Differential and Integral Equations, Volume 1, Number 1, January 1988, pp. 117–123.

DIFFERENTIAL EQUATIONS ASSOCIATED WITH COMPACT EVOLUTION GENERATORS

N.H. PAVEL
Department of Mathematics, University of Texas at Arlington
Arlington, Texas 76019, USA

(Submitted by C. Corduneanu)

Abstract. Let $U(t,s)$ be the evolution operator generated by a family of nonlinear, possibly multivalued operators $\{A(t),\ 0 \le t \le T\}$ of dissipative type acting in a Banach space X. We prove that if $x \to U(t,s)x,\ s < t \le T$ is compact and $F(t) : \overline{D(A(t))} \to X$ is continuous then the *Cauchy* Problem $u' \in A(t)u + F(t)u,\ u(s) = x_0 \in \overline{D(A(s))}$ has at least an integral solution. One extends the results of Pazy [12] and Vrabie [13] as well as the classical result on the behaviour of the solution as $t \uparrow t_{\max}$.

1. Statement of main results. Let X be a real space of norm $\|\cdot\|$. Denote by J the duality mapping of X and set

$$\begin{aligned}\langle y,x\rangle_i &= \inf\{x^*(y);\ x^* \in J(x)\};\ \langle y,x\rangle_- = \langle y,x\rangle_i \|x\|^{-1}\\ \langle y,x\rangle_{\bar s} &= \sup\{x^*(y);\ x^* \in J(x)\};\ \langle y,x\rangle_+ = \langle y,x\rangle_{\bar s} \|x\|^{-1}\end{aligned} \tag{1.1}$$

where $y,\ x \in X,\ x \ne 0$.

We shall be concerned with the abstract differential equation (inclusion)

$$u'(t) \in A(t)u(t) + F(t)u(t),\ u(s) = x_0 \in \overline{D(A(s))},\ 0 \le t < T \tag{1.2}$$

for some $T > 0$. The hypotheses on $\{A(t)\}$: For each $t \in [0,T]$, $A(t) : D(A(t)) \subset X \to 2^X$ satisfies the range condition

(C1) $R(I - hA(t)) = X$, for all $h > 0$ and $t \in [0,T]$ where I is the identity on X.

(C2) There are two continuous functions $f : [0,T] \to X$ and $L : [0, +\infty[\to [0, +\infty[$ such that:

$$\langle y_1 - y_2\,,\ x_1 - x_2\rangle_i \le \|f(t) - f(s)\|\, \|x_1 - x_2\|\, L\Big(\max\{\|x_1\|\,,\ \|x_2\|\}\Big)$$

for all $0 \le s \le t \le T$, $[x_1,\ y_1] \in A(t)$ and $[x_2,\ y_2] \in A(s)$.

(C3) If $t_n \uparrow t$, $x_n \in D(A(t_n))$ and $x_n \to x$, then $x \in \overline{D(A(t))}$ $\big(t_n,\ t \in [0,T]\big)$.

(C4) The evolution operator $U(t,s)$ generated by $\{A(t),\ 0 \le t \le T\}$ is compact (i.e. for every $0 \le s < t \le T$, the operator $x \to U(t,s)x$ maps bounded subsets of $\overline{D(A(s))}$ into relatively compact (precompact) subsets of X).

Received August 24, 1987.
AMS(MOS) Subject Classifications: 35K22, 34G20.

Actually, Condition (C1) can be replaced by a more general condition, namely

(C1.1) $R\big(I - hA(t+h)\big) \supset \overline{D(A(t))},\ 0 < h < T - t,\ t \in [0, T[$.

A strictly more general condition than (C1.1) is the following tangential condition

(C1.2) $\lim_{h \downarrow 0} h^{-1}\, d[x;\ R\big(I - hA(t+h)\big)] = 0,\ x \in \overline{D(A(t))},\ 0 < t < T$

which has been considered by the author in [7] as an extention of Kobayaski's condition from the autonomous case. An application of this condition to some partial differential equations can be found in the recent paper by Oharu and Takahashi [6].

