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COMPACT EVOLUTION OPERATORS

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(Submitted by G. Webb)

In this paper, we give a characterization of compactness of the Evolution Operator $U(t,s)$ generated by a family of nonlinear (possibly multivalued) operators $\{A(t), 0 \le t \le T\}$ of dissipative type. This is an extension of a result of Brezis [2] on the compactness of the semigroup $S_A(t)$ generated by an m-dissipative operator A via the exponential formula of Crandall-Liggett [3].

1.Preliminaries. Statements of the results. Let X be a real Banach space of norm $\| \ \|$. Recall some notations and definitions (for details we refer to [6], [7]).

$$\langle y,x\rangle_{\bar{s}} = \lim_{h\downarrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}\,, \quad \langle y,x\rangle_{+} = \lim_{h\downarrow 0} \frac{\|x+hy\| - \|x\|}{h}\,, \tag{1.1}$$

$$\langle y,x\rangle_{i} = \lim_{h\downarrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}\,, \quad \langle y,x\rangle_{-} = \lim_{h\downarrow 0} \frac{\|x+hy\| - \|x\|}{h}\,, \tag{1.2}$$

where x and y are elements of X. Clearly,

$$\langle y,x\rangle_{\bar{s}} = \|x\|\langle y,x\rangle_{+}\,;\,\langle y,x\rangle_{i} = \|x\|\langle y,x\rangle_{-}\,,\,\langle y,x\rangle_{+} = \frac{\|x+hy\| - \|x\|}{h}\,, \quad h>0.$$

It is known that

$$\langle y,x\rangle_{\bar{s}} = \sup\{x^*(y)\,;\,x^* \in J(x)\}\,, \quad \langle y,x\rangle_{i} = \inf\{x^*(y)\,;\,x^* \in J(x)\}\,, \tag{1.3}$$

where J is the duality mapping of X.

Now, let $\{A(t)\,;\,0 \le t \le T\}$ be a family of nonlinear (possibly multivalued) operators $A(t) : D(A(t)) \subset X \to 2^X$ satisfying the hypotheses

(C1) $R(I - hA(t)) = X$, for all $h > 0$ and $t \in [0,T]$

(C2) There are two continuous functions $f : [0,T] \to X$ and $L : R_+ \to R_+$ such that

$$\langle y_1 - y_2\,,\,x_1 - x_2\rangle_i \le \|f(t) - f(s)\|\,\|x_1 - x_2\|L(\max\{\|x_1\|\,,\,\|x_2\|\})$$

for all $0 \le s \le t \le T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$

(C3) If $t_n \uparrow t$, $x_n \in D(A(t_n))$ and $x_n \to x$, then $x \in \overline{D(A(t))}$.

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A few comments on the above conditions: (C1) is the well-known "Range Condition". It can be replaced by (a more general condition)

(C1)′ $R(I - hA(t+h)) \supset \overline{D(A(t))}\,, \quad 0 < h < T - t\,,\, t \in [0,T]$
where I is the identity on X (as in (C1)).

Another useful condition which is, in its turn, strictly more general than (C1)′, is the following tangential condition

(C1)″ $\lim_{h\downarrow 0} h^{-1} d[x; R(I - hA(t+h))] = 0\,, \quad x \in \overline{D(A(t))}\,, \quad s \le t < T$
which has been considered by the author in [5]. It is interesting to note that (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0$$

can be reduced to an abstract differential equation

$$u'(t) = A(t)u(t)\,, \quad u(s) = x_0 \in \overline{D(A(s))} \tag{1.4}$$

with some $A(t)$ satisfying tangential condition (C1)″ and (C2) but not the range condition (C1)′. This fact is carried out by Oharu and Takahashi [4]. We have also to note that (C1) and (C2) imply $\overline{D(A(t))}$ is independent of t ([7], Ch.1, Remark 4.2) while (C1)″ and (C2) do not ([7] Ch.3, Section 2). Obviously, for $t = s$, (C2) implies the dissipativity of each $A(t)$. Therefore (C1) and (C2) imply the m-dissipativity of each $A(t)$ so $\overline{D(A(t))}$ is independent of t. This is the case in which we are working here. Recall that if $A(t)$ satisfies (C1)–(C3), then for every

$$0 \le s \le t \le T\,, \quad x_0 \in \overline{D(A(s))} \quad \text{and} \quad x_0^n \in D(A(s)) \quad \text{with} \quad x_0^n \to x_0\,,$$

$$\lim_{n\to\infty} \prod_{j=1}^{n} \left(I - \frac{t-s}{n} A\big(s + j\,\frac{t-s}{n}\big)\right)^{-1} x_0^n \equiv U(t,s)x_0 \in \overline{D(A(t))} \tag{1.5}$$

exists and $U(t,s)$ is an evolution operator in the following sense:

1°) For each pair (s,t) with $0 \le s \le t \le T$, $U(t,s) : \overline{D(A(s))} \to \overline{D(A(t))}$; $U(s,s) = I$

2°) $U(t,s)U(s,r) = U(t,r)$, for every $0 \le r \le s \le t \le T$

3°) The function $(t,s,x) \to U(t,s)x$ is continuous.

In our case here, the dissipativity of $A(t)$ implies the nonexpansivity of $x \to U(t,s)x$; i.e.,

4°) $\|U(t,s)x - U(t,s)y\| \le \|x - y\|\,, \quad 0 \le s < t\,, \quad x, y \in \overline{D(A(s))}$.

The importance of $U(t,s)$ consists in the fact that the function $u(t) = U(t,s)x_0$ is the unique integral solution to the problem (1.4) ([5]). Moreover, if u is differentiable at t, then $U(t,s)x_0 \in D(A(t))$ and $\frac{d}{dt}U(t,s)x_0 \in A(t)U(t,s)x_0$, i.e., if $t \to U(t,s)$ is a.e. differentiable, then (1.4) is satisfied a.e. on $[0,T]$.

$U(t,s)$ is said to be compact (with $0 \le s < t \le T$) if it maps bounded subsets of $\overline{D(A(s))}$ into precompact sets of $\overline{D(A(t))}$.

The generators $\{A(t)\,;\, 0 \le t \le T\}$ of compact evolution operators (via (1.5)) are important in the study of perturbed differential equation

$$u'(t) \in A(t)u(t) + F(t, u(t))\,, \quad u(s) = x_0 \in \overline{D(A(s))} \tag{1.6}$$

where the function $(t,u) \to F(t,u)$ (with $u \in \overline{D(A(t))}$) is merely continuous and the domain of $F(t,\cdot)$ is $\overline{D(A(t))}$ (see [7, Ch.3] and [8]). We can define the notion of "evolution operator" independently of its generators $\{A(t)\}$, i.e., without Formula (1.5), as follows:

Let $\{D(t)\,;\quad 0 \le t \le T\}$ be a family of nonempty subsets of X. $U(t,s)$ is said to be an evolution operator with respect to $\{D(t)\}$ if Properties $1^o) - 3^o)$ holds with $D(t)$ in place of $\overline{D(A(t))}$ for each $t \in [0,T]$.

In order to arrive at the definition of integral solution to (1.4), let us suppose that u is a strong solution of the Cauchy problem (1.4) (i.e. $t \to u(t)$ is absolutely continuous on $[s,T]$, almost everywhere differentiable, $u(t) \in \overline{D(A(t))}$ and (1.4) is almost everywhere satisfied). Take an arbitrary $r \in [s,T]$,
$x \in D(A(r))$ and $y \in A(r)x$ (in short, $[x,y] \in A(r)$). There exists $x^* \in J(u(t)-x)$ such that

$$x^*(u'(t)-y) = \langle u'(t)-y, u(t)-x\rangle_i. \tag{1.7}$$

On the other hand (see e.g. [6], p.16),

$$x^*(u'(t)) = x^*(u(t)-x)') = \|u(t)-x\| \frac{d}{dt}\|u(t)-x\|, \text{ a.e. on } [s,T] \tag{1.7$'$}$$

On the basis of (1.7), (1.7)$'$ and C(2), from (1.4) we derive

$$\begin{aligned} \|u(t)-x\| \frac{d}{dt}\|u(t)-x\| &= x^*(A(t)u(t)) = x^*\big(A(t)u(t)-y\big) + x^*(y) \le \\ \|f(t)-f(r)\|\,\|u(t)-x\|\,L\,(\max\{\|u(t)\|,\, \|x\|\}) &+ \langle y, u(t)-x\rangle_{\bar{s}}\,, \quad \text{a.e. on } [s,T] \end{aligned} \tag{1.8}$$

