, A_n) \le \sigma(K^*x^*, w - \overline{\lim} A_n) = \sigma(x^*, K(w - \overline{\lim} A_n)).$$

Using Proposition 4.1 of [15], we finally have that

$$w - \overline{\lim} K(A_n) \subseteq \overline{\text{conv}}\, K(w - \overline{\lim} A_n).$$

∎

Now we can state the result on the topological properties of the set of admissible trajectories, denoted by $P(\phi)$.

Theorem 3.1. *If hypotheses H, $H(g)$, $H(K)$, $H(U)$, $H(f)$ and H_c hold, then $P(\phi)$ is compact in $C(T_b, X)$.*

Proof: Set $F(s,y) = f(s,y,U(s,y))$. Since $U(\cdot,\cdot)$ is $P_{kc}(z)$-valued and $f(s,y,\cdot)$ is continuous, we have that $f(s,y,U(s,y)) \in P_k(x)$.

From $H(U)$, we have that $(s,y) \to U(s,y)$ is measurable. So, using Castaing's representation result (see section 2), we can find $v_n : T \times X \to Z$ measurable such that $U(s,y) = \text{cl}\,\{v_n(s,y)\}_{n\geq 1}$. Exploiting the continuity of $f(s,y,\cdot)$, we get

$$f\big(s,y,\overline{\{v_n(s,y)\}}_{n\geq 1} \subseteq \overline{\{f(s,y,v_n(s,y))\}}_{n\geq 1} \Longrightarrow F(s,y) = \overline{\{f(s,y,v_n(s,y))\}}_{n\geq 1}\,.$$

Because of hypothesis $H(f)$, we have that $(s,y) \to f(s,y,v_n(y,s))$ is measurable. Hence, $(s,y) \to F(s,y)$ is measurable.

Next we will show that for fixed $t \in T$, $y \to F(t,y)$ is u.s.c. on X. We need to show that for every $V \subseteq X$ nonempty, open, $\{y \in X : F(t,y) \subseteq V\} = F^+(t,V)$ is open. So, let $y \in F^+(t,V)$. Then, by definition, $F(t,y) \subseteq V$ and so for all $u \in U(t,y)$, we have $f(t,y,u) \in V$. But recall that by hypothesis H(f) (2), $f(t,\cdot,\cdot)$ is continuous. So we can find neighborhoods $N(u)$, (y) of y and $N(u)$ of u such that for all $y' \in N(u)(y)$ and all $u' \in N(u)$ we have $f(t,y',u') \in V$. Since $U(\cdot,\cdot)$ is compact valued, we can find $u_1, \cdots, u_N$ such that

$$U(t,y) \subseteq \bigcup_{i=1}^{N} N(u_i) = W \quad \text{open in } Z\,.$$

Set $N(y) = \bigcap_{i=1}^{N} N(u_i)(y)$. This is a neighborhood of y. Since $U(t,\cdot)$ is u.s.c., there exists $N'(y)$ a neighborhood of y, such that $U(t,y') \in W$ for all $y' \in N'(y)$. Setting $\hat{N}(y) = N(y) \cap N'(y)$, we have that for every $y' \in \hat{N}(y)$ and for every $u' \in U(t,y')$

$$f(t,y',u') \in V \Longrightarrow F^+(t,V) \text{ is open } \Longrightarrow F(t,\cdot) \text{ is u.s.c.}$$

Now consider the following integrodifferential inclusion:

$$\begin{aligned} &\dot{x}(t) \in g(t,x_t) + \int_0^t K(t,s)F(s,x(s))\,ds \quad \text{a.e. on } T \\ &x(v) = \phi(v), \quad v \in T_0 \end{aligned} \qquad (*)'$$

By a solution of $(*)'$, we understand an absolutely continuous function $x : T_b \to X$, for which there exists $h(\cdot) \in L^1(X)$ such that $h(s) \in F(s,x(s))$ almost everywhere and

$$\dot{x}(t) = g(t,x_t) + \int_0^t K(t,s)h(s)\,ds, \quad \text{a.e. on } T, \;\; x(v) = \phi(v), \;\; v \in T_0.$$

We claim that systems $(*)$ and $(*)'$ are equivalent. Clearly, every solution of $(*)$ also solves $(*)'$. On the other hand, let $x(\cdot)$ be a solution of $(*)'$. Then, by definition, we have:

$$\begin{aligned} &\dot{x}(t) = g(t,x_t) + \int_0^t K(t,s)h(s)\,ds \quad \text{a.e. on } T \\ &x(v) = \phi(v), \quad v \in T_0 \text{ and } h \in S^1_{F(\cdot,x(\cdot))}\,. \end{aligned}$$

Let

$$S(t) = \{z \in U(t, x(t)) : h(t) = f(t, x(t), z)\}.$$

Define $p(t, z) = h(t) - f(t, x(t), z)$. Clearly, $p(\cdot, \cdot)$ is measurable in t and continuous in z, hence is jointly measurable. Thus

$$\operatorname{Gr} S = \{(t, z) \in T \times Z : p(t, z) = 0\} \cap \operatorname{Gr} U(\cdot, x(\cdot)).$$

Since by hypothesis, $(t, y) \to U(t, y)$ is measurable, we have that $t \to U(t, x(t))$ is measurable and so $\operatorname{Gr} U(\cdot, x(\cdot)) \in \Sigma(T) \times B(X) \times B(Z)$, where $\Sigma(T)$ = Lebesgue completion of the Borel σ-field $B(T)$. Apply Aumann's selection theorem (see Saint-Beuve [17]), to get $u : T \to Z$ measurable such that $u(t) \in S(t)$ for all $t \in T \Longrightarrow h(t) = f(t, x(t), u(t))$, $u(t) \in U(t, x(t))$, $t \in T \Longrightarrow x(\cdot)$ solves $(*)$. Hence, we have shown that systems $(*)$ and $(*)'$ are indeed equivalent.