Recall that if $\{A(t)\}$ satisfies the conditions (C1.2), (C2) and (C3) then $\{A(t)\}$ generates an evolution operator $U(t,s)$ in the following sense [Pavel [7]],

1°) $U(t,s) : \overline{D(A(s))} \to \overline{D(A(t))},\ U(s,s) = I,\ 0 \le s \le t \le T$

2°) $U(t,s)U(s,r) = U(t,r)$, for every $0 \le r \le s \le t \le T$

3°) The function $(t,s,x) \to U(t,s)x$ (with $t \ge s$) is continuous

4°) $x \to U(t,s)x$ is nonexpansive, i.e.

$$\|U(t,s)x - U(t,s)y\| \le \|x - y\|,\ 0 \le s \le t,\ x, y \in \overline{(A(s))}$$

The function $t \to U(t,s)x_0$ is an integral solution to the problem $v' \in A(t)v,\ v(s) = x_0,\ 0 \le s \le t$ and so on [10]. Note that (C2) contains some of the well-known conditions of Kato [4] and Crandall-Pazy [3]. For example, let us recall a condition of Kato's type: $A(t)$ is a single valued, dissipative operator with $D(A(t)) = D$ – independent of t and

$$\|A(t)x - A(s)x\| \le \|f(t) - f(s)\|\, L(\|x\|),\ \forall x \in D,\ t, s \in [0,T] \tag{1.3}$$

It is clear that (1.3) implies (C2). Indeed, let $x_1, x_2 \in D$ with $\|x_2\| \le \|x_1\|$. Then

$$\begin{aligned}\langle A(t)x_1 - A(s)x_2,\ x_1 - x_2\rangle_i &= \langle A(t)x_1 - A(s)x_1 + A(t)x_1 - A(s)x_2,\ x_1 - x_2\rangle_i \\ &\le \langle A(t)x_1 - A(s)x_1,\ x_1 - x_2\rangle_i \\ &\le \|A(t)x_1 - A(s)x_1\|\,\|x_1 - x_2\| \\ &\le \|f(t) - f(s)\|\,\|x_1 - x_2\|\, L(\|x_1\|)\end{aligned}$$

The hypotheses on $F(t)$:

(H1) The domain $D(F(t)) = \overline{D(A(t))},\ 0 \le t \le T$

(H2) $(t,x) \to F(t)x$ is continuous, (i.e., if $t_n \to t$ and $x_n \in D(F(t_n))$ is convergent to x then $x \in D(F(t))$ and $F(t_n)x_n \to F(t)x$

In what follows the notion of integral solution is needed.

Let $g : [0,T] \to X$ be a (continuous) function and $s \in [0,T[,\ s \le T_1 \le T$.

Definition 1.1. *A function $u : [s, T_1] \to X$ is said to be an integral solution to the problem*

$$u' \in A(t)u + g(t),\ u(s) = x_0 \in \overline{D(A(s))},\ s \le t \le T_1 \tag{1.4}$$

on $[s,T_1]$ if u is continuous on $[s,T_1]$, $u(t) \in \overline{D(A(t))}\ \forall t \in [s,T_1]$, $u(s) = x_0$ and the following inequality is satisfied

$$\|u(t) - x\| \le \|u(t_0) - x\| + \int_{t_0}^{t} \Big(\langle y + g(t),\ u(\tau) - x\rangle_+ + C\|f(\tau) - f(r)\|\Big)\, d\tau \tag{1.5}$$

for all $s \le t_0 \le t \le T_1$, $r \in [s, T_1]$, $x \in D(A(r))$ *and* $y \in A(r)x$, *with* $C = L\big(\max\{\|u\|, \|x\|\}\big)$, *where* $\|u\| = \sup_{s \le t \le T_1} \|u(t)\|$.

The existence (and the uniqueness) of the solution to (1.4) is known [7]; moreover, if v is the integral solution to the problem

$$v'(t) \in A(t)v + g_1(t), \; v(s) = y_0 \in \overline{D(A(s))} \tag{1.6}$$

on $[s, T_2]$, then

$$\|u(t) - v(t)\| \le \|u(\tau) - v(\tau)\| + \int_\tau^t \|g(\theta) - g_1(\theta)\| \, d\theta \tag{1.7}$$

for all $x \le \tau \le t \le T_0$, $T_0 = \min\{T_1, T_2\}$.

