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ON THE TWO DIMENSIONAL SINGULAR INTEGRAL EQUATIONS

HUANG XINMIN

Department of Mathematics, Guangxi University, Nanning, Guangxi, China

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Abstract. Applying the "equivalent curve" method provided by the author, this paper discusses general solutions for the two dimensional singular integral equation

$$\omega(z) - \frac{q(z)}{\pi} \int_G \int \frac{\bar{\omega}(\xi)}{(\bar{\xi} - \bar{z})^2} \, d\sigma_\xi = f(z),$$

here G is a simply connected region in the complex plane.

Suppose G is a simply connected region in the complex plane E, its boundary Γ is formed by finite arcs in C^1_α. In this paper we discuss general solutions for the following two dimensional singular integral equation on $\bar{G}$:

$$\omega(z) - \frac{q(z)}{\pi} \int_G \int \frac{\bar{\omega}(\xi)}{(\bar{\xi} - \bar{z})^2} \, d\sigma_\xi = f(z). \tag{1}$$

Here the solution $\omega(z)$ is in the space

$$V(\bar{G}) = \{\omega(z) : \omega \in L_1(\bar{G}) \cap L_p(\bar{G}/\{a_1, \cdots, a_Q\}), \;\; p > 2\},$$

and $q(z)$ satisfies conditions

(i) $|q(z)| = 1$, $z \in \bar{G}/\{a_1, \cdots, a_Q\}$;

(ii) in $\bar{G}/\{a_1, \cdots, a_Q\}$, $q(z)$ has continuous partial derivatives which may have poles with degree smaller than 2 at $a_1, \cdots, a_Q$;

(iii) $f(z) \in V(\bar{G})$.

The equation (1) is solved by N.H. Vekua [1] under the condition $|q(z)| < 1$ and is discussed by A. Džuraev [2] and N.N. Komjak [3], [4] in the case $|q(z)| > 1$ and $q(z) = e^{i\lambda}$ (here λ is a real constant).

Denote

$$(T\omega)(z) = -\frac{1}{\pi} \int_G \int \frac{\omega(\xi)}{\xi - z} \, d\sigma_\xi \quad \text{for } \omega \in V(G),$$

then $(T\omega)(z)$ has generalized derivatives [1]

$$\frac{\partial (T\omega)(z)}{\partial \bar{z}} = \omega(z), \quad \frac{\partial (T\omega)(z)}{\partial z} = -\frac{1}{\pi} \int_G \int \frac{\omega(\xi)}{(\xi - z)^2} \, d\sigma_\xi.$$

It is easy to prove that the function $(T\omega)(z)$ is analytic in $E/\bar{G}$, continuous in $E/\{a_1, \cdots, a_Q\}$ and $(T\omega)(\infty) = 0$.

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Lemma 1. *The equation (1) is solvable if and only if there exists a function $H(z)$ such that*

(i) $\bar{H}(z) = \bar{q}(z)H(z)$;

(ii) $\dfrac{H_{\bar{z}}(z) - f(z)}{q_{\bar{z}}(z)} = \dfrac{H_z(z) - q(z)\bar{f}(z)}{q_z(z)}, \quad z \in \bar{G}/\{a_1, \cdots, a_Q\}$;

(iii) *except for $a_1, \cdots, a_Q$ (if some of them are on Γ), $\psi(t)_{t\in\Gamma} = (\bar{H}_z(t) - \bar{f}(t))/\bar{q}_z(t)$ is a boundary-value of the function $\psi(z)$ that is analytic in $E/\bar{G}$ and $\psi(\infty) = 0$;*

(iv) *The function*

$$\omega(z) = -\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{H}_z(z) - \bar{f}(z)}{\bar{q}_z(z)}\right) \tag{2}$$

is one in $V(\bar{G})$.

If the solution exists, then it is precisely $\omega(z)$ as defined by (2).

