Differential and Integral Equations, Volume 1, Number 1, January 1988, pp. 79–96.

ON THE ORDER OF MAGNITUDE OF TITCHMARSH–WEYL FUNCTIONS

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Abstract. Upper and lower bounds are obtained for the absolute values of a family of Titchmarsh-Weyl m-coefficients, thereby determining their order of magnitude; only minimal restrictions on the second-order differential operator are imposed. The method also yields the asymptotic behaviour in a certain exceptional case. The results are applied to the estimation of spectral functions.

1. Introduction. Recent progress in the spectral theory of the second-order operator

$$-(py')' + qy = \lambda wy, \quad -\infty < a \leq x < b \leq \infty\,, \tag{1.1}$$

focussing on the twin concepts of a spectral function and an m-coefficient, has dealt largely with asymptotic approximation to these entities, necessarily with restrictive hypotheses on the coefficients in the differential operator. In the case of the m-coefficient the topic stems from the original order result of Hille [23], improved to an asymptotic formula by Everitt [10]. In one direction these have led the way to higher asymptotics, or indeed asymptotic series for the case $p = w = 1$ (see e.g. [18-20, 25-26]). Another type of development has been to extend the Everitt formula [10] to more general circumstances [1, 2, 11].

The thrust in this paper is in a third direction, going back to the aspect dealt with by Hille. We aim to obtain order-of-magnitude results covering the most general case of (1.1), imposing only the standard requirements for the "right-definite" case. We do not assume any specific asymptotic form for p, q and w as $x \to a$, and do not, in particular require p to be positive. We are concerned with results of the general form

$$C_1\psi(|\lambda|) \leq |m| \leq C_2\psi(|\lambda|) \tag{1.2-3}$$

as $\lambda \to \infty$ in a sector

$$\epsilon \leq \arg\lambda \leq \pi - \epsilon\,, \tag{1.4}$$

for fixed ϵ with $0 < \epsilon < \pi/2$. In (1.2-3), $\psi(\lambda)$ is a positive function to be specified, actually dependent only on $|\lambda|$.

Received September 10, 1987.

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AMS(MOS) Subject Classifications: 34B25.

Here we encounter the problem of the meaning to be attached to the term "Titchmarsh-Weyl function" for (1.1), and the question of its uniqueness. Bypassing this last point, we obtain, much as in [1-3], bounds for a general point in a certain region $D(X,\lambda)$, the "Weyl disc", to be defined later. The bounds will of course apply to the function $m(\lambda)$ in the case of uniqueness, that is to say the limit-point case.

Order-of-magnitude results for the Titchmarsh-Weyl function can be applied to obtain similar results for the spectral function; the parallel argument for asymptotic behaviour has been given in [2, 17]. They can also serve as an intermediate step in the proof of asymptotic formulae.

The present paper completes to a large degree an earlier paper [3], in which upper bounds of the form (1.3) were obtained, along with corresponding upper bounds for the spectral function. These upper bounds for $|m|$ could be verified as giving the correct order of magnitude in explicitly soluble cases of standard type, for example in which $a = 0$, $q = 0$ and p, w are powers of x (see [3], p.6). It turns out that the upper bounds referred to, while valid, sometimes give an over-estimate; an example of this involves the "rapidly oscillating" choice $p(x) = 1/\sin x^{-1}$ considered in more detail in §6. Our main aim here is to obtain upper and lower estimates which agree on the order of magnitude in a wide variety of cases, perhaps in all.

The results of this paper can be obtained in more than one way, and indeed have been so obtained. In an earlier version of the paper, written during a visit to the University of Birmingham, England, in 1985, I employed what may be termed the circle geometry method used earlier in [1]; in this method we estimate points in the "Weyl disc" by estimating some convenient reference point in the disc and also estimating the diameter of the disc. In this paper I use an entirely different method, based on the interpretation of the m-coefficient in terms of Riccati equations with solutions lying in the closed upper half-plane. This interpretation appears suitable for extensions to "half-linear" equations, and indeed was so derived. A general survey of the theory of the Titchmarsh-Weyl m-coefficient, covering developments up to the late 1970's is given in [9]. A substantial update of this survey is in preparation [5]. An important memoir of Bennewitz [6] provides a detailed account of new developments in the theory, both as regards order-of-magnitude aspects and also as regards asymptotic approximations, and is not confined to the right-definite case.

We pass to a brief review of the contents of individual sections of the paper.

We begin, in §2 with an apparently unrelated topic, that of estimates for solutions of scalar Riccati equations. The connection with the present topic lies in the characterization used here for elements of the Weyl disc $D(X,\lambda)$, that it consists of numbers m such that the solution of the initial-value problem, with $w \geq 0$, $\operatorname{Im}\lambda > 0$,

$$v' = -p^{-1} - (\lambda w - q)v^2, \quad v(a) = m, \tag{1.5}$$

satisfies $\operatorname{Im} v(X) \geq 0$. We discuss in §3 the relation between this and more conventional definitions. In §4 we give a preliminary upper bound for the Sturm-Liouville case, effective, roughly speaking, in cases of standard type. This general result for the Sturm-Liouville case is given in §5, with examples in the following section. In §7 we deal with a special case, which is remarkable in that the present methods yield an asymptotic formula for the Titchmarsh-Weyl coefficient, and not merely order-of-magnitude results.

We conclude the paper with brief discussions of two further aspects, firstly applications to the estimation of spectral functions, and secondly the asymptotics of m in the situation that $\lambda \to 0$.