We are now in a position to state the main result of this paper:

Theorem 1.1. *Assume that* $\{A(t)\}$ *satisfies the hypotheses (C1.2), (C2), (C3) and (C4) and that (H1)–(H2) are fulfilled. Then for every* $x_0 \in \overline{D(A(s))}$, *there exist* $T_1(s, x_0) \in [s, T[$ *and a continuous function* $u : [s, T_1] \to X$, *such that:* $u(s) = x_0$, $u(t) \in \overline{D(A(t))}$ *and*

$$\|u(t) - x\| \le \|u(\tau) - x\| + \int_\tau^t \Big(\langle y + F(\tau)u(\tau), u(\tau) - x\rangle_+ + C\|f(\tau) - f(r)\|\Big) \, d\tau \tag{1.8}$$

for all $s \le \tau \le t \le T_1$, $r \in [s, T_1]$, $x \in \overline{D(A(r))}$ *and* $y \in A(r)x$, *with* $C = L\big(\max\{\|u\|, \|x\|\}\big)$, *where* $\|u\| = \sup\{\|u(t)\|,\ t \in [s, T_1]\}$ *and* L *is as in (C2) (i.e., (1.2) has at least a local integral solution).*

In connection with the behaviour of u as $t \uparrow t_{\max}$ the following result (of classical nature) holds:

Theorem 1.2. *Let* $\{A(t)\}$ *satisfy Conditions (C2), (C3), and (C1.2) on* $[0, +\infty[$ *(in place of* $[0, T]$*). Suppose that* F *satisfies Hypotheses (H1) and (H2) on* $[0, +\infty]$ *and the problem (1.2) has a local integral solution for every* $s \ge 0$ *and* $x_0 \in \overline{D(A(s))}$. *Then the problem (1.2) (i.e., (1.8)) admits a maximal integral solution* $u : [s, t_{\max}[\to X$. *If* F *maps bounded subsets into bounded subsets, then either* $1^\circ)$ $t_{\max} = +\infty$ *or* $2^\circ)$ $t_{\max} < +\infty$ *and* $\lim_{t \uparrow t_{\max}} \|u(t)\| = +\infty$.

Note that (C1) and (C2) imply that $\overline{D(A(t))}$ is independent of t, while (C1.2) and (C2) do not ([6], [7]). A characterization of compactness of $U(t, s)$ is given in [11].

In some previous papers ([12], [13]) the property $2^\circ)$ in Theorem 1.2 was obtained in a weaker form, i.e. $\limsup_{t \uparrow t_m} \|u(t)\| = +\infty$ (in the autonomous case $A(t) = A$).

2. Proof of the results. First of all we note that for proving Theorem 1.1 we cannot apply the standard fixed point theorems. This is because $\overline{D(A(t))}$ is not convex in general. Fortunately, the method of steps (as in the theory of differential equations with delay) can be applied to our goal here, as below.

Proof of Theorem 2.1: The key of the proof is to show that for each $\lambda > 0$ sufficiently small, there exists an integral solution $u_\lambda : [s - \lambda, s + T_0]$ of the delay equation

$$u'_\lambda(t) \in A(t)u_\lambda(t) + F(t - \lambda)u_\lambda(t - \lambda), \; u_\lambda(\theta) = x_0 \quad \text{for } s - \lambda \le \theta \le s. \tag{2.1}$$

In other words, we shall prove that for each $\lambda \in]0, T]$ there exists a continuous function $u_\lambda : [s-\lambda,\, s+T_0]$, with the properties $u_\lambda(t) \in \overline{D(A(t))}$, $u_\lambda(\theta) = x_0$ for $\theta \in [s-\lambda,\, s]$ and

$$\|u_\lambda(t) - x\| \le \|u_\lambda(t_0) - x\| + \int_{t_0}^{t} \Big(\langle y + F(\tau-\lambda)u_\lambda(\tau-\lambda),\, u_\lambda(\tau-\lambda)\rangle_+ + C\|f(\tau) - f(r)\| \Big)\, d\tau \tag{2.2}$$

for all $0 \le s \le t_0 \le t \le s+T_0$, $r \in [s,\, s+T_0]$ and $[x,y] \in A(r)$ where $C = L\big(\max\{\|x\|,\, \rho + \|x_0\|\}\big)$ with ρ and T_0 as follows. The continuity of F at (s, x_0) guarantees the existence of some positive constants ρ, M and T_0 such that

$$\|F(t)z\| \le M, \quad \text{for } |t-s| \le T_0,\ \|z - x_0\| \le \rho,\ z \in \overline{D(A(t))} \tag{2.3}$$