Proof: Necessity. If there exists $\omega(z) \in V(\bar{G})$ that is a solution of equation (1), denote $W(z) = (T\omega)(z)$ and $\psi(z)_{z\in E/G} = W(z)$, then $\psi(z)$ is analytic in $E/\bar{G}$ and $\psi(\infty) = 0$. $W(z)$ is continuous in $E/\{a_1, \cdots, a_Q\}$ and the equation

$$W_{\bar{z}}(z) + q(z)\bar{W}_{\bar{z}}(z) = f(z)$$

holds in $\bar{G}$. Put

$$H(z) = W(z) + q(z)\bar{W}(z), \tag{3}$$

then $H(z)$ satisfies the condition (i) of Lemma 1 and there holds

$$H_{\bar{z}}(z) = f(z) + q_{\bar{z}}(z)\bar{W}(z). \tag{4}$$

Take the conjugate equation of (4) from (3) and

$$\bar{q}_z(z) = -\bar{q}(z)^2 q_z(z),$$

and we obtain

$$H_z(z) = q(z)\bar{f}(z) + q_z(z)\bar{W}(z), \tag{5}$$

thus, other conditions of Lemma 1 follow from (4) and (5).

Sufficiency. If the four conditions hold in Lemma 1 and the function $\omega(z)$ is defined by (2), it is known from [1] that

$$\overline{\frac{H_{\bar{z}}(z) - f(z)}{q_{\bar{z}}(z)}} = F(z) + (T\omega)(z), \quad z \in \bar{G}, \tag{6}$$

where $F(z)$ is analytic in G. By taking $z = t \in \Gamma$ in (6) and multiplying two sides of (6) by $1/(2\pi i(t - z))$, integrate (6) along Γ and obtain $F(z) = 0$. Hence,

$$\overline{\frac{H_{\bar{z}}(z) - f(z)}{q_{\bar{z}}(z)}} = -\frac{1}{\pi}\int_G\int \frac{\omega(\xi)}{\xi - z}\, d\sigma_\xi,$$

so

$$-\frac{1}{\pi}\int_G\int \frac{\bar{\omega}(\xi)}{(\bar{\xi} - \bar{z})^2}\, d\sigma_\xi = \overline{\frac{\partial}{\partial z}\left(\frac{\bar{H}_z(z) - \bar{f}(z)}{\bar{q}_z(z)}\right)}$$

$$= \overline{\frac{\partial}{\partial z}\left(\frac{(\bar{q}(z)H(z))_z - \bar{f}(z)}{\bar{q}_z(z)}\right)} = \overline{\frac{\partial}{\partial z}\left(H(z) - q(z)\left(\frac{H_z(z) - q(z)\bar{f}(z)}{q_z(z)}\right)\right)}$$

$$= \overline{q(z)\bar{f}(z) - q(z)\frac{\partial}{\partial z}\left(\frac{H_{\bar{z}}(z) - f(z)}{q_{\bar{z}}(z)}\right)} = \bar{q}(z)f(z) - \bar{q}(z)\omega(z),$$

concluding that $\omega(z)$ is a solution of the equation (1). ∎

Now, solving equation (1), is equivalent to finding the function $H(z)$ that satisfies four conditions of Lemma 1. For this purpose, the following definitions are needed.

Definition 1. *The curve ν is called an equivalent curve of $q(z)$ if $q(z)_{z\in\nu}$ = constant; the region D is called an equivalent region of $q(z)$ if $q(z)_{z\in D}$ = constant.*

Definition 2. *D_2 is called a second type set of $\bar{G}$ for $q(z)$ if any point of D_2 satisfies one of the following two conditions:*

(i) *It lies on an equivalent region of the function $q(z)$.*
(ii) *It has a neighborhood in which no point can be joined with the point of the boundary Γ by an equivalent curve of $q(z)$.*

$D_1 = \bar{G}/(D_2 \cup \{a_1, \cdots, a_Q\})$ is called a first type point set of $\bar{G}$ for $q(z)$. Equivalent curves of $q(z)$ from Γ are dense in D_1 and D_1 does not contain any region where $q(z)$ = constant. Points $a_1, \cdots, a_Q$ are neither in D_1 nor in D_2.