2. Lemmas on Riccati equations. We obtain in this section necessary conditions on the initial value $v(a)$ in order that the equation

$$v'(x) = -\alpha(x) - \beta(x)v(x) - \gamma(x)v^2(x), \quad a \le x \le c, \tag{2.1}$$

should have a solution satisfying

$$\operatorname{Im} v(c) \ge 0\,. \tag{2.2}$$

Here $c \in (a, b)$, and α, β, γ are in $L_{\mathrm{loc}}[a, b)$; they are in general complex valued.

We write

$$\alpha_0 = \int_a^c |\alpha(t)|\, dt\,, \qquad \alpha_1(x) = \int_a^x \alpha(t)\, dt\,, \tag{2.3}$$

and define similarly β_0, γ_0, $\beta_1(x)$, $\gamma_1(x)$.

We have then

Lemma 1. *Under the above conditions, if (2.1) has a solution satisfying (2.2), we have*

$$|v(a)| \ge \mathit{Im}\, \alpha_1(c) - \alpha_0(4\beta_0 + 16\alpha_0\gamma_0)\,, \tag{2.4}$$

$$|1/v(a)| \ge \mathit{Im}\, \gamma_1(c) - \gamma_0(4\beta_0 + 16\alpha_0\gamma_0)\,. \tag{2.5}$$

It will be sufficient to prove (2.4) only; we can then deduce (2.5) by applying (2.4) to the differential equation satisfied by $v^* = -1/v$.

Passing over trivial cases, we assume that

$$\alpha_0 > 0, \quad 4\beta_0 + 16\alpha_0\gamma_0 < 1, \quad |v(a)| < \alpha_0\,, \tag{2.6-8}$$

since the right of (2.4) does not exceed

$$\alpha_0(1 - 4\beta_0 - 16\alpha_0\beta_0) \le \alpha_0\,. \tag{2.9}$$

We first establish an upper bound for $|v(x)|$. We claim that

$$|v(x)| < 4\alpha_0, \qquad a \le x \le c\,. \tag{2.10}$$

In the contrary event, there would be an $x_1 \in (a, c]$ such that

$$|v(x_1)| = 4\alpha_0\,,$$

$$|v(x)| < 4\alpha_0 \quad \text{for} \quad a \le x < x_1\,. \tag{2.11}$$

We would then have

$$|v(x_1) - v(a)| > 3\alpha_0$$

while, by integration of (2.1) and use of (2.11), we would also have

$$|v(x_1) - v(a)| \le \alpha_0 + 4\beta_0\alpha_0 + 16\gamma_0\alpha_0^2 \le 2\alpha_0\,,$$

by (2.7). This gives a contradiction, and so proves (2.10).

We next integrate (2.1) over (a,c), and write the result in the form

$$v(a) = v(c) + \alpha_1(c) + \int_a^c (\beta v + \gamma v^2)\, dx\,.$$

Here we take imaginary parts, note that $\operatorname{Im} v(c) \geq 0$, and use (2.10). This gives

$$\operatorname{Im} v(a) \geq \operatorname{Im} \alpha_1(c) - 4\alpha_0\beta_0 - 16\gamma_0\alpha_0^2\,.$$

Since $|v(a)| \geq \operatorname{Im} v(a)$, this proves (2.4).

In the event that $\alpha(x)$ is real-valued, but not $\gamma(x)$, so that only (2.5) is informative, a transformation is useful. We give the details in the case that $\beta(x) \equiv 0$, which is relevant to the Sturm-Liouville application. We thus take it that v is a solution of

$$v' = -\alpha - \gamma v^2\,, \quad \operatorname{Im} v(x) \geq 0\,, \quad a \leq x \leq c\,, \tag{2.12}$$

with

$$\operatorname{Im} \alpha(x) \equiv 0\,, \quad \operatorname{Im} \gamma(x) \geq 0\,, \quad a \leq x \leq c\,. \tag{2.13}$$

We write

$$V(x) = v(x) + \alpha_1(x)\,, \tag{2.14}$$

so that

$$V' = -\gamma(V - \alpha_1)^2 \tag{2.15}$$

and also

$$V(a) = v(a)\,, \quad \operatorname{Im} V(x) \geq 0\,, \quad a \leq x \leq c\,. \tag{2.16}$$

By applying Lemma 1 to (2.15) we get

Lemma 2. *Defining $L \geq 0$ by*

$$L^2 = \gamma_0 \int_a^c |\gamma\alpha_1^2|\, dx\,, \tag{2.17}$$

we have

$$|v(a)| \geq Im \int_a^c \gamma\alpha_1^2\, dx - 10L \int_a^c |\gamma\alpha_1^2|\, dx\,, \tag{2.18}$$

$$|1/v(a)| \geq Im\, \gamma_1(c) - 10L\gamma_0\,. \tag{2.19}$$

In applying the results (2.4-5) to (2.15) we must replace α by $\gamma\alpha_1^2$ and β by $-2\gamma\alpha_1$, while γ remains unchanged. Since $v(a)$, $V(a)$ are the same we have from (2.4) that (with all integrals over (a,c)),

$$\begin{aligned} |V(a)| &\geq \operatorname{Im} \int \gamma\alpha_1^2\, dx - 8\int |\gamma\alpha_1^2|\, dx \int |\gamma\alpha_1|\, dx - 16\gamma_0\Big(\int |\gamma\alpha_1^2|\, dx\Big)^2 \\ &\geq \operatorname{Im} \int \gamma\alpha_1^2\, dx - (8L + 16L^2)\int |\gamma\alpha_1^2|\, dx\,, \end{aligned} \tag{2.20}$$

on using the Schwarz inequality. We observe that we may take it that $L < 1/10$, since otherwise (2.18-19) are trivial. We then get (2.18) from (2.20).

The proof of (2.19) is similar.