Moreover, we may choose $T_0 < T - s$ sufficiently small in order to satisfy condition

$$MT_0 + \sup_{t-s \le T_0} \|U(t,s)x_0 - x_0\| \le \rho \tag{2.4}$$

Set $T_1 = s+T_0$. Clearly, T_0 and T_1 depend on s and x_0. Now set $g(t) = F(t-\lambda)x_0$ and denote by u_λ^1 an integral solution of $u' \in A(t)u + g(t)$, $u(s) = x_0$, on $[s,\, s_\lambda)$. Define $u_\lambda^1(\theta) = x_0$ for $s - \lambda \le \theta \le s$. Therefore $g(t) = F(t-\lambda)x_0 = F(t-\lambda)\, u_\lambda^1(t-\lambda)$ on $[s,\, s+\lambda]$ and u_λ^1 is defined on $[s-\lambda,\, s+\lambda]$. Set

$$v(t) = U(t,s)x_0, \quad s \le t \le T \tag{2.5}$$

Since v is an integral solution of

$$v' \in A(t)v, \quad v(s) = x_0, \quad s \le t \le T$$

the inequality (1.7) yields

$$\begin{aligned} \|u_\lambda^1(t) - v(t)\| &\le \int_s^t \|F(\tau-\lambda)x_0\|\, d\tau \\ &\le M\lambda \\ &\le MT_0 \end{aligned} \tag{2.6}$$

for $s \le t \le s+\lambda \le s+T_0 = T_1$. We now have

$$\begin{aligned} \|u_\lambda^1(t) - x_0\| &\le \|u_\lambda^1(t) - v(t)\| + \|v(t) - x_0\| \\ &\le MT_0 + \sup_{0 \le t-s \le T_0} \|U(t,s)x_0 - x_0\| \le \rho \end{aligned} \tag{2.7}$$

i.e. $\|u_\lambda^1(t)\| \le \|x_0\| + \rho$ on $[s-\lambda,\, s+\lambda]$.

We conclude that (2.2) holds on $[s,\, s+\lambda)$ with u_λ^1 in place of u_λ. Of course, we may assume that L in (C2) is nondecreasing so $L\big(\max\{\|x\|,\, \sup\|u_\lambda^1(t)\|\}\big) \le L\big(\max\{\|x\|,\, \rho + \|x_0\|\}\big)$ a.s.a. Define $u_\lambda(t) = u_\lambda^1(t)$ for $t \in [s-\lambda,\, s+\lambda]$. Similarly we define u_λ on $[s+\lambda,\, s+2\lambda]$. Namely, set $g(t) = F(t-\lambda)u_\lambda^1(t-\lambda) = F(t-\lambda)u_\lambda(t-\lambda)$ for $t \in [s+\lambda,\, s+2\lambda]$ and denote by u_λ^2 the integral solution of

$$\begin{aligned} &(u_\lambda^2)'(t) \in A(t)\, u_\lambda^2(t) + F(t-\lambda)\, u_\lambda^1(s-\lambda) \\ &u_\lambda^2(s+\lambda) = u_\lambda^1(s+\lambda),\ t \in [s+\lambda,\, s+2\lambda]\,. \end{aligned}$$

We now extend u_λ on $[s+\lambda,\, s+2\lambda]$ by u_λ^2 i.e. $u_\lambda(t) = u_\lambda^2(t)$ for $t \in [s+\lambda,\, s+2\lambda]$. It follows that u_λ is an integral solution to (2.2) on $[s,\, s+2\lambda]$. Arguing as for (2.6), one has $\|u_\lambda(t) - v(t)\| \le \int_s^t \|F(\tau-\lambda)u_\lambda(\tau-\lambda)\|\,d\tau \le M(t-s) \le 2\lambda M \le MT_0$ for $s \le t \le s+2\lambda \le s+T_0$. It is now clear that (2.7) holds with u_λ in place of u_λ^1 on $[s,\, s+2\lambda]$. Actually we have $\|u_\lambda(t) - x_0\| \le \rho$, for all $t \in [s-\lambda,\, s+2\lambda]$. In such a manner we extend u_λ to $[s,\, s+T_0]$ as an integral solution to (2.1) with the additional property that

$$\|u_\lambda(t) - x_0\| \le \rho \quad \text{for all} \quad t \in [s-\lambda,\, s+T_0] \tag{2.8}$$

(and for all $\lambda > 0$ sufficiently small, with $\rho > 0$ independent of λ). It remains to prove that (2.2) implies (1.8) (i.e. that $\{u_\lambda\}$ is precompact in $C\big([s,\, s+T_0];\, X\big)$ and that we can pass to the limit in (2.2)).