We may assume that

$$L\,(\max\{\|u(t)\|,\, \|x\|\}) \le L\big(\max\{\sup_t \|u(t)\|,\, \|x\|\}\big) = C\,(\|x\|,\, \|x_0\|,\, s) \tag{1.8$'$}$$

and that C is bounded on bounded subsets containing x and x_0. Therefore, dividing the inequality (1.8) by $\|u(t)-x\|$ and then integrating over $[h,t]$ we get

$$\|u(t)-x\| \le \|u(h)-x\| + \int_h^t (\langle y, u(\tau)-x\rangle_+ + C(\|x\|,\, \|x_0\|)\,\|f(\tau)-f(r)\|)\,d\tau \tag{1.9}$$

for all $0 \le s \le h \le t \le T$, $r \in [s,T]$ and $[x,y] \in A(r)$, with C as in (1.8)$'$.

It is now reasonable to give

Definition 1.1. *A function $u : [s,T] \to X$ is said to be an integral solution to the problem* (1.4) *if:*

1) *u is continuous on $[s,T]$,*
2) *$u(s) = x_0$, $u(t) \in \overline{D(A(t))}$ and*
3) *u satisfies the inequality* (1.9) *with $C(\|x\|,\, \|x_0\|,\, s)$ as defined by* (1.8)$'$.

Definition 1.1 was first given in [5]. In the case $A(t) = A$ - independent of t, it is just the notion of integral solution in the sense of Benilan [1]. The fact that $u(t) = U(t,s)x_0$ given by (1.5) is the unique integral solution to (1.4) is proved in [5] and [7].

Since (C1) and (C2) imply the closure of $D(A(t))$ independent of t, set

$$\overline{D(A(t))} = \overline{D(A(0))} = \overline{D}\,, \quad t \in [0,T].$$

We are now in a position to state the main result of this paper.

Theorem 1.1. *Suppose that* (C1)–(C3) *are fulfilled. Let* $U(t,s)$ *be the evolution operator generated by* $\{A(t)\}$ *via the formula* (1.5). *Then* $U(t,s)$ *is compact (for every* $0 \le s < t \le T$*) if and only if the following two conditions below hold:*

(I) *For each* $s \in [0,T]$ *and* $\lambda > 0$ *the operator* $J_\lambda(s) = (I - \lambda A(s))^{-1}$ *from* X *into* $D(A(s))$ *is compact.*

(II) *For each* $t_0 \in]s,T]$, *the functions* $\{t \to U(t,s)x,\ x \in Y\}$ *are equicontinuous at* t_0 *on bounded subsets* $Y \subset \overline{D(A(s))}$, *i.e., for every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, t_0, s, Y) > 0$ *such that* $\|U(t,s)y - U(t_0,s)y\| < \varepsilon$, $\forall t \in [t_0 - \delta,\ t_0 + \delta]$, $y \in Y$.

In short, in place of (II) we shall say "$U(t,s)$ is equicontinuous at $t_0 > s$, on bounded subsets of $\overline{D(A(s))}$". Condition $t_0 > s$ is essential (in (II)). In the autonomous case $A(t) = A$, independent of t, Theorem 1.1 reduces to

Theorem Brezis'. ([2] or [7, p.89). *Let* $S(t)$ *be the semigroup generated by* A *via the exponential formula of* Crandall-Liggett. *Then* $S(t)$ *is compact for every* $t > 0$ *if and only if*

1^0) $(I - A)^{-1}$ *is compact and*

2^0) *for every bounded subset* Y *of* $\overline{D(A)}$, *the functions* $\{t \to S(t)x,\ x \in Y\}$ *are equicontinuous at every* $t_0 > 0$.

In the proof of (sufficiency part) of Theorem 1.1, the result below is needed

Lemma 1.1. *Suppose that the hypotheses of Theorem 1.1 are fulfilled. Then,*

$$\begin{aligned}\|J_\lambda(s)x - x\| \le \frac{2}{t-s}\int_s^t \|U(\tau,s)s - s\|\,d\tau + \frac{\lambda}{t-s}\|U(t,s)x - x\| \\ + \frac{\lambda C(\|x\|)}{t-s}\int_s^t \|f(\tau) - f(s)\|\,d\tau\end{aligned} \tag{1.10}$$

for all $\lambda > 0$, $0 \le s \le t \le T$, $x \in \overline{D}$, *where* $C = C(\|x\|,\ s)$ *is bounded on* x*- bounded sets.*

