Differential and Integral Equations, Volume 1, Number 1, January 1988, pp. 71–78.

ON RADIAL SOLUTIONS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

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Dedicated to Professor L.K. Jackson on the occassion of his 65th birthday

Abstract. This paper is concerned with the existence of radial positive solutions, which satisfy certain boundary conditions at infinity, of a class of semilinear elliptic equations. Our methods are a combination of differential inequalities techniques for ordinary and partial differential equations.

I. Introduction. In the analysis in $\mathbf{R}^n$ of nonlinear partial differential equations of the form

$$\frac{1}{b^2(r)}\nabla\cdot(b^2(r)\nabla y)+h(r,y)=0, \tag{1.1}$$

with r the radial variable, one is naturally led to the consideration of radially symmetric solutions which satisfy an associated second order nonlinear ordinary differential equation of the form

$$\frac{1}{a^2(r)}\frac{d}{dr}(a^2(r)\frac{dy}{dr})=k(r,y),\quad 0<r<\infty, \tag{1.2}$$

along with certain boundary conditions. In particular, in the case, $b(r)=1$, $a(r)=r^{(n-1)/2}$, so that (1.2) is a singular differential equation at $r=0$. For the case $n=3$ this equation becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dy}{dr})=k(r,y). \tag{1.3}$$

Gregus [G] has considered this equation for the case $k(r,y)=f(y)-g(r)$, where $f(y)=\sinh y$, $g(r)=\frac{1}{\alpha\beta^3}\exp(-r/\beta)$, and showed that (1.3) has a positive solution y satisfying

$$y(0)=y_0\geq 0,\quad y'(0)=0,\quad y(\infty)=y'(\infty)=0, \tag{1.4}$$

Received August 20, 1987.
Keywords. Semilinear elliptic problem, radial solutions, singular boundary value problems.
AMS(MOS) Subject Classifications: 34B15, 35J25, 35J65.

whenever $\alpha,\ \beta \geq 0, \quad \beta \leq e$. In addition comparison and uniqueness results were obtained in [G], and the techniques used there involved applications of results of Kiguradze, [K], on the existence of solutions of certain singular initial value problems, and in addition the special nature of the functions f and g.

In this paper we shall analyze the more general equation (1.2), subject to certain boundary conditions, by means of differential inequality techniques for ordinary and partial differential equations. In fact, we obtain a general result (theorem 2.1) which will guarantee the existence of certain radial solutions of

$$\Delta y + h(r, y) = 0, \tag{1.5}$$

in all space dimensions, except $n = 2$ and $n = 4$, where $a(r) = \sqrt{r}$, respectively, $a(r) = \sqrt{r^3}$. However, we shall be able to treat these exceptional cases nevertheless, (theorem 2.2), by using techniques from the theory of second order elliptic partial differential equations.

II. Existence theory. We consider the nonlinear differential equation

$$\frac{1}{a^2(r)}\frac{d}{dr}(a^2(r)\frac{dy}{dr}) = f(y) - g(r), \tag{2.1}$$

subject to one or more of the following boundary conditions

$$y(0) = y_0 \geq 0, \quad y'(0) = 0, \tag{2.2}$$

$$\lim_{r\to\infty} y(r) = 0 = \lim_{r\to\infty} y'(r). \tag{2.3}$$

We shall assume that $a \in C^2(0,\infty) \cap C[0,\infty)$ with

$$a(0) = 0, \quad a(r) > 0, \quad a'(r) \geq 0, \quad \text{if } r > 0 \tag{2.4}$$

and

$$\lim_{r\to\infty} a(r) = +\infty. \tag{2.5}$$

In addition, we shall suppose that $f, g : \mathbf{R} \to \mathbf{R}$ are such that

$$f(0) = 0, \quad f \in C^1(\mathbf{R}), \quad f(y) > 0, \quad \text{if } y > 0, \tag{2.6}$$

and

$$g(r) > 0, \quad g'(r) < 0, \quad g \in C^1[0,\infty). \tag{2.7}$$

Using the substitution $y = u/a(r)$, we may write (2.1) as

$$u'' = F(r, u), \tag{2.8}$$

where

$$F(r, u) = a(r)f(\frac{u}{a(r)}) - a(r)g(r) + \frac{a''(r)}{a(r)}u. \tag{2.9}$$

We observe that F is continuously differentiable with respect to u and is continuous with respect to r, where

$$\frac{\partial F}{\partial u} = f'(\frac{u}{a(r)}) + \frac{a''(r)}{a(r)}, \quad r > 0, \quad u \in \mathbf{R}. \tag{2.10}$$