We first prove that for each $t \in [s,\, s+T_0]$, $\{u_\lambda(t),\ \lambda > 0\}$ is precompact in X. For $t = s$ this fact is trivial since $u_\lambda(s) = x_0$. Therefore let $t > s$ and let $\epsilon > 0$ be such that $s < t - \epsilon < t$. Set $v_\lambda(t) = U(t,\, t-\epsilon)u_\lambda(t-\epsilon)$. Since $U(t,\, t-\epsilon)$ is compact and $\{u_\lambda(t-\epsilon),\ \lambda > 0\}$ is a bounded subset of $\overline{D(A(t-\epsilon))}$, the subset $K_\epsilon(t) = \{v_\lambda(t),\ \lambda > 0\}$ is precompact in X (see (C4)). But we have to observe that v_λ is the integral solution to the problem $v_\lambda'(\tau) \in A(\tau)v_\lambda(\tau)$, $v_\lambda(t-\epsilon) = u_\lambda(t-\epsilon)$, $t - \epsilon \le \tau$ and therefore, by (1.7) we derive

$$\|u_\lambda(t) - v_\lambda(t)\| \le \int_{t-\epsilon}^{t} \|F(\tau-\lambda)u_\lambda(\tau-\epsilon)\|\,d\tau \le \epsilon M \tag{2.9}$$

which shows that the precompactness of $K_\epsilon(t)$ implies the precompactness of $K(t)$.

We now prove the equicontinuity of $\{u_\lambda,\ \lambda > 0 \text{ sufficiently small}\}$. Take an arbitrary $t_0 \in [s,\, s+T_0]$. If $t_0 = s$, then according to (2.5) and (2.9) with s in place of $t - \epsilon$ we have,

$$\begin{aligned}\|u_\lambda(t) - u_\lambda(s)\| &\le \|u_\lambda(t) - v(t)\| + \|v(t) - x_0\| \\ &\le (t-s)M + \|v(t) - v(s)\|\end{aligned}$$

so $\{u_\lambda\}$ are equicontinuous at $t_0 = s$. If $s < t_0 < s+T_0 = T_1$ choose $0 \le d \le \min\{\epsilon/6M\,,\ t_0 - s,\ T_1 - t_0\}$ and set

$$v_d^\lambda(t) = U(t,\, t_0-d)u_\lambda(t_0-d), \quad t_0 - d \le t$$

Then,

$$\|u_\lambda(t) - v_d^\lambda(t)\| \le \int_{t_0-d}^{t} \|F(\tau-\lambda)u_\lambda(t-\lambda)\|\,d\tau \le \big(|t-t_0| + d\big)M \le 2dM \le \frac{\epsilon}{3} \tag{2.10}$$

for all $|t - t_0| \le d$.
On the other hand, $\{v_d^\lambda\,,\ \lambda > 0\}$ are equicontinuous at t_0. This fact can be proved as follows:

$$\|v_d^\lambda(t) - v_d^\lambda(t_0)\| = \|U(t,\, t_0-d)u_\lambda(t_0-d) - U(t_0,\, t_0-d)u_\lambda(t_0-d)\| \tag{2.11}$$

But the compactness of $U(t,s)$ for $t > s$ implies the equicontinuity of $t \to U(t,\, t_0-d)z$ at $t = t_0$ on the bounded subset $S = \{u_\lambda(t_0-d),\ \lambda > 0\}$ of $\overline{D(A(t_0-d))}$, i.e. there exists $n = n(\epsilon,\, t_0,\, d) < d$ such that $\|v_d^\lambda(t) - v_d^\lambda(t_0)\| < \epsilon$, for all $|t - t_0| < n$ and $\lambda > 0$. Finally, the inequality