In the following, we will prove that if a point z can be joined with a point $t \in \Gamma$ by an equivalent curve of $q(z)$, then we can determine $H(z)$ from $H(t)$ restricted on Γ. Since $H(z)$ is continuous we can determine all $H(z)$ for $z \in D_1$.

Lemma 2. *If $z \in G$ is joined with initial point $t \in \Gamma$ by the equivalent curve ν of $q(z)$, then*

$$H(z) = H(t) + \int_{\nu} f(\xi)\, d\bar{\xi} + q(\xi)\bar{f}(\xi)\, d\xi. \tag{7}$$

Proof: Since $q_{\bar{z}}(z)d\bar{z} + q_z(z)dz = 0$ on ν, (7) will be proved by (4) and (5).

Lemma 3. *If Γ has parametric representation $t = t(s)$ (here s is arc length), then finding $H(t)_{t\in\Gamma}$ of Lemma 2 is equivalent to finding $\psi(z)$ which satisfies the following conditions:*

(i) *$\psi(z)$ is analytic in $E/\bar{G}$ and $\psi(\infty) = 0$;*
(ii) *$\psi(z)$ is continuous in E/G except $a_1, \cdots, a_Q$ (if some of them are on Γ) and there holds the boundary-value condition on Γ:*

$$t'(s)\psi'(t) + \bar{t}'(s)q(t)\bar{\psi}'(t) = \bar{t}'(s)f(t) + t'(s)q(t)\bar{f}(t). \tag{8}$$

Proof: Assume ν is an arbitrary arc and has parametric $z = z(s)$ (here s is arc length). Denote the tangential direction of ν by T. Denote the normal direction of ν by N and call the positive normal direction the one which points to the right of T. Directional derivatives of $H(z)$ with respect to T and N are

$$H_T(z) = H_{\bar{z}}(z)\bar{z}'(s) + H_z(z)z'(s)$$
$$H_N(z) = i\big(H_{\bar{z}}(z)\bar{z}'(s) - H_z(z)z'(s)\big).$$

So, we have

$$H_{\bar{z}}(z) = \frac{z'(s)}{2}(H_T(z) - iH_N(z))$$
$$H_z(z) = \frac{\bar{z}'(s)}{2}(H_T(z) + iH_N(z)),$$

and it follows from the condition (ii) of Lemma 1 that

$$\frac{z'(s)(H_T(z)-iH_N(z))-2f(z)}{z'(s)(q_T(z)-iq_N(z))}=\frac{\bar{z}'(s)(H_T(z)+iH_N(z))-2q(z)\bar{f}(z)}{\bar{z}'(s)(q_T(z)+iq_N(z)};$$

i.e.,

$$\frac{H_T(z)-(\bar{z}'(s)f(z)+z'(s)q(z)\bar{f}(z))}{q_T(z)}=\frac{H_N(z)-i(\bar{z}'(s)f(z)-z'(s)q(z)\bar{f}(z))}{q_N(z)}.$$

Thus,

$$\bar{W}(z)=\frac{H_{\bar{z}}(z)-f(z)}{q_{\bar{z}}(z)}=\frac{H_T(z)-(\bar{z}'(s)f(z)+z'(s)q(z)\bar{f}(z))}{q_T(z)}.$$

If ν is the boundary $\Gamma: t=t(s)$ then the last formula gives

$$\bar{\psi}(t)_{t\in\Gamma}=\frac{H_{\bar{z}}(t)-f(t)}{q_{\bar{z}}(t)}=\frac{H_s(t)-(\bar{t}'(s)f(t)+t'(s)q(t)\bar{f}(t))}{q_s(t)},$$

Lemma 3 will be proved from this formula and $H(t)=\psi(t)+q(t)\bar{\psi}(t)$.