Since $F(r,0) = -a(r)g(r) \leq 0$, $r > 0$, we conclude that $\alpha(r) \equiv 0$ is a lower solution of (2.8) (cf [S]). Also for any $\sigma > 0$, $F(r,\sigma) \geq 0$ if and only if

$$f(\frac{\sigma}{a(r)}) + \frac{\sigma}{a(r)}\frac{a''(r)}{a(r)} \geq g(r), \quad r > 0, \tag{2.11}$$

so that $\beta(r) \equiv \sigma$ is an upper solution of (2.8), whenever (2.11) holds. Furthermore, if $\psi(r) \equiv ka(r)$, $k > 0$, then

$$F(r,\psi(r)) = a(r)f(k) - a(r)g(r) + ka''(r) \geq \psi''(r) = ka''(r)$$

if and only if

$$f(k) \geq g(r), \quad 0 < t < \infty. \tag{2.12}$$

Hence, if (2.11) and (2.12) hold, then

$$\eta(r) \equiv \min\{ka(r), \sigma\} \tag{2.13}$$

is also an upper solution of (2.8) for $r > 0$, (cf. [S]).

We are now in position to state and prove a preliminary result.

Theorem 2.1. *Assume that $a \in C^2[0,\infty)$ and that (2.4) - (2.7) hold. Further let there exist constants $\sigma > 0$ and $k > 0$ such that (2.11) and (2.12) hold. Then the boundary value problem*

$$u'' = F(r,u), \quad u(0) = 0 \tag{2.14}$$

has a solution $u_0 \in C^2[0,\infty)$ with $0 \leq u_0(r) \leq \eta(r), \quad 0 \leq r < \infty$.

Proof: For each $n \geq 1$ there exists a solution $u_n \in C^2[1/n, +\infty)$ of the boundary value problem

$$u'' = F(r,u),\ u(1/n) = ka(1/n),\ 0 \leq u_n(r) \leq \eta(r),\ r \geq 1/n. \tag{2.15}$$

We claim there exists a subsequence of $\{u_n\}$ converging to a solution of (2.14) with $0 \leq u_0'(0) \leq ka'(0)$. Since $u_n'(1/n) \leq ka'(1/n)$ and since $|u_n'(r)| \leq M$ for some $M > 0$, $r \geq 1/n$, for all n, it follows from the Ascoli Arzela theorem and a diagonalization argument that $\{u_n\}$ has a subsequence which converges uniformly on compact subintervals of $(0,\infty)$ to a solution u_0 of (2.14). It follows that $\lim_{r\to 0+} u_0'(r) = u_0'$ exists (by the Cauchy criterion) and that $0 \leq u_0'(0) \leq ka'(0)$ so that $u_0 \in C^2[0,\infty)$ holds.

We note therefore that the substitution $y(r) = u_0(r)/a(r)$ yields a solution of the differential equation (2.1) which by (2.5) must satisfy $\lim_{r\to\infty} y(r) = 0$. In order that the other boundary conditions will be satisfied also, additional conditions will be imposed on $a(r)$ (see theorem 2.3, below). In particular, if, as in [G], $f(y) = \sinh y$, $g(r) = (1/(\alpha\beta^3)\exp(-r/\beta)$, $a(r) = r$, $\alpha, \beta > 0$, it is easy to check that the conditions of theorem 2.1 hold for some $\sigma > 0$, $k > 0$. Hence, we may conclude the existence of a solution of (1.3) for all α, $\beta > 0$, whereas in [G] it was assumed that $\beta \leq e$. Using the equation (2.8) and l'Hôpital's rule, we conclude that the solution $y(r) = u_0(r)/r$ satisfies $y(0) = u_0'(0) \geq 0$ and $y'(0) = 0$. It also easily follows that $\lim_{r\to\infty} y'(r) = 0$. In this particular example we have found, in dimension 3, a positive radially symmetric solution of (1.1) in the case that $b(r) \equiv 1$, and we note that (for this particular situation) the hypothesis that $a \in C^2[0,\infty)$ is satisfied in all dimensions, except, $n = 2$ and $n = 4$. For these cases, we proceed directly, by considering the partial differential equation. We are thus able to prove an existence result for the partial differential equation

$$\Delta y = f(y) - g(r), \tag{2.17}$$

where f, g satisfy (2.6), (2.7), in all space dimensions.