It should be noted that the argument yields slightly stronger results than those stated. The left-hand sides of (2.18-19), may be replaced respectively by $\mathrm{Im}\, v(a)$, $\mathrm{Im}\,(-1/v(a))$. We omit the details but note a possible application at the end of §5.

3. Various definitions of the Weyl disc. In this section we list for comparison various definitions of the "Weyl disc" $D(X, \lambda)$, associated with any $X \in (a, b)$ and any λ with $\mathrm{Im}\,\lambda > 0$; this will be a set of complex numbers m having various equivalent properties. Such an object can be set up for (1.1) and any homogeneous initial conditions (see e.g. [9]), though it is essentially sufficient to take one case; as usual, we take here the Dirichlet initial conditions.

We make in the sequel the following general assumptions regarding (1.1):

(i) $1/p$, q, w are real and Lebesgue-integrable on compact subsets of $[a, b)$,

(ii) $w(x) \geq 0$, and

$$\int_a^x w(t)\, dt > 0 \quad \text{for} \quad x \in (a, b)\,, \tag{3.1}$$

(iii) writing

$$r(x) = 1/p(x)\,, \quad r_1(x) = \int_a^x r(t)\, dt, \tag{3.2}$$

we require that

$$\int_a^x w r_1^2\, dt > 0 \quad \text{for} \quad x \in (a, b). \tag{3.3}$$

We denote as usual by $\theta(x)$, $\phi(x)$ a pair of solutions of (1.1), fixed by the initial data

$$\theta = 0, \;\; p\theta' = 1, \;\; \phi = -1, \;\; p\phi' = 0, \;\; \text{at } x = a. \tag{3.4}$$

The second and fourth of these are to be interpreted, if necessary, in a limiting sense as $x \to a$, or as values of "quasi-derivatives". We write occasionally $\theta(x, \lambda)$, $\phi(x, \lambda)$; dependence on λ is in any case to be understood.

For any $X \in (a, b)$ and any λ with $\mathrm{Im}\,\lambda > 0$ we can then set up a number of equivalent definitions of a "Weyl disc". The first three of these are concerned with properties of a linear combination

$$y = \theta + m\phi \tag{3.5}$$

of the solutions determined by (3.5). We write $y = y(x) = y(x, \lambda)$; we could write also $y = y(x, \lambda, m)$. Although we are dealing with the scalar case, we follow matrix usage in using (*) to indicate the complex conjugate.

Definition 1. *The set of m such that*

$$Im\,\big(p(X)y'(X)y^*(X) - (p(X)y'(X))^* y(X)\big) \geq 0. \tag{3.6}$$

Definition 2. *The set of m such that either $y(X) = 0$ or*

$$Im\,\big(p(X)y'(X)/y(X)\big) \geq 0. \tag{3.7}$$

Definition 3. *The set of m such that*

$$Im\, m \geq Im\, \lambda \int_a^X |y|^2\, dt\,. \tag{3.8}$$

Definition 4. *The image of the lower half-plane $Im\, \chi \leq 0$ under the map*

$$\chi :\rightarrow m = -\big(\theta(X) + \chi p(X)\theta'(X)\big)/\big(\phi(X) + \chi p(X)\phi'(X)\big)\,. \tag{3.9}$$

Definition 5. *The set of m such that the Riccati equation*

$$v' = -r - (\lambda w - q)v^2 \tag{3.10}$$

with initial condition

$$v(a) = m \tag{3.11}$$

exists over $[a, c)$ and satisfies

$$Im\, v(x) \geq 0, \quad for \quad a \leq x < c\,. \tag{3.12}$$

This last definition may be modified by replacing (3.12) equivalently by

$$\text{Im}\, v(c) \geq 0\,. \tag{3.12$'$}$$

The first four of these definitions are linked in a well-known manner with the identity

$$2i \text{ Im } m = 2i \text{ Im } \lambda \int_a^X w|y|^2\, dt + \big(y^* py' - y(py')^*\big)|_x\,. \tag{3.13}$$

The last, less standard definition, which is our main tool, arises from the differential equation (3.10) satisfied by

$$v(x) = -y/(py')\,. \tag{3.14}$$

Any of these definitions of the Weyl disc $D(X, \lambda)$ can also serve as a basis for defining a Titchmarsh-Weyl function $m(\lambda)$, namely as a function holomorphic in the upper half-plane $\text{Im}\, \lambda > 0$ and lying in $D(X, \lambda)$ for all $X \in (a, b)$. For the order of magnitude of such a function, whether unique or not, it turns out to be sufficient to estimate elements of $D(X, \lambda)$ when $X = X(\lambda) \rightarrow a$ in a controlled manner as $|\lambda| \rightarrow \infty$.

4. A one-sided bound for points of the Weyl disc. In a preliminary use of this machinery, we use Lemma 1 to get an upper bound for points m of a certain Weyl disc; the result is very close to one of [3], obtained by the circle geometry method. We prove

Theorem 1. *For some ϵ in $(0, \pi/2)$ let λ satisfy*

$$\epsilon \leq \arg\lambda \leq \pi - \epsilon\,. \tag{4.1}$$

For large λ, let $c = c(\lambda) = c(|\lambda|)$ be determined so that

$$|\lambda| \int_a^c w\,dt \int_a^c |p^{-1}|\,dt \leq 32^{-1}\,\sin\epsilon\,. \tag{4.2}$$

Then, for large λ, any $m \in D(c, \lambda)$ satisfies

$$|m| \leq 2(\csc\epsilon)\Big(|\lambda| \int_a^c w\,dt\Big)^{-1}. \tag{4.3}$$

It will clearly be sufficient to prove the result for the case that equality holds in (4.2), since that will yield the strongest form of (4.3).