Let $x(\cdot) \in P(\phi)$. For $t \in T$ we have:

$$\begin{aligned}\|x(\tau)\| &\leq \|\phi(0)\| + \int_0^\tau \|g(t, x_t)\|\, dt + \int_0^\tau \int_0^t \|K(t, s) f(s, x(s), u(s))\|\, ds\, dt \\ &\leq \|\phi(0)\| + \int_0^\tau \big(a_1(t) + b_1(t)\|x_t\|_\infty\big)\, dt + \int_0^\tau \int_0^t M\big(a_2(s) + b_2(s)\|x(s)\|\big)\, ds\,.\end{aligned}$$

Let $k(t) = \| x_t \|_\infty$. Then we can write:

$$k(\tau) \leq \big(\| \phi(0) \| + \| a \|_1 + M \| a_2 \|_1 \big) + \int_0^\tau b_1(t) k(t)\, dt + \int_0^\tau \int_0^t M b_2(s) k(s)\, ds\,.$$

Define $\hat{b}(t) = \max(b_1(t), M, 1)$. Clearly, $\hat{b}(\cdot) \in L^1$. Also, let $\eta_0 = \|\phi(0)\| + \|a_1\|_1 + M\|a_2\|_1$. Then we have:

$$k(\tau) \leq \eta_0 + \int_0^\tau \hat{b}(t) k(t)\, dt + \int_0^\tau \hat{b}(t) \int_0^t b_2(s) k(s)\, ds\, dt\,.$$

Apply Theorem 1 of Pachpatte [14], to get that

$$\begin{aligned}&k(\tau) \leq \eta_0\big(1 + \int_0^\tau \hat{b}(t) \exp(\|\hat{b}\|_1 + \|b_2\|_1)\, dt\big) \\ &\Longrightarrow k(\tau) \leq \eta_0\big(1 + \|\hat{b}\|_1 \exp(\|\hat{b}\|_1 + \|b_2\|_1)\big) = L \Longrightarrow \|x\|_\infty \leq L \quad \text{for all } x(\cdot) \in P.\end{aligned}$$

Now, let τ, $\tau' \in T_b$, $\tau' > \tau$. We consider the following cases:

$\tau < \tau' \leq 0$: We have $\| x(\tau') - x(\tau) \| = \| \phi(\tau') - \phi(\tau) \|$.

$\tau \leq 0 < \tau'$: In this case, we can write

$$\begin{aligned}&\| x(\tau') - x(\tau) \| \\ &= \| \phi(0) + \int_0^{\tau'} g(t, x_t)\, dt + \int_0^{\tau'} \int_0^t K(t, s) f(s, x(s), u(s))\, ds\, dt - \phi(\tau) \| \\ &\leq \| \phi(0) - \phi(\tau) \| + \int_0^{\tau'} \| g(t, x_t) \|\, dt + \int_0^{\tau'} \int_0^t \| K(t, s) \| \| f(s, x(s), u(s)) \|\, ds \\ &\leq \| \phi(0) - \phi(\tau) \| + \int_0^{\tau'} (a_1(t) + b_1(t) \| x_t \|_\infty)\, dt + \int_0^{\tau'} \int_0^t M(a_2(s) + b_2(s) \| x(s) \|)\, ds\, dt \\ &\leq \| \phi(0) - \phi(\tau) \| + \int_0^{\tau'} (a_1(t) + b_1(t) L)\, dt + \int_0^{\tau'} \int_0^t M(a_2(s) + b_2(s) L)\, ds\, dt.\end{aligned}$$

$0 \leq \tau < \tau'$: In this final case, we have:

$$\| x(\tau') - x(\tau \| \leq \int_{\tau}^{\tau'} \| g(t, x_t) \| \, dt + \int_{\tau}^{\tau'} \int_0^t \| K(t,s) \| \cdot \| f(s, x(s), u(s)) \| \, ds \, dt$$
$$\leq \int_{\tau}^{\tau'} (a_1(t) + b_1(t)L) \, dt + \int_{\tau}^{\tau'} \int_0^t M(a_2(s) + b_2(s)L) \, ds \, dt.$$

From all these estimates, we deduce that $P(\phi)$ is equicontinuous. Invoking the Arzela-Ascoli theorem, we get that $P(\phi)$ is relatively compact in $C(T_b, X)$. We will show that $P(\phi)$ is, in fact, compact in $C(T_b, X)$. So we only need to show that it is closed. Hence, let $\{x_n(\cdot)\}_{n\geq 1} \subseteq P(\phi)$ be such that $x_n \to x$ in $C(T_b, X)$. For all $n \geq 1$, we have

$$\dot{x}_n(t) \in g(t, (x_n)_t) + \int_0^t K(t,s) F(s, x_n(s)) \, ds.$$

We saw earlier in the proof that $F(t, \cdot)$ is u.s.c. Therefore we have that

$$H(t) = \Big(\bigcup_{n\geq 1} F(t, x_n(t)) \cup F(t, x(t)) \in P_k(X) \quad t \in T.$$

Also, from hypothesis H(U) (3), we have

$$|H(t)| \leq a(t) + b(t) \sup_{n\geq 1} \| x_n \|_{\infty} \Longrightarrow t \to H(t) \text{ is integrably bounded.}$$

Since $F(t, x_n(t))$, $F(t, x(t)) \subseteq H(t)$, $n \geq 1$, we can apply Theorem 3.2 of [15] and get that

$$\overline{\lim} \int_0^t K(t,s) F(s, x_n(s)) \, ds \subseteq \int_0^t \overline{\lim} \, K(t,s) F(s, x_n(s)) \, ds.$$