In order to find the function $\psi(z)$ which satisfies the two conditions in Lemma 3, we conformally map $E/\bar{G}$ onto $K=\{|\xi|>1\}$. If the mapping function is

$$z=\phi(\xi),\quad \xi\in K,\quad \phi(\infty)=\infty,$$

denote

$$Q(\xi)=\psi(\phi(\xi)),$$

then it follows from (8) that (taking into account the boundary K is $\tau(s)=e^{is}$)

$$\tau Q'(\tau)-q(\phi(\tau))\bar{\tau}\bar{Q}'(\tau)=q(\phi(\tau))\phi'(\tau)\tau\bar{f}(\phi(\tau))-\bar{\phi}'(\tau)\bar{\tau}f(\phi(\tau)). \tag{9}$$

Obviously, finding $\psi(z)$ which satisfies the two conditions of Lemma 3 is equivalent to finding the function $Q(\xi)$ which satisfies conditions

(i) $Q(\xi)$ is analytic in K and $Q(\infty)=0$;

(ii) $Q(\xi)$ is continuous in $\bar{K}$ except finite points (if some of $a_1,\cdots,a_Q$ are on Γ) and the boundary-value condition (9) holds on $|\tau|=1$.

For the purpose of finding $Q(\xi)$, we represent $q(\phi(\tau))$ in the form [1]

$$q(\phi(\tau))=\tau^n e^{2iP(\tau)+P_1(\tau)},\quad |\tau|=1,$$

here $n=\mathrm{Ind}_\Gamma q(t)$ is called an index of $q(z)$ to Γ, $P(\tau)$ is the boundary-value of the function $P(\xi)$ that is analytic in K, $|P(\infty)|<\infty$ and $P_1(\tau)=\mathrm{Im}\,2P(\tau)=i(\bar{P}(\tau)-P(\tau))$; hence, we may further represent $q(\phi(\tau))$ in the form

$$q(\phi(\tau))=\tau^n e^{i\bar{P}(\tau)+iP(\tau)}$$

and (9) as

$$e^{i\bar{P}(\tau)}\bar{Q}'(\tau)-\tau^{2-n}e^{-iP(\tau)}Q'(\tau)=f_1(\tau), \tag{10}$$

here, $f_1(\tau)$ satisifies

$$\bar{f}_1(\tau) = -\tau^{n-2} f_1(\tau). \tag{11}$$

If we assume

$$h(\xi) = \begin{cases} e^{i\bar{P}(\frac{1}{\xi})}\bar{Q}'(\frac{1}{\xi}), & \xi \in E/\bar{K}, \\ e^{-iP(\xi)}Q'(\xi), & \xi \in K, \end{cases} \tag{12}$$

then finding $Q(\xi)$ is equivalent to finding a piecewise analytic function $h(\xi)$ that satisfies the following three conditions:

(i) $h^+(\tau) = \tau^{2-n}h^-(\tau) + f_1(\tau)$, $|\tau| = 1$, here $f_1(\tau)$ satisfies (11);

(ii) $\bar{h}(\frac{1}{\xi}) = h(\xi)$, $(|\xi| \neq 1)$;

(iii) At $\xi = \infty$, $h(\xi)$ has zero of order at least 1.

Lemma 4. *For the piecewise analytic function $h(\xi)$ satisfying the above-mentioned three conditions, consider the following cases:*

I. The case of the homogeneous equation (10), i.e. $f_1(\tau) = 0$.

(i) *if $n \geq -1$, then the problem does not have non-trivial solution;*

(ii) *if $n \leq -2$, then the problem has $|n| - 1$ solutions which are linearly independent over the real field.*

II. The case of the inhomogeneous equation (10).

(i) *if $n \leq 0$, then the problem is solvable without any condition;*

(ii) *if $n \geq 1$, then one has $[\frac{n}{2}] + 1$ solvability conditions of the problem (here $[\frac{n}{2}]$ is the largest integer not larger than $\frac{n}{2}$) :*

$$\int_{|\tau|=1} \tau^k f_1(\tau)\, d\tau = 0, \quad (k = -2, -1, \cdots, [\frac{n}{2}] - 2).$$

Proof: It follows from [5] and condition (i) that

$$h(\xi) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{|\tau|=1} \frac{f_1(\tau)}{\tau - \xi}\, d\tau + R(\xi), & \xi \in E/\bar{K}, \\ \xi^{n-2}\Big(\dfrac{1}{2\pi i}\displaystyle\int_{|\tau|=1} \frac{f_1(\tau)}{\tau - \xi}\, d\tau + R(\xi)\Big), & \xi \in K. \end{cases} \tag{13}$$