We apply Definition 5 and Lemma 1. Identifying the differential equation (2.1) with (3.10), we have

$$\alpha = p^{-1}\,, \quad \beta = 0\,, \quad \gamma = \lambda w - q\,. \tag{4.4}$$

Since α is real, only (2.5) will be informative, and this shows that for any $c \in (a, b)$ we have

$$|1/m| \geq \operatorname{Im}\lambda \int_a^c w\,dt - 16\Big(\int_a^c |\lambda w - q|\,dt\Big)^2 \int_a^c |p^{-1}|\,dt \tag{4.5}$$

for any $m \in D(c, \lambda)$.

We then choose c to satisfy (4.2), with equality, which will be possible if λ is sufficiently large. To see this we note that both integrals on the left of (4.2) are continuous, positive and non-decreasing functions of c on (a, b), by our hypotheses (3.1), (3.3). We can make $c(\lambda)$ unique by specifying it to be the least value satisfying (4.2) with equality.

It is clear that we must have

$$c(\lambda) \to a\,, \qquad \text{as} \qquad |\lambda| \to \infty\,. \tag{4.6}$$

Since the second integral in (4.2) then tends to 0, we must also have

$$|\lambda| \int_a^c w\,dt \to \infty\,, \tag{4.7}$$

and hence

$$\int_a^c |\lambda w - q|\,dt \sim |\lambda| \int_a^c w\,dt\,. \tag{4.8}$$

Using this in (4.5) we have

$$\Big(\int_a^c |\lambda w - q|\,dt\Big)^2 \int_a^c |p^{-1}|\,dt \sim 32^{-1}\,\sin\epsilon|\lambda| \int_a^c w\,dt\,.$$

This gives the required result.

As it happens, Theorem 1 gives the correct order of magnitude in certain straightforward, explicitly calculable cases, but may give an over-estimate in cases of rapidly oscillating $p(x)$. Examples of both kinds will be presented in §6.

5. Two-sided bounds for the Weyl disc. We show now that by use of Lemma 2 we can replace the result of Theorem 1 by upper and lower bounds which agree in cases so far known. In the notation of Lemma 2, we have now

$$\alpha_1(x) = r_1(x), \quad \gamma(x) = \lambda w(x) - q(x), \tag{5.1-2}$$

and, for some $c \in (a, b)$ to be determined,

$$\gamma_0 = \int_a^c |\lambda w - q| \, dt, \quad \gamma_1(x) = \int_a^x (\lambda w - q) \, dt. \tag{5.3}$$

Following (2.17), we define $L \geq 0$ by

$$L^2 = \int_a^c |\lambda w - q| \, dt \int_a^c |\lambda w - q| r_1^2 \, dt. \tag{5.4}$$

Restricting λ by (4.1), we deduce from (2.18-19) that any $m \in D(c, \lambda)$ will satisfy

$$|m| \geq |\lambda| \sin\epsilon \int_a^c w r_1^2 \, dt - 10L \int_a^c |\lambda w - q| r_1^2 \, dt, \tag{5.5}$$

$$|1/m| \geq |\lambda| \sin\epsilon \int_a^c w \, dt - 10L \int_a^c |\lambda w - q| \, dt. \tag{5.6}$$

To derive from (5.5-6) a complete determination of the order of magnitude of m in the general Sturm-Liouville case we have, it seems, to supplement the general hypotheses of §3 with a special hypotheses involving q, namely

$$\Big(\int_a^x |q r_1^2| \, dt\Big)^2 \Big(\int_a^x w \, dt\Big) = o\Big(\int_a^x w r_1^2 \, dt\Big) \tag{5.7}$$

as $x \to a$. In particular, (5.7) will hold if

$$\int_a^x w \, dt \, \sup r_1^4 = o\Big(\int_a^x w r_1^2 \, dt\Big), \tag{5.8}$$

with "sup" over $[a, x]$. Obviously, this is true if $q = 0$. Indeed, it seems difficult to construct examples in which (5.7) is not satisfied.

With our general assumptions, including (4.1) and (5.7) we have

Theorem 2. *For large λ let $c = c(\lambda) \in (a, b)$ be determined subject to*

$$|\lambda|^2 \int_a^c w \, dt \int_a^c w r_1^2 \, dt \leq 400^{-1} \sin^2\epsilon. \tag{5.9}$$

Then any $m \in D(c, \lambda)$ satisfies, for sufficiently large λ,

$$|m| > (1/2)|\lambda| \sin \epsilon \int_a^c w r_1^2 \, dt \tag{5.10}$$

$$|1/m| > (1/2)|\lambda| \sin \epsilon \int_a^c w \, dt . \tag{5.11}$$

As in the case of Theorem 1, it is sufficient to prove (5.10-11) with $c(\lambda)$ chosen so that equality holds in (5.9). We have as previously that $c(\lambda) \to a$ as $|\lambda| \to \infty$. We shall have also

$$\int_a^x w r_1^2 \, dt = \mathrm{o} \{ \int_a^x w \, dt \} , \tag{5.12}$$

since $r_1(x) \to 0$ as $x \to a$. We can then deduce (4.7), and so also (4.8).

We claim in addition that

$$\int_a^c |\lambda w - q| r_1^2 \, dt \sim |\lambda| \int_a^c w r_1^2 \, dt . \tag{5.13}$$

It is here that we use the special hypothesis (5.7). A property equivalent to (5.13) is

$$\int_a^c |q| r_1^2 \, dt = \mathrm{o} \left(|\lambda| \int_a^c w r_1^2 \, dt \right) .$$

This follows from (5.7), if $|\lambda|$ be determined from (5.9) with equality. It then follows that

$$\int_a^c |\lambda w - q| \, dt \int_a^c |\lambda w - q| r_1^2 \, dt \to 400^{-1} \sin^2 \epsilon .$$

Hence, by (5.4),

$$L \to 20^{-1} \sin \epsilon .$$

We can now deduce the required results (5.10-11) from (5.5-6).