Using Lemma 3.1 and Theorem 3.1 of [10], we get that

$$\int_0^t \overline{\lim} \, K(t,s) F(s, x_n(s)) \, ds \subseteq \int_0^t K(t,s) \, \overline{\lim} \, F(s, x_n(s)) \, ds.$$

Since $F(t, \cdot)$ is u.s.c., we have $\overline{\lim} \, F(t, x_n(s)) \subseteq F(t, x(s))$. Thus, finally,

$$\overline{\lim} \int_0^t K(t,s) F(s, x_n(s)) \, ds \subseteq \int_0^t K(t,s) F(s, x(s)) \, ds.$$

Also, since

$$\dot{x}_n(t) \in g(t, (x_n)_t) + \int_0^t K(t,s) F(s, x_n(s)) \, ds,$$

we have

$$\| \dot{x}_n(t) \| \leq a_1(t) + b_1(t)L + \int_0^t M(a_2(s) + b_2(s)L) \, ds \ \text{ a.e.}$$

So, because of the Dunford-Pettis compactness criterion, by passing to a subsequence, if necessary, we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1(T, X)$. Using Theorem 3.1 of [15], we have:

$$\dot{x}(t) \in \text{conv } \overline{\lim} \{\dot{x}_n(t)\}_{n\geq 1} \text{ a.e.} \Longrightarrow \dot{x}(t) \in g(t, x_t) + \text{conv} \int_0^t K(t,s) F(s, x(s)) \, ds.$$

But from [10] (see also Hiai-Umegaki [9]), we know that

$$\text{conv}\int_0^t K(t,s)F(s,x(s))\,ds = \int_0^t K(t,s)F(s,x(s))\,ds$$

$$\Longrightarrow \dot{x}(t) \in g(t,x_t) + \int_0^t K(t,s)F(s,x(s))\,ds \Longrightarrow x(\cdot) \in P(\phi).$$

Therefore, $P(\phi)$ is indeed compact in $C(T_b, X)$. ∎

The attainable (reachable) set of system (*) at time instant $t \in T$ is defined by:

$$R(t) = \{z \in X : z = x(t), \quad x(\cdot) \in P(\phi)\};$$

i.e., $R(t) = P(\phi)(t)$.

An immediate, interesting consequence of Theorem 3.1 is the following result concerning the structure of the attainable sets $R(t)$, $t \in T$.

Theorem 3.2. *If the hypotheses of Theorem 3.1 hold, then for every $t \in T$, $R(t) \in P_k(X)$.*

Proof: Note that for every $t \in T$, $R(t) = e_t(P(\phi))$, where $e_t(\cdot)$ is the evaluation map at $t \in T$, defined on $C(T_b, X)$. From topology (see for example Kuratowski [12]), we know that $e_t(\cdot)$ is continuous. Since by Theorem 3.1, $P(\phi)$ is compact in $C(T_b, X)$, we conclude that $R(t) = e_t(P(\phi)) \in P_k(X)$. ∎

In the next result, we examine the dependence of the attainable set on the time variable.

Theorem 3.3. *If the hypotheses of Theorem 3.1 hold, then $t \to R(t)$ is h-continuous.*

Proof: Let $\epsilon > 0$ be given and choose $\delta > 0$ such that if $|\tau' - \tau| < \delta$, then $\| x(\tau') - x(\tau \| < \epsilon$ for all $x(\cdot) \in P(\phi)$. This is possible due to the equicontinuity of the set $P(\phi)$ (see Theorem 3.1). Let $z \in R(\tau)$. Then by definition, there exists admissible pair (x, u) such that

$$z = x(\tau) = \phi(0) + \int_0^\tau g(t,x_t)\,dt + \int_0^\tau \int_0^t K(t,s)f(s,x(s),u(s))\,ds\,dt$$

So we have

$$d(z, R(\tau')) \le \| x(\tau) - x(\tau') \| < \epsilon \Longrightarrow \sup\{d(z, R(\tau')), \;\; z \in R(\tau)\} \le \epsilon.$$

Similarly, we get that

$$\sup\{d(z', R(\tau)), \;\; z' \in R(\tau')\} \le \epsilon.$$

So finally, we have: $h(R(\tau), R(\tau')) < \epsilon$ for $|\tau' - \tau| < \delta$. Thus, $R(\cdot)$ is indeed h-continuous. ∎

A useful consequence of Theorem 3.3 is the following property of the set of all reachable points at all time instants.

Corollary. *If the hypotheses of Theorem 3.1 hold, then $\hat{R} = \cup_{t \in T} R(t)$ is compact in X.*

Proof: From Theorem 3.3, we get that $R(T) = \hat{R}$ is compact in $(P_k(X), h)$. Recalling that in $(P_k(X), h)$, the Vietoris and Hausdorff topologies coincide, we can apply Proposition 2.3.2 of Klein-Thompson [11], and get that $\hat{R} \in P_k(X)$. ∎

With additional hypotheses on the data, we can say more about the structure of the trajectory set $P(\phi)$. Namely assume that hypotheses H, H(g), H(K), H_c remain as before, while hypotheses H(U) and H(f) take the following form:

H(U)$'$: $U : T \to P_{kc}(X)$ is u.s.c.;

H(f)$'$: $f : T \times X \times Z \to X$ is a function such that

(1) $(t, y, u) \to (f(t, y, u)$ is continuous;

(2) $\| f(t, y, u) \| \le N$ for all $(t, y, u) \in T \times X_L \times V$, where $X_L = \{x \in X : \|x\| \le L\}$ (L as in the proof of Theorem 3.1) and $V = u(T)$.