$$\|u_\lambda(t) - u_\lambda(t_0)\| \le \|u_\lambda(t) - v_d^\lambda(t)\| + \|v_d^\lambda(t) - v_d^\lambda(t_0)\| + \|v_d^\lambda(t_0) - u_\lambda(t_0)\| < \epsilon$$

for all $|t-t_0| < n$ (and sufficiently small $\lambda > 0$) means just the equicontinuity of $\{u_\lambda\}$ at t_0. Therefore $\{u_\lambda\}$ is precompact in $C([s,\ s+T_0];\ X)$. We may assume (relabeling if necessary), that even u_λ is convergent, to a function u. This implies that $F(t-\lambda)u_\lambda(t-\lambda) \to F(t)u(t)$ as $\lambda \downarrow 0$, uniformly on $[s,\ s+T_0]$. Since the function $(y,x) \to \langle y,x\rangle_+$ is upper semicontinuous we can pass to the limit in (2.2). Thus, letting $\lambda \downarrow 0$, (2.2) yields just (1.8), which completes the proof.

Proof of Theorem 1.2. *If u_1 is an integral solution of (1.2) on $[s,T_1]$ and u_2 is an integral solution of $u_2' \in A(t)u_2 + F(t)u_2$ on $[T_1,T_2]$ with $u_1(T_1) = u_2(T_1)$, then the function $u : [s,T_2] \to X$ defined by $u(t) = u_1(t)$ on $[s,T_1]$ and $u(t) = u_2(t)$ on $[T_1,T_2](s < T_1 < T_2)$ is an integral solution of (1.2) on $[s,T_2]$ (i.e. it satisfies (1.8) on $[s,T_2]$). This property of integral solutions can be proved via DS–approximate solutions [5, 10]. Now the existence of maximal solutions $u : [s,\ t_{\max}[\to X$ follows from Zorn's lemma. We will prove that if $t_{\max} \equiv t_m < +\infty$ then $\lim_{t\uparrow t_m}\|u(t)\| = +\infty$. In the first step we prove that in this case (i.e. $t_m < +\infty$) u is unbounded on $[s,t_m[$. Indeed, if u were bounded on $[s,t_m[$ (i.e. $\|u(t)\| \le K$, $\forall t \in [s,t_m[0)$ then we would have $\|F(t)u(t)\| \le K_1$, $\forall t \in [s,t_m[$. Given $\epsilon > 0$ choose $\delta = \delta(\epsilon) > 0$ such that $\delta K_1 < \epsilon/3$ and set $v(t) = U(t,\ t_m-\delta)u(t_m-\delta)$, $t_m - \delta \le t$. We have*

$$\|u(t)-v(t)\| \le \int_{t_m-\delta}^{t} \|F(\theta)u(\theta)\|\,d\theta \le \delta K_1 < \epsilon/3, \qquad t_m-\delta \le t < t_m \tag{2.12}$$

Since v is continuous on $[s,t_m]$ we may assume that $\|v(t)-v(\tau)\| \le \epsilon/3$ for $t,\ \tau \in [t_m-\delta, t_m]$. Therefore $\|u(t)-u(\tau)\| < \epsilon$ for all $t,\tau \in [t_m-\delta,\ t_m[$ so $\lim_{t\uparrow t_m} u(t)$ exists. This implies the absurdity that u can be extended to the right of t_m and thus it follow that u is unbounded on $[s,t_{\max}[$. It remains to prove that

$$\liminf_{t\uparrow t_m}\|u(t)\| = +\infty \tag{2.13}$$

The proof will be carried out by reduction and absurdum. Therefore, suppose that $\liminf_{t\uparrow t_m}\|u(t)\| < +\infty$. Then there exist $r > 0$ and $t_k \uparrow t_m$ such that

$$\|u(t_k)\| < r, \;\; s < t_k < t_{k+1},\ ,\ k = 1,2,\ldots \tag{2.14}$$

Since $(t,s,x) \to U(t,s)x$ is bounded on bounded sets, there exists $c > 0$ with the properties

$$\|U(t_k+h,\ t_k)u(t_k)\| \le c, \;\; k=1,2,\ldots,\;\; h \in [0,b],\; r \le c \tag{2.15}$$

where b is a given positive number ($0 < b < t_m$). Set $B = \sup\{\|F(t)y\|,\ t \in [s,t_m],\ y \in \overline{D(A(t))},\ \|y\| \le 3c\}$ and assume that $bB < c$. Then