As indicated at the end of §2 we can sharpen (5.10-11) by replacing the left-hand sides by $\mathrm{Im}\, m$, $\mathrm{Im}\,(-1/m)$. Multiplying the results we deduce the remark that

$$\sin^2 \arg m \geq 1600^{-1} \sin^4 \epsilon , \tag{5.14}$$

for sufficiently large λ; the latter qualification can be dropped if $q \equiv 0$.

6. Examples and discussion. We take in these examples $a = 0$. We suppose first that p, w have power-type behaviour as $x \to 0$; it will be sufficient that this be in an integral, and not necessarily in a pointwise sense. Specifically, let us assume that

$$C_{11} x^{n(1)} \leq \int_0^x w \, dt \leq C_{12} x^{n(1)} , \tag{6.1}$$

$$C_{21} x^{n(2)} \leq r_1(x) \leq C_{22} x^{n(2)} \tag{6.2}$$

for small $x > 0$. Here the C_{ij}, $n(1)$, $n(2)$ are positive constants. The requirement (5.7) for q is now equivalent to

$$\int_0^x |q| t^{2n(2)}\, dt = \mathrm{o}\,(x^{n(2)}) \tag{6.3}$$

as $x \to 0$, which is certainly satisfied if, as we assume, $|q|$ is integrable at $x = 0$. Applying Theorem 2, we suppose that $c(\lambda)$ is chosen so as to ensure equality in (5.9) and find that $c(\lambda)$ is of order

$$\lambda^{-1/(n(1)+n(2))}\,. \tag{6.4}$$

Since we have equality in (5.9), the bounds (5.10-11) agree as to the order of magnitude of m. Using (5.10) we find that m is exactly of order

$$|\lambda|(c(\lambda))^{n(1)+2n(2)}$$

or

$$|\lambda|^{-n(2)/(n(1)+n(2))}\,. \tag{6.5}$$

Thus, to take the simplest example, if

$$w = p = 1, \tag{6.6}$$

we have $n(1) = n(2) = 1$, and so m is exactly of order

$$|\lambda|^{-1/2}\,, \tag{6.7}$$

as is known from the original Hille result [23]. One-sided order results were discussed in [3], on the basis of a result similar to Theorem 1.

If (6.1-2) are sharpened to asymptotic equality as $x \to 0$, asymptotic expressions become available for m; we cite [9], [13], [25, 26]. If (6.1-2) are further sharpened to equality over the half-axis $[0, \infty)$ exact expressions become available [13] (see also [8]) for the limiting $m \in D(\infty, \lambda)$.

We recall here the general remark that our estimates for $m \in D(c(\lambda), \lambda)$ automatically apply to any $m \in D(X, \lambda)$ if $X \geq c(\lambda)$, in view of the nesting circle property.

We give next an example in which Theorem 2 yields a more accurate result than Theorem 1. This is

$$p = \csc x^{-1}\,, \quad w = 1,\ \ q = 0,\ \ 0 < x < \infty\,. \tag{6.8}$$

Here

$$\int_0^x |p^{-1}|\, dt = \mathrm{o}\,(x)$$

and so, using Theorem 1, we find that for any fixed $X \in (0, \infty)$, we have that the general $m \in D(X, \lambda)$ is of order at most (6.7) as in the case (6.6). Using Theorem 2, however, we have

$$r_1(x) = x^2 \cos x^{-1} + \mathrm{o}\,(x^3)\,,$$

and so

$$\int_0^x w r_1^2\, dt \sim x^5/10\,. \tag{6.9}$$

Used in (5.9), this leads to $c(\lambda)$ of order $|\lambda|^{-1/3}$, and so to the sharper estimate

$$m = \mathrm{o}\left(|\lambda|^{-2/3}\right), \tag{6.10}$$

with a corresponding estimate for $1/m$. Thus (6.10) gives the precise order of magnitude of m.

In a forthcoming paper an asymptotic formula along these lines will be derived.

The constant 400^{-1} appearing on the right of (5.9) is chosen so as to validate the inequalities (5.10-11). If however this constant is replaced by any other positive number, and again (5.9) is solved for $c(\lambda)$ with equality, this will not affect the order of magnitude of the right of (5.10-11).

7. A special example. We show here that for the case

$$p \equiv 1, \quad q \equiv 0, \quad w = 1/(x \log^2 x), \quad 0 < x < 1, \tag{7.1}$$

a development of the above methods yields the asymptotic behaviour of m, rather than just its order of magnitude. The relevant feature of this case is that the weight-function is concentrated near the initial point. The result can be extended to some other weight-functions of this nature, and indeed with other choices of p, q [4]. Here we prefer to give a sharpened version of the result for the special case (7.1), which we cite as

Theorem 3. *For fixed $X \in (0,1)$ and λ subject to (4.1), any $m \in D(X,\lambda)$ satisfies as $|\lambda| \to \infty$*

$$m = -\lambda^{-1}\left(\log|\lambda| - 2\log\log|\lambda|\right) + o\left(|\lambda|^{-1}\log\log\log|\lambda|\right). \tag{7.2}$$

It turns out to be convenient to prove the equivalent result for $-1/m$, namely that

$$-1/m = \lambda\left(\log|\lambda| - 2\log\log|\lambda|\right)^{-1} + \mathrm{o}\left(|\lambda|\log\log\log|\lambda|\right)/\log^2|\lambda|. \tag{7.3}$$