Theorem 3.4. *If hypotheses H, H(g), H(K), H′(u), H′(f) and H_c hold and for all $(t,y) \in T \times X$, $f(t,y,U(t))$ is convex then $P(\phi)$ is a compact, connected subset of $C(T_b, X)$.*

Proof: As before, let $F(t,y) = f(t,y,U(t))$. Working as in the proof of Theorem 3.1 and using hypotheses $H'(U)$ and $H'(f)$, we get that $(t,y) \to F(t,y)$ is u.s.c. Invoking Theorem 4.5 of DeBlasi [7], we get $F_n : T \times X_L \to P_{kc}(X)$ such that $F(t,y) \subseteq \cdots \subseteq F_n(t,y) \subseteq F_{n+1}(t,y) \subseteq \ldots$, $F_n(t,y) \xrightarrow{h} F(t,y)$, $|F_n(t,y)| \leq N$ and $F_n(t,y) = \sum_{k \in I_n} p_k^n(t,y) G_k^n$, where I_n is a finite index set $\{p_k^n(\cdot,\cdot)\}_{k \in I_n}$ is a locally Lipschitz partition of unity of $T \times X_L$ and $\{G_k^n\}_{k \in I_n} \subseteq P_{kc}(X)$.

Consider the following approximating multivalued problems:

$$\dot{x}(t) \in g(t,x_t) + \int_0^t K(t,s) F_n(s,x(s))\, ds \text{ a.e. on } T \qquad (*)'_n$$
$$x(v) = \phi(v), \quad v \in T_0 .$$

Let $P_n(\phi)$ be the solution set of $(*)'_n$. Clearly, $P \subseteq \cdots \subseteq P_{n+1} \subseteq \ldots$. Also, let $S_n = \prod_{k \in I_n} S^1_{G_k^n}$. Clearly this is a convex subset of $\prod_{k \in I_n} L^1(X)$. Let $v^n \in S_n$. Then $v^n = (v_k^n)_{k \in I_n}$, with $v_k^n \in S^1_{G_k^n}$. Consider the following single valued integrodifferential equation:

$$\dot{x}(t) = g(t,x_t) + \int_0^t K(t,s) \sum_{k \in I_n} p_k^n(s,x(s)) v_k^n(s)\, ds, \quad x(0) = \phi(0).$$

This has a unique solution $x(\cdot, v^n)$. We claim that $v \to x(\cdot, v)$ is continuous from S_n into $C(T,X)$. To simplify the already cumbersome notation, assume that $|I_n| = 1$. Then, let $v_m^n \to v^n$ in $L^1(X)$ as $m \to \infty$. We have:

$$\begin{aligned}
\| x_m(\tau) - x(\tau) \| \leq & \int_0^\tau \| g(t,(x_m)_t) - g(t,x_t) \| \; dt \\
& + \int_0^\tau \int_0^t \| K(t,s) \| \, \| p(s,x_m(s)) v_m^n(s) - p(s,x(s)) v_m^n(s) \\
& + p(s,x(s)) v_m^n(s) - p(s,x(s)) v^n(s)) \| \; ds\, dt \\
& \leq \int_0^\tau \psi(t) \| (x_m)_t - x_t \|_\infty \; dt + \int_0^\tau \int_0^t M \cdot l(s) \| x_m(s) - x(s) \| \\
& + p(s,x(s)) \| v_m^n(s) - v^n(s) \|) \, ds\, dt.
\end{aligned}$$

Let $h_m(t) = \| (x_m)_t - x_t \|_\infty$. We have:

$$h_m(\tau) \leq \| v_m^n - v^n \| + \int_0^\tau \psi(t) h_m(t)\, dt + M \int_0^\tau \int_0^t l(s) h_m(s)\, ds\, dt.$$

Set $\hat{\psi}(t) = \max(\psi(t), 1) \in L^1_+$. Then,

$$h_m(\tau) \leq \| v_m^n - v^n \|_1 + \int_0^\tau \hat{\psi}(t) h_m(t)\, dt + M \int_0^\tau \hat{\psi}(t) \int_0^t l(s) h_m(s)\, ds\, dt.$$

Applying Theorem 1 of Pachpatte [14], we get that

$$h_m(\tau) \leq \| v_m^n - v^n \|_1 \, (1 + \| \hat{\psi} \|_1 \exp(\| \hat{\psi} \|_1 + \| l \|_1)) \to 0$$

uniformly in τ. Therefore, we have that

$$x(\cdot, v_m^n) \to x(\cdot, v^n) \quad \text{in } C(T, X)$$

$\Longrightarrow q : v \to x(\cdot, v)$ is continuous from S_n into $C(T, X) \Longrightarrow q(S_n) = P_n(\phi))|_T$ is connected since S_n is convex $\Longrightarrow P_n(\phi)$ is connected and, in fact, compact in $C(T_b, X)$ (the latter property can be obtained as in the proof of Theorem 3.1).

Now we claim that $P(\phi) = \cap_{n\geq 1} P_n(\phi)$. Clearly, $P(\phi) \subseteq \cap_{n\geq 1} P_n(\phi)$. Let $x \in \cap_{n\geq 1} P_n(\phi)$. Then, for all $n \geq 1$, we have

$$\dot{x}(t) = g(t, x_t) + \int_0^t K(t,s) F_n(s, x(s))\, ds, \quad t \in T.$$

But recall that $F_n(s, x(s)) \xrightarrow{h} F(s, x(s))$. From Proposition 4.4 of [16] we get, that $K(t,s)F_n(s,x(s)) \xrightarrow{h} K(t,s)F(s,x(s))$. Invoking Theorem 3.5 of [15], we get that

$$\int_0^t K(t,s) F_n(s, x(s))\, ds \xrightarrow{h} \int_0^t K(t,s) F(s, x(s))\, ds$$

$$\Longrightarrow \dot{x}(t) \in g(t, x_t) + \int_0^t K(t,s) F(s, x(s))\, ds \quad t \in T.$$

Clearly, $x(v) = \phi(v)$, $v \in T_0$. Hence, $x \in P(\phi) \Longrightarrow P(\phi) = \cap_{n\geq 1} P_n(\phi)$. But the intersection of a decreasing sequence of compact, connected sets is connected. Therefore, $P(\phi)$ is compact, connected in $C(T_b, X)$. ∎

Again, this property of connectedness passes to the attainable sets.