2. Proof of the results.

The property of $U(t,s)$ which plays an essential role in (1.10) is that $t \to U(t,s)x_0$ is the (unique) integral solution to (1.4), i.e., (1.9) holds with $u(t) = U(t,s)x_0$ [5], [7]. Therefore, with $r = s$, (1.9) yields

$$\|U(t,s)x_0 - x\| \le \|x_0 - x\| + \int_s^t (\langle y, U(\tau,s)x_0 - x\rangle_+ + C\|f(\tau) - f(s)\|)\,d\tau \tag{2.1}$$

for all $x \in D(A(s))$, $y \in A(s)x$ and $x_0 \in \overline{D}$, with $C = C(\|x\|, \|x_0\|, s)$ bounded on bounded x-subsets.

Proof of Lemma 1.1: We first recall that

$$\langle y, U(t,s)x_0 - x\rangle_+ \le \lambda^{-1}(\|U(t,s)x_0 - x + \lambda y\| - \|U(t,s)x_0 - x\|), \lambda > 0$$

Replacing in (2.1) x by $J_\lambda(s)x_0$ and $y = A_\lambda(s)x_0 \in AJ_\lambda(s)x_0$ and taking into account $J_\lambda(s)x_0 - \lambda A_\lambda(s)x_0 = x_0$, one obtains

$$\begin{aligned}\|U(t,s)x_0 - J_\lambda(s)x_0\| \le \|J_\lambda(s)x_0 - x_0\| + \overline{C}(\|J_\lambda(s)x_0\|)\int_s^t \|f(\tau) - f(s)\|\,d\tau \\ + \int_s^t \lambda^{-1}(\|U(\tau,s)x_0 - x_0\| - \|U(\tau,s)x_0 - J_\lambda(s)x_0\|)\,d\tau,\end{aligned} \tag{2.2}$$

with $\overline{C}(\|x\|) = C(\|x\|, \|x_0\|, s)$

It is now an elementary exercise to check that (2.2), along with

$$\|J_\lambda(s)x_0 - x_0\| \le \|J_\lambda(s)x_0 - U(t,s)x_0\| + \|U(t,s)x_0 - x_0\|$$

lead us to (1.10). The fact that $\overline{C}$ in (2.2) implies the existence of a C as required in (1.10) follows from the inequality

$$\begin{aligned}\|J_\lambda(s)x_0\| &\le \|J_\lambda(s)x_0 - J_\lambda(s)\tilde{x}_0\| + \|J_\lambda(s)\tilde{x}_0 - \tilde{x}_0\| + \|\tilde{x}_0\| \\ &\le \|x_0 - \tilde{x}_0\| + \|\tilde{x}_0\| + \lambda|A(s)\tilde{x}_0|,\end{aligned}$$

where $\tilde{x}_0 \in D(A(s))$ and $|A(s)\tilde{x}_0| = \inf\{\|z\|, z \in A(s)x_0\}$. This completes the proof. ∎

The result given below is independent of the generators of $U(t,s)$.

Proposition 2.1. *Let $U(t,s)$ be an evolution operator with respect to the family $\{D(t),\ 0 \le t \le T\}$ in the sense above. If $U(t,s)$ is compact from $D(s)$ into $D(t)$ (for every $0 \le s < t \le T$), then $U(t,s)$ is equicontinuous at every $t_0 \in [s,T]$ on bounded subsets of $D(s)$ in the sense of Condition (II) in Theorem 1.1.*

Proof: Let $\varepsilon > 0$, $s < t_0 \le T$ and let Y be a bounded subset of $D(s)$. If $s < \bar{t} < t_0$, then by hypothesis $U(\bar{t},s)Y$ is a precompact subset of $D(t)$. Since $(t,z) \to U(t,\bar{t})z$ is continuous on $[\bar{t},T] \times \overline{U(t,s)Y}$, it is uniformly continuous on this set, so there is $\delta = \delta(\varepsilon,t,s,Y) > 0$ such that

$$\|U(t,\bar{t})z - U(t_0,\bar{t})z\| < \varepsilon, \quad |t - t_0| < \delta, \quad z \in U(\bar{t},s)Y \tag{2.3}$$

Clearly, $z = U(\bar{t},s)y$ with $y \in Y$ and $U(t,\bar{t})U(\bar{t},s)y = U(t,s)y$ for $s < \bar{t} < t \le T$ and thus the proof is complete. ∎

In what follows, the following elementary (classical) result is needed:

Lemma 2.1. *Let K be a nonempty subset of the Banach space X. If, for every $\varepsilon > 0$, there is a precompact subset K_ε with the property that for each $x \in K$ there is $x_\varepsilon \in K_\varepsilon$ such that $\|x - x_\varepsilon\| < \varepsilon$, then K is precompact.*

Proposition 2.2. *Let $U(t,s)$ be as indicated by Theorem 1.1. If $U(t,s)$ is compact (for every $0 \le s < t \le T$), then for every $s \in [0,T]$ and $\lambda > 0$, $J_\lambda(s) = (I - \lambda A(s))^{-1} : X \to D(A(s))$ is compact.*

Proof: The key to the proof is the following relationship between $U(t,s)$ and its generators $A(t)$ (cf. [7], Prop. 5.2, Ch.1)

$$\|U(t,s)x - x\| \le (t-s)(\|z\| + M(\|x\|, s)), \quad x \in D(A(s)), \quad z \in A(s)x \tag{2.4}$$

with $x \to M(\|x\|, s)$ bounded on bounded subsets of $D(A(s))$.

Now let Y be a bounded subset of X. Then, $J_\lambda(s)Y$ is a bounded subset of $D(A(s))$ (this is because $y \to J_\lambda(s)y$ is Lipschitz continuous with Lipschitz constant 1). Substituting $x = J_\lambda(s)y$ and $z = A_\lambda(s)y \in A(s)J_\lambda(s)y$ in (2.4), we get

$$\|U(t,s)J_\lambda(s)y - J_\lambda(s)y\| \le (t-s)(\|A_\lambda(x)y\| + M(\|J_\lambda(s)y\|, s), \quad \forall y \in Y. \tag{2.5}$$

Let ε be an arbitrary positive number. Since $A_\lambda(s)y = \lambda^{-1}(J_\lambda(s)y - y)$ is also bounded with respect to $y \in Y$, it follows from (2.5) that there is a $d = d(\varepsilon,\lambda,s,Y) \equiv d(\varepsilon) > 0$, such that

$$\|U(s+d,s)J_\lambda(s)y - J_\lambda(s)y\| \le \varepsilon, \quad \forall y \in Y. \tag{2.6}$$

Finally, the precompactness of $K_\varepsilon = U(s+d(\varepsilon), s)J_\lambda(s)Y$, Lemma 2.1 and inequality (2.6) lead to the precompactness of $J_\lambda(s)Y$. ∎

Proof of Theorem 1.1 (Sufficiency part). It has remained to prove that Conditions (I) and (II) imply the compactness of $U(t,s)$. Therefore, let $0 \le r < s < t \le T$ and let Y be a bounded subset of $\overline{D}$. We will prove that $U(s,r)Y$ is precompact. To this goal, replace x in (1.10) by $U(s,r)y$ with $y \in Y$. Thus, for all $t \in [s,T]$ we have

$$\begin{aligned} \|J_\lambda(s)U(s,r)y - U(s,r)y\| &\le \frac{2}{t-s}\int_s^t \|U(\tau,r)y - U(s,r)y\|\,d\tau \\ &+ \frac{\lambda}{t-s}\|U(t,r)y - U(s,r)y\| + \frac{\lambda C}{t-s}\int_s^t \|f(\tau)-f(s)\|\,d\tau, \quad y \in Y \end{aligned} \tag{2.7}$$

where $C = C(s,r,Y) > 0$ can be chosen independent of $y \in Y$.

On the basis of Condition(II), for every $\varepsilon > 0$, there is $\delta = d(\varepsilon, s, r, Y) > 0$ such that

$$\|U(\tau,r)y - U(s,r)y\| < \varepsilon \quad \text{for all } \tau \in [r,T] \text{ with } |\tau - s| < \delta \text{ and } y \in Y. \tag{2.8}$$

Combining (2.7) and (2.8), we conclude that there exists $\lambda_\varepsilon = \lambda(\varepsilon, r, s, Y)$ such that

$$\|J_{\lambda_\varepsilon}(s)U(s,r)y - U(s,r)y\| \le M\varepsilon \tag{2.9}$$

where M is a constant independent of $y \in Y$. According to Lemma 2.1 with $K = U(s,r)Y$ and $K_\varepsilon = J_{\lambda_\varepsilon}(s)U(s,r)Y$, we get the precompactness of $U(s,r)$. ∎

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