We apply Theorem 2 (or Lemma 2) to the differential equations

$$v' = -1 - \lambda v^2/(x\log^2 x), \tag{7.4}$$

$$V' = -\lambda/(x\log^2 x) - V^2. \tag{7.5}$$

Their relationship to the problem at hand is that the solution of (7.4) with $v(0) = m \in D(X,\lambda)$ must satisfy $\operatorname{Im} v(X) \geq 0$, and likewise the solution of (7.5) with $V(0) = -1/m$ must satisfy $\operatorname{Im} V(X) \geq 0$; we have necessarily also that $\operatorname{Im} v(x) \geq 0$, $\operatorname{Im} V(X) \geq 0$ for $0 \leq x \leq X$. Here $X \in (0,1)$ will be fixed throughout our discussion, as will ϵ in (4.1). We define a function $C(\lambda)$ with the properties that $C(\lambda) > 0$, $C(\lambda) \to 0$ as $|\lambda| \to \infty$, so that $C(\lambda) < X$ for large λ. We will therefore have $\operatorname{Im} v(C(\lambda)) \geq 0$, $\operatorname{Im} V(C(\lambda)) \geq 0$. We choose

$$C(\lambda) = \left(K\log^2|\lambda|\right)/\left(|\lambda|\log\log|\lambda|\right), \tag{7.6}$$

where

$$K = 400^{-1}\sin^2\epsilon. \tag{7.7}$$

We integrate (7.5) over $(0, C(\lambda))$, and write the result in the form

$$V(0) = -\lambda/\big(\log C(\lambda)\big) + V(C(\lambda)) + \int_0^{C(\lambda)} V^2\,dt\,. \tag{7.8}$$

Here the first term on the right has the form given by the right of (7.3), so that we need to show that the last two terms in (7.8) can be accomodated within the error bound in (7.3). Specifically, we must show that, writing ρ for $|\lambda|$,

$$V(C(\lambda)) = \mathrm{o}\left(\rho \log\log\log \rho/(\log^2 \rho)\right), \tag{7.9}$$

and also

$$\int_0^{C(\lambda)} V^2\,dt = \mathrm{o}\left(\rho \log\log\log \rho/(\log^2 \rho)\right). \tag{7.10}$$

We first find a bound for $V(0)$. We apply Theorem 2 with $a = 0$ and p, q, w as in (7.1), and use (5.10) to get an upper bound for $|1/m| = |V(0)|$. In (5.9) we replace c by the function given by

$$c_1(\lambda) = 20^{-1}(\sin\epsilon)\rho^{-1}(\log\rho)^{3/2}, \tag{7.11}$$

always for large ρ. Verifying (5.9), we find that, as $|\lambda| \to \infty$,

$$|\lambda| \int_0^{c_1} w\,dt = |\lambda|/\log c_1 \sim \rho/\log\rho, \tag{7.12}$$

while

$$|\lambda| \int_0^{c_1} w r_1^2\,dt = |\lambda| \int_0^{c_1} t\,dt/\log^2 t \sim 2^{-1}|\lambda| c_1^2/\log^2 c_1 \sim 800^{-1}(\sin^2\epsilon)\rho^{-1}\log\rho. \tag{7.13}$$

It follows from (7.12-13) that (5.9) is satisfied for large λ, so that substituting (7.13) in (5.10) we can deduce that, again for large λ,

$$|V(0)| < 2000(\sin^{-3}\epsilon)\rho/\log\rho. \tag{7.14}$$

We claim next that a similar bound holds over $(0, 1/\rho)$, or that

$$|V(x)| < 3000(\sin^{-3}\epsilon)\rho/\log\rho, \quad 0 \le x \le 1/\rho \tag{7.15}$$

for large λ. While this can be proved in the same way as (7.14), it can also be proved by the argument of Lemma 2 (see (2.10)). We denote by x_1 a supposed first value in $(0, 1/\rho)$ for which equality holds in (7.15) and integrate (7.5) over $(0, x_1)$, using (7.15) over $(0, x_1)$, and derive a contradiction. We omit the details.

We deduce that

$$\int_0^{1/\rho} V^2\,dt = \mathrm{o}\left(\rho/\log^2\rho\right), \tag{7.16}$$

in partial verification of (7.10).

To complete the proof of (7.9-10) we need to estimate

$$V(x), \qquad 1/\rho \le x \le C(\lambda). \tag{7.17}$$

We write

$$\sigma = \rho^{-1} \log^2 \rho \,, \tag{7.18}$$

and claim that

$$V(x) = \mathrm{o}\left(\sigma^{-1} \log(\sigma/x)\right) \tag{7.19}$$

in this interval, as $\rho \to \infty$.

We apply Theorem 2 once more, this time with

$$a = x\,, \qquad c = y = K^{1/2}\sigma\big(\log(\sigma/x)\big)^{-1/2}\,. \tag{7.20}$$

Checking the condition (5.9) needed for Theorem 2, we have

$$\int_x^y w\,dt = 1/|\log x| - 1/|\log y| = \log(y/x)/\big(|\log x|\,|\log y|\big)\,. \tag{7.21}$$