Theorem 3.5. *If the hypotheses of Theorem 3.4 hold, then for all $\tau \in T$, $R(\tau)$ is connected.*

Proof: Again, note that $R(\tau) = e_\tau(P(\phi))$ and recall that the evaluation map $e_\tau(\cdot)$ is continuous, while by Theorem 3.4, $P(\phi)$ is connected.

Remarks. (1) To get convexity of the values of $R(\cdot)$, we need to impose additional hypotheses on $f(\cdot,\cdot,\cdot)$ which are close to linearity of the system.

(2) If $t_n \to t$, then it is easy to check that $R(t_n) \to R(t)$ in the Kuratowski sense (see Kuratowski [12]). Since $R(t)$ is connected (Theorem 3.5), from Corollary 3A of Salinetti-Wets [18], we get that $R(t_n) \xrightarrow{h} R(t)$. So we recover, from a different path, the h-continuity of the attainability multifunction $R(\cdot)$, proved in Theorem 3.3.

4. Relaxation and "bang-bang" results. The next result, shows that convexification of the velocity field of $(*)$ (see also $(*)'$), does not change the attainability multifunction.

Often an optimization problem has no optimal solution, but the mathematical problem and the corresponding set of solutions can be modified in such a way that an optimal solution exists, and yet neither the system of trajectories nor the value of the optimization problem are essentially modified. This new system and its solutions are very important from a theoretical point of view and often have a significant physical interpretation.

Here, we will assume that the control constraint multifunction is state independent (open loop). So, we have

H″(U): $U : T \to P_f(z)$ is u.s.c.

Following Angell [2] and Berkovitz [4], to system $(*)$, we associate the following "relaxed system:"

$$\left.\begin{aligned} &\dot{x}(t) = g(t, x_t) + \int_0^t \sum_{k=1}^{n+1} \lambda_k(s) f(s, x(s),\, u_k(s))\, ds \quad \text{a.e. on } T \\ &x(v) = \phi(v), \quad v \in T_0\,. \end{aligned}\right\} \qquad (*)_c$$

Here, $n = \dim X$, $\lambda_k : T \to R_+$ are measurable such that

$$\sum_{k=1}^{n+1} \lambda_k(t) = 1 \ \text{ a.e. and } \ u_k(t) \in U(t) \ \text{ a.e. } \ k \in \{1, \ldots, n+1\}.$$

As we will see, this augmented system $(*)_c$, corresponds to the convexification of the original velocity field $F(t,x) = f(t,x,U(t))$. Let $R_c(t) = \{y \in X : \ y = x(t), \ x \in P_c(\phi)\}$, where $P_c(\phi)$ is the set of admissible trajectories of $(*)_c$. Clearly, $P(\phi) \subseteq P_c(\phi)$ and therefore, $R(t) \subseteq R_c(t)$ for all $t \in T$.

Theorem 4.1. *If hypotheses H, H(g), H(K), H''(U), H(f) and H_c hold, then for all $\tau \in T$, $R(\tau) = R_c(\tau)$.*

Proof: We claim that $(*)_c$ is equivalent to the following integrodifferential inclusion:

$$\left.\begin{aligned} &\dot{x}(t) \in g(t, x_t) + \int_0^t \operatorname{conv} K(t,s) F(s, x(s))\, ds \quad \text{a.e. on } T \\ &x(v) = \phi(v), \quad v \in T_0. \end{aligned}\right\} \qquad (*)_c'$$

Clearly, a solution of $(*)$ also solves $(*)_c'$. Next, let $x(\cdot)$ be a trajectory of $(*)_c'$. Then, since $\operatorname{conv} K(t,s)F(s,x(s)) = K(t,s) \operatorname{conv} F(s,x(s))$, we have:

$$x(\tau) = \phi(0) + \int_0^\tau g(t, x_t)\, dt + \int_0^\tau \int_0^t K(t,s) h(s)\, ds\, dt, \quad \tau \in T$$

for some $h \in S^1_{\operatorname{conv} F(\cdot, x(\cdot))}$. Let S^n be the n-simplex and define $L : T \to 2^{S^n \times X^{n+1}}$ by

$$L(t) = \Big\{(\lambda, \eta) \in S^n \times X^{n+1} : h(t) = \sum_{k=1}^{n+1} \lambda_k \eta_k, \ \ \eta_k \in F(t, x(t))\Big\}.$$

From Caratheodory's theorem, we know that for all $t \in T$, $L(t) \neq \emptyset$. Set

$$F^{n+1}(t, x(t)) = \prod_{k=1}^{n+1} F(t, x(t)).$$

Clearly, $t \to F^{n+1}(t, x(t))$ is measurable. Define

$$c_1(t, \lambda, \eta) = h(t) - \sum_{k=1}^{n+1} \lambda_k \eta_k$$

and

$$c_2(t, \eta) = d(\eta, F^{n+1}(t, x(t))).$$

Note that $t \to c_1(t,\lambda,\eta)$ is measurable and $(\lambda,\eta) \to c_1(t,\lambda,\eta)$ is continuous. Similarly, $t \to c_2(t,\eta)$ is measurable, while $\eta \to c_2(t,\eta)$ is continuous. Therefore, both are jointly measurable. Then observe that

$$GrL = \{(t,\lambda,\eta) \in T\times S^n \times X^{n+1} : c_1(t,\lambda,\eta) = 0,\ \ c_2(t,\eta) = 0\} \in \Sigma(T)\times B(S^n)\times B(X^{n+1}).$$