If $n \geq 1$, then from the condition (iii), $R(\xi) = 0$ is obtained. If $n \leq 0$, then

$$R(\xi) = \sum_{k=0}^{|n|} C_k \xi^k.$$

If equation (10) is homogeneous, then it follows from condition (ii) that

$$\sum_{k=0}^{|n|} \bar{C}_k \xi^{-k} = \sum_{k=0}^{|n|} C_k \xi^{k-(|n|+2)} = \sum_{j=2}^{|n|+2} C_{|n|+2-j}\xi^{-j},$$

so the homogeneous problem does not have a non-trivial solution when $n = 0$ or $n = -1$. When $n \leq -2$ it has $|n|-1$ solutions which are linearly independent over the real field. $h(\xi)$ can be represented as

$$h(\xi) = \begin{cases} \sum_{k=2}^{|n|} C_k \xi^k, & \xi \in E/\bar{K}, \\ \sum_{k=2}^{|n|} \bar{C}_{|n|+2-k} \xi^{-k}, & \xi \in K; \end{cases}$$

where $C_k = \bar{C}_{|n|+2-k}$ $(k = 2, \cdots, |n|)$.

In the case of the inhomogeneous equation (10), consider (11), then it follows from condition (ii) that

$$\bar{h}(\frac{1}{\bar{\xi}}) = -\frac{1}{2\pi i}\int_{|\tau|=1} \frac{\bar{f}_1(\tau)}{\bar{\tau} - 1/\xi}\, d\bar{\tau} = \frac{\xi}{2\pi i}\int_{|\tau|=1} \frac{\tau^{n-3} f_1(\tau)}{\tau - \xi}\, d\tau$$
$$= -\frac{1}{2\pi i}\int_{|\tau|=1} \tau^{n-3} f_1(\tau)\, d\tau - (\frac{1}{2\pi i})^* \frac{1}{\xi}\int_{|\tau|=1} \tau^{n-2} f_1(\tau)\, d\tau + \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\tau^{n-1} f_1(\tau)}{\xi(\tau - \xi)}\, d\tau,$$

here $n \geq 1$ and $\xi \in K$, now we turn to condition (iii) and obtain

$$\int_{|\tau|=1} \tau^{n-3} f_1(\tau)\, d\tau = \int_{|\tau|=1} \tau^{n-2} f_1(\tau)\, d\tau = 0.$$

Take the conjugate at both sides of the above equality and apply the formula (11); again, it follows that

$$\int_{|\tau|=1} \tau^{-2} f_1(\tau)\, d\tau = \int_{|\tau|=1} \tau^{-1} f_1(\tau)\, d\tau = 0.$$

Observe the represent form of $h(\xi)$ in (12), we obtain

$$\int_{|\tau|=1} \frac{\tau^{n-1} f_1(\tau)}{\xi(\tau - \xi)}\, d\tau = \int_{|\tau|=1} \frac{\xi^{n-2} f_1(\tau)}{\tau - \xi}\, d\tau$$
$$= -\sum_{k=0}^{n-2} \xi^{n-1-k} \int_{|\tau|=1} \tau^k f_1(\tau)\, d\tau + \int_{|\tau|=1} \frac{\tau^{n-1} f_1(\tau)}{\xi(\tau - \xi)}\, d\tau,$$

so

$$\int_{|\tau|=1} \tau^k f_1(\tau)\, d\tau = 0, \quad (k = 0, 1, \cdots, n-2),$$

and if $n = 1$, then the solvable condition is

$$\int_{|\tau|=1} \tau^{-2} f_1(\tau)\, d\tau = \int_{|\tau|=1} \tau^{-1} f_1(\tau)\, d\tau = 0.$$

But it is easy to verify from (11) that

$$\overline{\int_{|\tau|=1} \tau^k f_1(\tau)\, d\tau} = \int_{|\tau|=1} \tau^{n-k-4} f_1(\tau)\, d\tau,$$

so there are only $[\frac{n}{2}]+1$ solvable conditions

$$\int_{|\tau|=1} \tau^k f_1(\tau)\,d\tau = 0, \quad (k=-2,-1,\cdots,[\frac{n}{2}]-2).$$