Since, by (7.17),

$$1/\rho \le x \le (K\log^2 \rho)/(\rho \log\log \rho) = (K\sigma)/(\log\log \rho)\,, \tag{7.22}$$

we have

$$|\log x| \sim \log \rho \tag{7.23}$$

as $\rho \to \infty$. This is true also with y replacing x. We have

$$\log y = \log \sigma - (1/2)\log\log(\sigma/x) + (1/2)\log K\,, \tag{7.24}$$

and here

$$\log \sigma = 2\log\log \rho - \log \rho\,, \tag{7.25}$$

while, by (7.22),

$$K^{-1}\log\log\rho \le \sigma/x \le \log^2 \rho\,, \tag{7.26}$$

so that

$$\log(\sigma/x) = \mathrm{o}\,(\log\log\rho)\,. \tag{7.27}$$

Hence

$$|\log y| \sim \log \rho\,. \tag{7.28}$$

We have also that

$$\log|y/x| = \log(\sigma/x) - (1/2)\log\log(\sigma/x) + (1/2)\log K\,, \tag{7.29}$$

and so have from (7.21) that

$$\int_x^y w\,dt \sim \log(\sigma/x)/\log^2 \rho\,. \tag{7.30}$$

Passing to the next factor in (5.9), we have now $r_1(t) = t - x$, and so

$$\begin{aligned}\int_x^y w r_1^2\,dt &< \int_x^y t\,dt/(\log^2 t) \le y^2/(2\log^2 y) \\ &\sim K\sigma^2/\big(2\log(\sigma/x)\log^2\rho\big) \sim K(\log^2\rho)/\big(2\rho^2\log(\sigma/x)\big)\,,\end{aligned} \tag{7.31}$$

by (7.20), (7.28). Combining this with (7.30) we have, for large λ, that

$$\int_x^y |\lambda|^2 w\,dt \int_x^y w r_1^2\,dt \le (1/2)K(1+\mathrm{o}(1))$$

and by the choice (7.7) of K this shows that (7.10) holds for large λ.

We deduce from (5.10) that

$$V(x) = \mathrm{o}\,\Big(\frac{1}{\rho \int_x^y w r_1^2\,dt}\Big)\,, \tag{7.32}$$

so that we now need a lower bound for

$$\int_x^y w r_1^2\,dt = \int_x^y \frac{(t-x)^2}{t\log^2 t}\,dt\,. \tag{7.33}$$

We start by noting that, by (7.20),

$$\frac{y}{x} = \frac{\sigma/x}{\log(\sigma/x)} \to \infty$$

as $\rho \to \infty$, by (7.26). Thus, for large λ, we can bound (7.33) from below by taking the integral over $(2x, y)$ instead of over (x, y). Using the bounds

$$|\log t| > |\log x|\,, \quad t - x \ge t/2\,, \quad (t-x)^2/t \ge t/4\,,$$

for $t \ge 2x$, and the bound $y > 4x$ for large λ, we get as a lower bound for (7.33)

$$\frac{y^2-(2x)^2}{8\log^2 x} \sim \frac{y^2}{8\log^2 \rho} = K\frac{\log^2\rho}{8\rho^2\log(\sigma/x)}\,.$$

Hence

$$\frac{1}{\rho\int_x^y w r_1^2\,dt} = \mathrm{o}\left(\sigma^{-1}\log(\sigma/x)\right).$$

Using this in (7.32) we get the required result (7.19).

We can now prove (7.9-10). In the case of (7.9) we have

$$\sigma/C(\lambda) = K^{-1}\log\log\rho\,,$$

so that (7.19) yields

$$V(C(\lambda)) = \mathrm{o}\left((\rho/\log^2\rho)\log(K^{-1}\log\log\rho)\right),$$

which is equivalent to the required result.

In proving (7.10) we take account of (7.16), and so need to show that

$$\int_x^y V^2(t)\,dt = \mathrm{o}\left(\rho(\log\rho)^{-2}\log\log\log\rho\right). \tag{7.34}$$

Using (7.19), we have that the left of (7.34) is of order

$$\sigma^{-2}\int_x^y \log^2(\sigma/t)\,dt = \sigma^{-1}\int_{1/(\rho\sigma)}^{C(\lambda)/\rho} \log^2 u\,du\,,$$

and here the integral on the right is o(1), the limits of integration being

$$1/(\rho\sigma) = 1/\log^2\rho\,, \qquad C(\lambda)/\rho = K/\log\log\rho\,,$$

both of which are o(1) as $\rho \to \infty$. We deduce that

$$\int_{1/\rho}^{C(\lambda)} V^2(t)\,dt = \mathrm{o}\,(\rho/\log^2\rho)\,,$$

which proves (7.34), and so completes the proof of Theorem 3.

8. The case of small λ. Throughout the foregoing the emphasis was on asymptotics as $|\lambda| \to \infty$, this making it possible in Sections 4, 5 to eliminate the influence of the potential q. If however $q \equiv 0$ we can use the same arguments to study asymptotics of the m-coefficient as $\lambda \to 0$. The details will however vary more from case to case. We confine attention here to a few examples.

We note first one general difference in this case. In the case $\lambda \to \infty$, the imposition of (5.9) with equality implied that $C(\lambda) \to a$; the results thus depended only on the behaviour of the coefficients near the initial point, so to speak the germs of these functions, and were applicable to $D(X, \lambda)$ for any fixed $X \in (a, b]$. If $\lambda \to 0$, we can solve (5.9) with equality if

$$\int_a^b w\,dt = \infty\,, \tag{8.1}$$

and will then have $c(\lambda) \to b$; the results will be applicable to $D(b, \lambda)$.

Passing to examples, we note first the power-type case of §6 in which $a = 0$, $b = \infty$, with (6.1-2) being assumed to hold over $(0, \infty)$. The order results for $c(\lambda)$ and m are formally the same, now with $\lambda \to 0$.