Apply Aumann's selection theorem to find $\lambda : T \to S^n$ and $\eta : T \to X^{n+1}$ measurable such that $(\lambda(t),\eta(t)) \in L(t)$ for all $t \in T$. For each $k \in \{1,\dots,n+1\}$, we have $\eta_k(t) \in F(t,x(t))$. Define

$$L_k'(t) = \{u \in U(t) : \eta_k(t) = f(t,x(t),u)\}.$$

Set $c_3^k(t,u) = \eta_k(t) - f(t,x(t),u)$. Then, $t \to c_3^k(t,u)$ is measurable, while $u \to c_3^k(t,u)$ is continuous. Thus, $(t,u) \to c_3^k(t,u)$ is measurable and so

$$GrL_k' = \{(t,u) \in T \times Z : c_3^k(t,u) = 0\} \in \Sigma(T) \times B(Z).$$

Once again, through Aumann's selection theorem (see Saint-Beuve [17]), we get $u_k : T \to Z$ measurable such that $u_k(t) \in L_k'(t)$, for all $t \in T$. Then, $\eta_k(t) = f(t,x(t),u_k(t))$. So finally, we have

$$h(t) = \sum_{k=1}^{n+1} \lambda_k(t) f(t,x(t),u_k(t)) \Longrightarrow x(\cdot) \in P_c(\phi).$$

Therefore, $(*)_c$ and $(*)_c'$ are equivalent. Through Theorem 3.1 of [10] (see also Hiai-Umegaki [9]), we get

$$\int_0^t K(t,s)\,\text{conv}\,F(s,x(s))\,ds = \int_0^t \text{conv}\,K(t,s)F(s,x(s))\,ds = \int_0^t K(t,s)F(s,x(s))\,ds.$$

Recalling that $(*)$ is equivalent to the multivalued system $(*)'$, we conclude that $R(\tau) = R_c(\tau)$, $\tau \in T$. ∎

Another result in this direction is the following "bang-bang" type theorem. It says that essentially all reachable points can be attained by using only "bang-bang" controls. These are controls moving only through the extremal points of the control constraint set. So let $R_e(\tau) = \{y \in X : y = x(u)(\tau),\ x(u)(\cdot) \in P(\phi), \text{ with } u(t) \in \text{ext}\,U(t) \text{ a.e. }\}$. We will need the following modifications of hypotheses H(f) and H(u).

H(f)$''$: $f : T \times X \times X \times Z \to X$ is a function such that

(1) $t \to f(t,x,u)$ is measurable;

(2) $(x,u) \to f(t,x,u)$ is continuous and linear in u;

(3) $\| f(t,x,u) \| \le a_2(t) + b_2(t) \| x \|$ a.e. with $a_2(\cdot), b_2(\cdot) \in L_+^1$, $u \in U(t)$;

(4) $\| f(t,x_1,u) - f(t,x_2,u) \| \le \psi_2(t) \| x_1 - x_2 \|$ a.e., for all $u \in U(t)$ and with $\psi_2(\cdot) \in L_+^1$.

H(u)$'''$: $U : T \to P_{kc}(X)$ is integrably bounded.

Theorem 4.2. *If hypotheses H, H(g), H(k), H(U)$'''$, H(f)$''$ and H_c hold, then for every $\tau \in T$, $R(\tau) = \overline{R_e(\tau)}$.*

Proof: Let $z \in R(\tau)$. Then, by definition, there exists admissible pair (x,u) such that

$$z = x(\tau) = \phi(0) + \int_0^\tau g(t,x_t)\,dt + \int_0^\tau \int_0^t K(t,s)f(s,x(s),u(s))\,ds\,dt.$$

Note that $u \in S^1_U$ and from Corollary II of [10], we know that $S^1_U = \overline{S^1_{\text{ext}\,U}}^w$. Since S^1_U is w-compact in $L^1(x)$ and the latter is separable, the relative weak topology on S^1_U is metrizable. Thus, we can work with sequences and find $u_n \in S^1_{\text{ext}\,U}$ such that $u_n \xrightarrow{w} u$ in $L^1(X)$. Set

$$z_n = \phi(0) + \int_0^\tau g(t,(x_n)_t)\,dt + \int_0^\tau \int_0^t K(t,s)f(s,x_n(s),u_n(s))\,ds\,dt.$$

Observe that $\{x_n(\cdot)\}_{n\geq 1} \subseteq P(\phi)$ and the latter is compact in $C(T_b,X)$ (Theorem 3.1). So, by passing to a subsequence if necessary, we may assume that $x_n \to x \in P(\phi)$ in $C(T_b,X)$. So $g(t,(x_n)_t) \to g(t,x_t)$ and from the dominated convergence theorem, we have that:

$$\int_0^\tau g(t,x_n)_t)\,dt \to \int_0^\tau g(t,x_t)\,dt \quad \text{as} \quad n \to \infty.$$

Furthermore, let

$$\Phi(x,u)(t) = \int_0^t K(t,s)f(s,x(s),u(s))\,ds.$$

Then, from hypothesis H(f)$''$(2), we see that $u \to \Phi(x,u)(t)$ is continuous, linear, while $x \to \Phi(x,u)(t)$ is Lipschitz, namely for $x_1,x_2 \in C(T,X)$, we have

$$\| \Phi(x_1,u)(t) - \Phi(x_2,u)(t) \| \leq \| \psi_2 \|_1 \| x_1 - x_2 \|_\infty .$$

Then, for every $t \in T$, we have

$$\begin{aligned}\| \Phi(x_n,u_n)(t) - \Phi(x,u)(t) \| &\leq \| \Phi(x_n,u_n)(t) - \Phi(x,u_n)(t) \| + \| \Phi(x,u_n)(t) - \Phi(x,u)(t) \| \\ &\leq \| \psi_2 \|_1 \| x_1 - x_2 \|_\infty + \| \Phi(x,u_n)(t) - \Phi(x,u)(t) \| .\end{aligned}$$

Since a continuous linear map is weakly continuous, we have that

$$\| \Phi(x,u_n)(t) - \Phi(x,u)(t) \| \to 0 \quad \text{as} \quad n \to \infty \Longrightarrow \| \Phi(x_n,u_n)(t) - \Phi(x,u)(t) \| \to 0 .$$

So from the dominated convergence theorem, we get that

$$\int_0^\tau \Phi(x_n,u_n)(t)\,dt \to \int_0^\tau \Phi(x,u)(t)\,dt \Longrightarrow z_n \to z .$$

Since $z_n \in R_e(\tau)$, we conclude that $\overline{R_e(\tau)} = R(\tau)$.