If $n \le 0$, then it follows from condition (iii) that

$$R(\xi) = \sum_{k=0}^{|n|} C_k \xi^k.$$

Let $\xi \in K$, then condition (ii) gives (taking into account (13))

$$\begin{aligned}\bar{h}(\frac{1}{\bar{\xi}}) &= -\frac{1}{2\pi i}\int_{|\tau|=1} \frac{\bar{f}_1(\tau)}{\bar{\tau}-1/\xi}\,d\bar{\tau} + \sum_{k=0}^{|n|} \bar{C}_k \xi^{-k} \\ &= \frac{\xi}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau^{|n|+3}(\tau-\xi)}\,d\tau + \sum_{k=0}^{|n|} C_k \xi^{k-(|n|+2)} = h(\xi).\end{aligned}$$

Since condition (iii) holds, then

$$\bar{C}_0 = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau^{|n|}+3}\,d\tau = -\overline{\frac{1}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau}\,d\tau},$$

$$\bar{C}_1 = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau^{|n|+2}}\,d\tau = -\overline{\frac{1}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau^2}\,d\tau}.$$

If we take C_k such that

$$C_{|n|+2-k} = \bar{C}_k - \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f_1(\tau)}{\tau^{|n|-k+3}}\,d\tau, \quad (k=2,\cdots,|n|),$$

then the function $h(\xi)$ in the form (13) satisfies the above-mentioned three conditions. This completes the proof of Lemma 4. ∎

After $h(\xi)$ is found we can obtain $Q(\xi)$ from (12) and $Q(\infty)=0$. It is easy to obtain $\psi(z) = Q(\phi^{-1}(z))$, so $\psi(t)$, satisfying (8) on Γ, will be found and solvable conditions (if $n \ge 1$) will be obtained from Lemma 4. The number of linear independent solutions is the same, too. With this $\psi(t)$, we put

$$H(t) = \psi(t) + q(t)\bar{\psi}(t). \tag{14}$$

Now we can prove

Theorem 1. *If $D_2 = \emptyset$; i.e., $D_1 = \bar{G}/\{a_1,\cdots,a_Q\}$, then the equation (1) is solvable if and only if the following conditions hold:*

(i) In the case of $n = \mathrm{Ind}_\Gamma q(t) > 0$, there are $[\frac{n}{2}]+1$ solvability conditions which may be obtained from Lemma 4.

(ii) If the equivalent curve ν of $q(z)$ joins t_1 with t_2 (t_1 is initial point and t_1, $t_2 \in \Gamma$), then the function defined by (14) satisfies

$$H(t_2) - H(t_1) = \int_\nu f(\xi)\,d\bar{\xi} + q(\xi)\bar{f}(\xi)\,d\xi;$$

(iii) $\omega(z) = \frac{\partial}{\partial \bar{z}}\left(\frac{\bar{H}_z(z)-\bar{f}(z)}{\bar{q}_z(z)}\right) \in V(\bar{G})$, *here* $H(z)$ *is defined by (7).*

In the case $n \geq -1$, *the solution is unique; in the case* $n \leq -2$, *the solution depends on* $|n| - 1$ *arbitrary real constants.*

Proof: Necessary conditions already have been obtained from previous discussion. Now we prove that the above conditions are sufficient. In the case of $n = \mathrm{Ind}_\Gamma q(t) > 0$, condition (i) guarantees that we can find a function $\psi(z)$ which is analytic in $E/\bar{G}$ and satisfies (8) on Γ, $\psi(\infty) = 0$. Through $\psi(t)$, the function $H(t)_{t\in\Gamma}$ or $H(z)$ is obtained by (14) or by (7). It can be proved from (7) that conditions (i) and (ii) of Lemma 1 are true. At last condition (iii) of Lemma 1 will be proved from condition (iii) of Theorem 1 and $\overline{(T\omega)(z)} = \big(H_{\bar{z}}(z) - f(z)\big)/q_{\bar{z}}(z)$. Hence, $\omega(z) = (\partial/\partial\bar{z})\big(\bar{H}_z(z) - \bar{f}(z)\big)/\bar{q}_z(z)$ is the general solution of the equation (1). The number of linearly independent solutions follows from Lemma 4. ∎