$$\|u(t)\| \le 3c, \;\; \forall t \in [t_{k_0},\ t_m[\tag{2.16}$$

where k_0 is such that $t_{k_0} > t_m - b$. The proof of (2.16) is again by contradiction. Namely, if (2.16) were not true, there would exist $h_1 > 0$ satisfying

$$h_1 < b,\;\; \|u(t_{k_0}+t)\| < 3c,\;\; \forall t \in [0,h_1],\;\; \|u(t_{k_0}+h_1)\| = 3c \;\; t_{k_0}+h_1 < t_m\,. \tag{2.17}$$

This time set $v(t) = U(t, t_{k_0})u(t_{k_0}),\ t \geq t_{k_0}$. *We know that*

$$\|u(t_{k_0} + h_1) - v(t_{k_0} + h_1)\| \leq \int_{t_{k_0}}^{t_{k_0}+h_1} \|F(\tau)u(\tau)\|\, d\tau \leq h_1 B < bB < c \tag{2.18}$$

On the basis of (2.15) and (2.18), we have reached the following contradiction

$$\begin{aligned} 3c = \|u(t_{k_0} + h_1)\| &\leq \|u(t_{k_0} + h_1) - v(t_{k_0} + h_1)\| + \|v(t_{k_0} + h_1)\| \\ &< c + \|U(t_{k_0} + h_1\,,\, t_{k_0})u(t_{k_0})\| \leq 2c \end{aligned}$$

which proves the estimate (2.16). But (2.16) contradicts the unboundedness of u *on* $[s, t_m[$ *so the hypothesis* $\liminf_{t \uparrow t_m} \|u(t)\| < +\infty$ *fails. Therefore (2.13) holds and the proof is complete.*

Remark. In the case $X = R^n$ and $A(t) = 0$ the existence and precompactness of u_λ satisfying $u'_\lambda(t) = F(t)u_\lambda(t - \lambda)$, $u_\lambda(\theta) = x_0$, $t_0 - \lambda \leq \theta \leq t_0$ are obvious. Now Peano's theorem follows by passing to the limit as $\lambda \downarrow 0$. This proof (by the method of steps) is perhaps the simplest proof of Peano's existence theorem.

If X is of infinite dimension, then $A(t) = 0$ is not allowed in Theorem 1.1. This is because in this case $U(t, s) = I$ (the identity on X) which is not compact.

REFERENCES

[1] Ph. Benilan, *Solutions intégrales d'équations d'évolution dans un espace de Banach*, C.R. Acad. Sci. Paris 274 (1972), 47-50.

[2] M.G. Crandall & L.C. Evans, *On the relation of the operator* $\partial/\partial s + \partial/\partial \tau$ *to evolution governed by accretive operators*, Israel J. Math. 21 (1975), 261-278.

[3] M.G. Crandall & A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. 11 (1972), 57-94.

[4] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, 19 (1967), 508-520.

[5] K. Kobayasi, Y. Kobayashi & S. Oharu, *Nonlinear evolution operators in Banach spaces*, Osaka J. Math. 21 (1984), 281-310.

[6] S. Oharu & T. Takahashi, *On nonlinear evolution operators associated with some nonlinear dispersive systems*, Proc. AMS 97 (1986), 139-145.

[7] N.H. Pavel, *Nonlinear evolution equations governed by f-quasi-dissipative operators*, Nonlinear Anal. TMA. 5 (1981), 449-468.

[8] N.H. Pavel, *Some problems on nonlinear semigroups and the blow-up of integral solutions*, Lecture Notes in Math. Vol. 1076 (1984), Springer-Verlag.

[9] N.H. Pavel, *Differential equations, flow-invariance and applications*, Pitman Research Notes in Math. Vol. 113 (1984).

[10] N.H. Pavel, *Nonlinear Evolution Operators and Semigroups*, Lecture Notes in Math. Vol. 1260 (1987), Springer-Verlag.

[11] N.H. Pavel, *Compact evolution operators*, (to appear).

[12] A. Pazy, *A class of semilinear equations of evolution*, Israel J. Math. 20 (1975), 26-36.

[13] I.I. Vrabie, *The nonlinear version of Pazy's local existence theorem*, Israel J. Math. 32 (1979), 221-235.