5. Optimal control problems. In this section, we solve some optimization problems involving the control system $(*)$.

We start with a time optimal control problem. So let $L : T \to P_f(x)$ be a moving target set and assume that $\phi(0) \in L(0)$. Also assume that

$H_0: \;C = \{t \in T : L(t) \cap R(t) \neq \emptyset\} \neq \emptyset.$

This is a controllability type hypothesis with respect to $L(\cdot)$. For $L(\cdot)$, we will make the following hypothesis:

$H(L): \;L : T \to P_f(x)$ is u.s.c.

Our goal is to find an admissible control so as to hit the moving target in minimum time moving along admissible trajectories of system $(*)$.

An admissible pair $(\hat{x},\hat{u})$ solving this problem is known as an optimal pair. In particular $\hat{u}(\cdot)$ is the time optimal control and $\hat{x}(\cdot)$ is the corresponding time optimal trajectory.

Theorem 5.1. *If hypotheses H, H(g), H(K), H(U), H(f), H_c, H_0 and H(L) hold, then there exists time optimal control.*

Proof: Let $\tau = \inf C$. Let $t_n \in C$ and let $t_n \downarrow \tau$. By definition, we can find $\{x_n(\cdot)\}_{n\geq 1} \subseteq P(\phi)$ such that $x_n(t_n) \in L(t_n)$. But from Theorem 3.1, we know that P is compact in $C(T_b, X)$. So we may assume that $x_n \to x \in P(\phi)$ in $C(T_b, X)$. Then $x_n(t_n) \to x(\tau) \Longrightarrow x(\tau) \in \overline{\lim} L(t_n) \subseteq L(\tau)$, since $L(\cdot)$ is u.s.c. Let $\hat{u}(\cdot)$ be the admissible control generating the admissible trajectory $x(\cdot)$. Clearly, this is the desired time optimal control. ∎

Next we will consider a Lagrange problem involving $(*)$. So consider the following integral functional:

$$J(x,u) = \int_0^b h(t, x(t), u(t))\, dt.$$

We will make the following hypothesis concerning the cost integrand $h(\cdot,\cdot,\cdot)$.

H(h): $h : T \times X \times Z \to \bar{R} = RU\{+\infty\}$ is a function such that

(1) $(t, x, u) \to h(t, x, u)$ is measurable;
(2) $(x, u) \to h(t, x, u)$ is l.s.c. and convex in u;
(3) $h(t, x, u) \geq \lambda(t) - a \parallel x \parallel$ almost everywhere for all $u \in U(t)$, with $a \geq 0$, $\lambda(\cdot) \in L^1$;
(4) there exists $(x, u) \in A(\phi)$ such that $J(x, u) < +\infty$.

Let $m = \inf\{J(x,u) : (x,u) \in A(\phi)\}$. Because of hypothesis H(h)(4), $m < \infty$.

Theorem 5.2. *If hypotheses H, H(g), H(K), H(U), H(f)″, H_c and H(h) hold, then there exists $(\hat{x}, \hat{u}) \in A(\phi)$ such that $m = J(\hat{x}, \hat{u})$.*

Proof: Let $\{(x_n, u_n)\}_{n\geq 1} \subseteq A(\phi)$ be a minimizing sequence of admissible pairs, i.e., $J(x_n, u_n) \downarrow m$. From Theorem 3.1, we have that $\overline{\{x_n\}_{n\geq 1}}$ is compact in $C(T_b, X)$. So we may assume that $x_n \to \hat{x} \in P(\phi)$. Also set $H(t) = \left[\cup_{n\geq 1} U(t, x_n(t))\right] \cup U(t, x(t))$. Exploiting the upper semicontinuity of $U(t,\cdot)$, we get that $H(\cdot)$ is a $P_k(X)$-valued, integrably bounded multifunction. So from the Dunford-Pettis theorem we have that $S^1_{\overline{\text{conv}\, H}}$ is w-compact in $L^1(X)$. Thus, we may assume that $u_n \xrightarrow{w} \hat{u}$ in $L^1(X)$. Let $\hat{\lambda}(t) = \lambda(t) - a \sup \parallel x_n \parallel_\infty$. Then, $h(t, x_n(t), u_n(t)) \geq \hat{\lambda}(t)$ almost everywhere and so we can apply Theorem 2.1 of Balder [3], to get that

$$J(\hat{x}, \hat{u}) \leq \underline{\lim}\, J(x_n, u_n) = m.$$

If we can show that $(\hat{x}, \hat{u}) \in A(\phi)$, we are done. But for all $n \geq 1$,

$$x_n(\tau) = \phi(0) + \int_0^\tau g(t, (x_n)_t)\, dt + \int_0^\tau \int_0^t K(t,s) f(s, x_n(s), u_n(s))\, ds\, dt.$$

As in the proof of Theorem 4.2, in the limit as $n \to \infty$, we get

$$\hat{x}(\tau) = \phi(0) + \int_0^\tau g(t, (\hat{x})_t)\, dt + \int_0^\tau \int_0^t K(t,s) f(s, \hat{x}(s), \hat{u}(s))\, ds.$$