Now we discuss the case $D_2 \neq \emptyset$. Suppose ν is a connected component of the boundary of D_2. As equivalent curves whose initial point on Γ exists in an arbitrary neighborhood of ν on the side of D_1, we can prove

Lemma 5. *Each connected component of the boundary of* D_2 *is an equivalent curve of* $q(z)$. *The boundary of the arbitrarily connected component of* D_2 *has only finite equivalent curves of* $q(z)$.

The last result of Lemma 5 is obtained due to the fact that the end point of each equivalent curve in G must be $a_1, \cdots, a_Q$ unless it is a closed contour.

Theorem 2. *If* $D_2 \neq \emptyset$ *then the equation (1) is solvable if and only if the following conditions hold:*

(i) All conditions of Theorem 1 are true in D_1.

(ii) For any closed equivalent curve ν *of* $q(z)$

$$\int_\nu f(z)\,d\bar{z} + q(z)\bar{f}(z)\,dz = 0.$$

If equation (1) is solvable then its linear independent solutions are infinite.

Proof: The necessity of condition (i) is obvious. Since $H(z)$ is a single-valued function of z, so the condition (ii) holds.

Sufficiency. If condition (i) holds, then we can decide $H(z)$ in D_1 from the boundary-value $H(t)$ by (7) and obtain $H(z)$ on the boundary of D_2 consequently. In D_2 we can construct a few curves C_k, which intersect all equivalent curves of $q(z)$ at only one point and define a continuous differentiable function $H(t)$ on C_k, this function must satisfy $\bar{H}(t)_{t\in C_k} = \bar{q}(t)H(t)$ and be continuous to the boundary of D_2. Using (8) iterately (here $t \in C_k$), we may construct a single-valued function $H(z)$ which satisfies all conditions of Lemma 1. If there is an equivalent region in D_2, then in that region integral curve ν is an arbitrary one because condition (ii) guarantees that integral in (7) is independent of the form of ν.

In the case that $D_2 \neq \emptyset$ and equation (1) is solvable, we will prove that equation (1) has infinite solutions which are linearly independent over the real field. It is sufficient to prove *that the homo*geneous equation (1) has infinite solutions. If there is an equivalent region in D_2, we may suppose there is a disk $D = \{|z - z_0| < p\} \subset D_2$ and $q(z)_{z\in D} = e^{i\alpha}$ (α is a real constant), define

$$W(z)=\begin{cases} ie^{i\frac{\alpha}{2}}\sum_{k=0}^{n}(|z-z_0|^{2\alpha_k}-p^{2\alpha_k}), & z\in D,\\ 0, & z\in \bar{G}/D,\end{cases}$$

$$\omega(z)=W_{\bar z}(z),$$

here, α_k is an arbitrary real constant larger than 1 and n is an arbitrary positive integer. It is easy to check that $\omega(z)$ is a solution of the equation (1). Suppose there is not any equivalent region in D_2 and D is a connected component of D_2. It is known that the boundary of the connected component D of D_2 has only finite equivalent curves of $q(z)$. Denote these curves by $\nu_1,\cdots,\nu_n$, choose $z_k\in\nu_k$ $(k=1,\cdots,n)$ and define

$$H(z)=\begin{cases} (1+q(z))\sum_{k=1}^{n}(q(z)-q(z_k)+\bar q(z)-\bar q(z_k))^{\alpha_k}\,, & z\in D,\\ 0,\quad z\in\bar G/D; \end{cases}$$

here, α_k is an arbitrary real constant which is larger than 2. It is easy to prove that this $H(z)$ satisfies all four conditions of Lemma 1. Since α_k is arbitrary, we may construct infinite solutions of the homogeneous equation (1) from this $H(z)$ and Lemma 1. ∎