Taking next the case (6.8) of rapidly oscillating $p(x)$, we note first that (8.1) holds, and that the equation is in the limit-point case at ∞, so that there is a unique $m(\lambda)$. In applying Theorem 2, we now have

$$r_1(x) = \int_a^x \sin t^{-1}\,dt \sim \log x\,, \tag{8.2}$$

as $x \to \infty$. We then find that (5.9) leads to a choice of c of the form

$$c(\lambda) \sim K(\epsilon)/|\lambda \log|\lambda||\,, \tag{8.3}$$

where $K(\epsilon) > 0$ depends on the ϵ in (4.1). We then deduce from (5.10-11) that $m(\lambda)$ is precisely of order $|\log|\lambda||$ as $\lambda \to 0$.

Finally, we take the case (7.1). Again (8.1) holds, and the limit-point condition prevails at the upper end $x = 1$. We have here, as $c \to 1-0$,

$$\int_a^c w\,dx = 1/|\log c| \sim 1/(1-c)$$
$$\int_a^c wr_1^2\,dx = \int_a^c x/\log^2 x\,dx \sim 1/(1-c),$$

and (5.9) leads to

$$|\lambda|^2/(1-c)^2 \sim K(\epsilon).$$

The conclusion is now that $m(\lambda)$ remains bounded, and also bounded from zero as $\lambda \to 0$ in a sector (4.1).

9. Application to the spectral function. We now specialize to the study of the limiting Weyl disc $D(b,\lambda)$, $\operatorname{Im}\lambda > 0$, which is the intersection of all $D(X,\lambda)$, $a < X < b$. By the term "m-coefficient" we understand a function $m(\lambda)$, analytic in the open upper half-plane, and satisfying there $m(\lambda) \in D(b,\lambda)$. Of course, in the limit-point case when $D(b,\lambda)$ reduces to a single point, $m(\lambda)$ is unique. In any case, in view of the Nevanlinna property

$$\operatorname{Im} m(\lambda) > 0 \qquad \text{if} \quad \operatorname{Im}\lambda > 0, \tag{9.1}$$

there is a non-decreasing right-continuous function $\tau(\mu)$, the spectral function, with the property that, for $\operatorname{Im}\lambda > 0$,

$$m(\lambda) = A + B\lambda + \int_{-\infty}^{\infty} \left((\mu-\lambda)^{-1} - \mu(1+\mu^2)^{-1}\right) d\tau(\mu) \tag{9.2}$$

which often can be simplified to

$$m(\lambda) = \int_{-\infty}^{\infty} (\mu-\lambda)^{-1}\,d\tau(\mu). \tag{9.3}$$

This spectral function is unique to within an additive constant, for the m-coefficient in question. As is well-known, it can be constructed without reference to $m(\lambda)$, by means of a sequence of Sturm-Liouville problems over (a, b_n) with $b_n \to b$. Some discussion and references are given in [2].

It is clear that (9.2-3) can be used to translate asymptotic information about $\tau(\mu)$ into similar information about $m(\lambda)$. The reverse type of argument was considered in [2], by means of a quantitative version of the Stieltjes inversion formula (see also [16]). Applying this procedure to Theorem 2 we get

Theorem 4. *With the hypotheses of Theorem 2 and for large real $\mu > 0$, let $c(\mu)$ be determined by*

$$\mu^2 \int_a^c w\,dt \int_a^c wr_1^2\,dt = K, \tag{9.4}$$

for some fixed $K > 0$. Then, as $\mu \to \infty$,

$$\tau(\mu) - \tau(-\mu) = O\big(1/\int_a^c w\,dt\big)\,. \tag{9.5}$$

The choice of K does not affect this order-result. The result gives only an upper bound for the growth of the left of (9.5), and does not deal with whether the spectrum is concentrated more on the positive or negative half-axes; the latter question will sometimes be determined by the sign of $p(x)$, when that is fixed.

To indicate the proof, we choose some convenient value of ϵ, say $\pi/4$, and use Theorem 2 to obtain an estimate of $m(\lambda)$ in the sector

$$\pi/4 \le \arg\lambda \le 3\pi/4\,.$$

We then apply Theorem 2.2.1 (or 2.3.1) of [2] (see also [27] for a partial result in the same direction) with

$$\Lambda_1 = -\mu + i\mu\,, \qquad \Lambda_2 = \mu + i\mu\,.$$

A rather similar result was obtained in [3], in which c in (9.5) was determined by

$$\mu\int_a^c w_1(t)|r(t)|\,dt = 1/8\,,$$

where

$$w_1(x) = \int_a^x w(t)\,dt\,;$$

the choice of the constant on the right does not affect the order result. It is not clear that either of these results includes the other.

Similar reasoning can be applied to the case (7.1), for which the asymptotic formula for m leads to an asymptotic formula for $\tau(\mu)$, namely

$$\tau(\mu) \sim \log\mu \qquad \text{as} \qquad \mu \to \infty\,.$$

Again we use Theorem 2.2.1 of [2], but now make $\epsilon \to 0$ as $\mu \to \infty$. We omit the details. In this case, since $p(x) > 0$, $q(x) = 0$, there will be no negative spectrum.

Acknowledgements. An earlier version of this paper was completed in 1985 during a period as visitor to the Department of Mathematics, University of Birmingham, England; the author is grateful for the hospitality of the Department and of Professor W.N. Everitt, and for the support of the Science and Engineering Research Council (U.K.). The present version, which uses quite different reasoning, was written during a visit to the Mathematics and Computer Science Division, Argonne National Laboratory (host Dr. H.G. Kaper). Many helpful discussions were had with Dr. C. Bennewitz (Uppsala), Professor Everitt and Dr. Kaper, who provided comments on earlier versions of this paper. The author is grateful for the opportunity to see pre-publication copies of extensive work of Dr. Bennewitz. Appreciation is also expressed for the continuing support of the National Sciences and Engineering Research Council of Canada, under Grant # A-3979.

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