Also, from Theorem 3.1 of [15], we have

$$\hat{u}(t) \in \overline{\text{conv}}\, \overline{\lim}\, U(t, x_n(t)) \subseteq U(t, \hat{x}(t)) \text{ a.e.}$$

since $U(t,\cdot)$ is u.s.c. Therefore, $(\hat{x}, \hat{u}) \in A(\phi) \Longrightarrow m = J(\hat{x}, \hat{u})$. ∎

Sometimes it is required to regulate the system so that the period of its stay inside a certain region is maximized. Such optimal control problems, with dynamics described by (*), appear often in mathematical economics in connection with growth models, where the rate of growth of the capital depends on the past history of the capital (state $x(t)$) and of the investment (control or input $u(t)$).

Let $L : T \to P_f(x)$ be the target multifunction. We want to find an admissible control so as to maximize $\lambda\{t \in T : x(t) \in L(t))\}$ where $x \in P(\phi)$ and λ is the Lebesgue measure on T. For $L(\cdot)$ we will assume that

$H(L)'$: $L : T \to P_f(X)$ is measurable.

Theorem 5.3. *If hypotheses H, H(g), H(K), H(U), H(f), H_c and H(L)' hold, then there exists an optimal control solving the above problem.*

Proof: Consider the function $k : C(T, X) \to R_+$ defined by

$$k(x) = \lambda\{t \in T : x(t) \in L(t)\}.$$

Let $x_n \to x$ in $C(T, X)$. Set $B_n = \{t \in T : x_n(t) \in L(t)\}$ and $B = \{t \in T : x(t) \in L(t)\}$.

Note that $B_n = \{t \in T : d(x_n(t), L(t)) = 0\}$. Since $(t, y) \to d(y, L(t))$ is a Caratheodory function (i.e., measurable in t, continuous in y), is jointly measurable. Hence, $t \to d(x_n(t), L(t))$ is measurable $\Longrightarrow B_n \in \Sigma(T) =$ Lebesgue σ-field of T. Similarly for B. Also from their definitions is clear that

$$\limsup B_n = \bigcap_{k\geq 1} \bigcup_{n \geq k} B_n \subseteq B.$$

Therefore from Chung [6, p. 74], we have that

$$\overline{\lim}\, \lambda(B_n) \leq \lambda(\limsup B_n) \leq \lambda(B) \Longrightarrow \overline{\lim}\, k(x_n) \leq k(x) \Longrightarrow k(\cdot) \text{ u.s.c.}$$

Also, $P(\phi)$ is compact in $C(T_b, X)$ (Theorem 3.1). Now note that our problem is to maximize $k(\cdot)$ over $P(\phi)\big|_T$. From Weierstrass theorem we can find $\hat{x} \in P(\phi)$ such that $\sup\{k(x) : x \in P(\phi)\} = k(\hat{x})$. Let $\hat{u}(\cdot)$ be the corresponding control. Then, $(\hat{x}, \hat{u})$ is the desired optimal pair.

REFERENCES

[1] T. Angell, *Existence theorems for optimal control problems involving functional differential equations,* J. Optim. Th. Appl., 7 (1971), 148-169.

[2] T. Angell, *Existence of optimal control without convexity and a bang-bang theorem for linear Volterra equations,* J. Optim. Th. Appl., 19 (1976), 63-79.

[3] E. Balder, *Lower closure for orientor fields by lower semicontinuity of outer integral functionals,* Ann. Mat. Pura ed Appl., 139 (1985), 349-360.

[4] L. Berkovitz, "Optimal Control Theory," Applied Math. Sci. No. 12, Springer, New York, (1974).

[5] L. Cesari, *Existence theorems for optimal solutions in Pontryagin and Lagrange problems,* SIAM J. Control, 3 (1965), 475-498.

[6] K.L. Chung, "A Course in Probability Theory," Academic Press, New York (1974).

[7] F. DeBlasi, *Characterizations of certain classes of semicontinuous multifunctions by continuous approximations,* J. Math. Anal. Appl., 106 (1985), 1-18.

[8] H. Hermes and J. LaSalle, "Functional Analysis and Time Optimal Control," Academic Press, New York (1969).

[9] F. Hiai and H. Umegaki, *Integrals, conditional expectations and martingales of multivalued functions,* J. Multiv. Anal., 7 (1977), 149-182.

[10] D. Kandilakis and N.S. Papageorgiou, *On the properties of the Aumann integral with applications to differential inclusions and control systems,* Czechosl. Math. Jour., to appear.

[11] E. Klein and A. Thompson, "Theory of Correspondences," Wiley, New York (1984).
[12] K. Kuratowski "Topology I," Academic Press, New York (1966).
[13] M.N. Oğuztoreli, "Time Lag Control Systems," Academic Press, New York (1966).
[14] B.G. Pachpatte, *A note on the Gronwall-Bellman inequality,* J. Math. Anal. Appl., 44 (1974), 758-762.
[15] N.S. Papageorgiou, *Convergence theorems for Banach space valued integrable multifunctions,* Inter. J. Math. and Math. Sci., 10 (1987), 433-442.
[16] N.S. Papageorgiou, *Volterra integral inclusions in Banach spaces,* J. Integral Eq. and Applications, 1 (1988), 65-81.
[17] M.-F. Saint-Beauve, *On the extension of Von Neumann-Aumann's theorem,* J. Funct. Anal., 17 (1974), 112-129.
[18] G. Salinetti and R. Wets, *On the convergence of sequences of convex sets in finite dimensions,* SIAM Rev., 21 (1979), 18-33.
[19] J. Warga, "Optimal Control of Differential and Functional Equations," Academic Press, New York (1972).