Remark. (i) If the function $f(z)$ and $q(z)$ have stronger conditions, we can derive the representation formula of $W(z)=(T\omega)(z)$ by $q(z)$ and $f(z)$ directly. Actually, it follows that from (4) and (5)

$$H_{\bar z z}(z)=f_z(z)+q_{\bar z z}(z)\bar W(z)+q_{\bar z}\bar W_z(z),$$
$$H_{z\bar z}(z)=(q(z)\bar f(z))_{\bar z}+q_{z\bar z}(z)\bar W(z)+q_z(z)\bar W_{\bar z}(z),$$

so

$$q_{\bar z}(z)\bar W_z(z)-q_z(z)\bar W_{\bar z}(z)=(q(z)\bar f(z))_{\bar z}-f_z(z).$$

If $\nu: z=z(s)$ is an equivalent curve in $\bar G/\{a_1,\cdots,a_Q\}$, then along ν,

$$q_T(z)=q_{\bar z}(z)\bar z'(s)+q_z(z)z'(s)=0,$$

$$q_N(z)=i(q_{\bar z}(z)\bar z'(s)-q_z(z)z'(s))=2iq_{\bar z}(z)\bar z'(s)=-2iq_z(z)z'(s),$$

so along ν :

$$\frac{d\bar W(z(s))}{ds}=\bar W_{\bar z}(z)\bar z'(s)+\bar W_z(z)z'(s)$$
$$=2i\frac{q_{\bar z}(z)\bar W_z(z)-q_z(z)\bar W_{\bar z}(z)}{q_N(z)}=2i\frac{(q(z)\bar f(z))_{\bar z}-f_z(z)}{q_N(z)}.$$

If the point z joins with $t\in\Gamma$ by the equivalent curve ν (here the point t is initial), then we obtain

$$\bar W(z)=\bar\psi(t)+2i\int_\nu\frac{(q(\xi)\bar f(\xi))_{\bar z}-f_z(\xi)}{q_N(\xi)}\,ds. \tag{15}$$

On the contrary, if $W(z)$ is defined by (15), $H(t)$ and $H(z)$ is defined by (14) and (3) separately, and it can be proved that the function $H(z)$ satisfies all four conditions of Lemma 1 and we can obtain a solution of the equation (1).

(ii) For the conjugate equation of the equation (1)

$$\omega^*(z) - \frac{1}{\pi}\int_G\int \frac{\bar{q}(\xi)\bar{\omega}^*(\xi)}{(\xi - z)^2}\, d\sigma_\xi = g(z),$$

if we multiply by $q(z)$ both sides of this equation and define

$$\omega(z) = \bar{q}(z)\bar{\omega}^*(z), \quad f(z) = \bar{q}(z)\bar{g}(z),$$

then it can be transformed to the same form as the equation (1),

$$\omega(z) - \frac{\bar{q}(z)}{\pi}\int_G\int \frac{\bar{\omega}(\xi)}{(\bar{\xi} - \bar{z})^2}\, d\sigma_\xi = f(z),$$

and we need not give further detailed discussion.

Example. Suppose $G = \{|z| < 1\}$, $\Gamma = \{|t| = 1\}$, $q(z) = \frac{\bar{z}}{|z|}$, $f(z) = \frac{3}{2}|z| + \frac{z}{2}$. The equivalent curve of $q(z)$ is

$$z(s) = se^{i\theta} \ (1 \geq s \geq r,\ \theta \text{ is a real constant}).$$

It is easy to obtain $\psi(t) = \frac{1}{t}$, so there is $\bar{\psi}(t) = \frac{z}{|z|}$ on this equivalent curve of $q(z)$. Notice $(q(z)\bar{f}(z))_{\bar{z}} - f_z(z) = 1$ and $q_N(z) = i(\bar{z}'(s)q_{\bar{z}}(z) - z'(s)q_z(s)) = \frac{i}{z}$ on this equivalent curve. It follows from (15) that $\bar{W}(z) = z|z|$, so $\omega(z) = \frac{3}{2}|z|$ is the solution of equation (1).

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