If $y : \mathbf{R}^n \to \mathbf{R}$ is a spherically symmetric solution of (2.17), then $y = y(r)$, where as before r denotes the radial variable, solves the equation

$$r^{1-n}\frac{d}{dr}(r^{n-1}\frac{dy}{dr}) = f(y) - g(r), \quad 0 < r < \infty. \tag{2.18}$$

Therefore, to obtain a radial solution of (2.18), we solve (2.17). The following theorem covers this situation.

Theorem 2.2. *Let f, g satisfy (2.6), (2.7). Then (2.18) has a solution $y = y(r)$ satisfying the boundary conditions*

$$y(0) = y_0, \quad y'(0) = 0, \quad \lim_{r\to\infty} y(r) = 0.$$

Proof: Let us set

$$\alpha(x) \equiv 0, \quad \beta(x) \equiv k, \quad r = |x|, \quad x \in \mathbf{R}^n, \tag{2.20}$$

then

$$\Delta\alpha \geq f(\alpha) - g(r), \qquad \Delta\beta \leq f(\beta) - g(r),$$

as follows from (2.6), (2.7), and (2.12). Hence, α and β are lower and upper solutions of the Dirichlet problem

$$\begin{cases} \Delta y = f(y) - g(r), & |x| < R \\ y = 0, & |x| = R, \end{cases} \tag{2.21}$$

and so (2.21) has a classical maximal solution y_R with $0 \leq y_R(x) \leq k$ (see [S]). Because of maximality, we conclude in fact that y_R must be spherically symmetric. Now letting $R \to \infty$, we obtain, via a compactness argument, that there exists a spherically symmetric solution y of (2.17). If $R_2 > R_1$ one can easily show that the function

$$\alpha(x) = \begin{cases} y_{R_1}(x), & |x| \leq R_1 \\ 0, & |x| \geq R_1, \end{cases}$$

is a lower solution of (2.21) for $R = R_2$. We hence conclude the monotonicity condition

$$R_2 > R_1 \Rightarrow y_{R_2} \geq y_{R_1}, \quad |x| \leq R_1,$$

and hence also

$$y(x) \geq y_R(x), \quad |x| \leq R.$$

Also y is a solution of (2.18). Since $\lim_{r\to\infty} g(r) = 0$, we conclude by a simple indirect argument, that $y(r) \to 0$ as $r \to 0$. Moreover, since y is spherically symmetric, it follows that $y'(0) = 0$, completing the proof.

Imposing additional assumptions on a, we can analyze the behavior of the solution of (2.1) in a neighborhood of $r = 0$ and $r = \infty$ and obtain that the other boundary conditions (2.2), (2.3) also may be satisfied. These are the contents of our next result.

Theorem 2.3. *Assume that the hypotheses of theorem 2.1 are satisfied and that $a'(r)/a(r)$ is bounded as $r \to \infty$ and that $\lim_{r\to 0} a(r)/a'(r) = 0$. Then equation (2.1) has a solution satisfying (2.2) and (2.3).*

Proof: Since $a(0) = 0$, we consider the three cases

(1) $a'(0) > 0$
(2) $a'(0) = 0 < a''(0)$
(3) $a'(0) = 0 = a''(0)$.

Suppose that $a'(0) > 0$. Using l'Hôpital's rule we obtain

$$\lim_{r\to 0+} y(r) = \lim_{r\to 0+} \frac{u_0(r)}{a(r)} = \frac{u_0'(0)}{a'(0)} = y_0. \tag{2.22}$$

Furthermore, we have

$$\begin{cases} \lim\limits_{r\to 0+} \dfrac{y(r)-y_0}{r} = \lim\limits_{r\to 0+} \dfrac{a(r)u_0'(r) - u_0(r)a'(r)}{a^2(r)} \\ \qquad = \lim\limits_{r\to 0+} \dfrac{a(r)u_0''(r) - u_0(r)a''(r)}{2a(r)a'(r)} \\ \qquad = \lim\limits_{r\to 0+} \dfrac{a(r)f(\frac{u_0(r)}{a(r)}) - a(r)g(r)}{2a'(r)} \\ \qquad = 0, \end{cases} \tag{2.23}$$

so that (2.2) holds. Since $u_0(r)$ and $u_0''(r)$ are both bounded on $[0,\infty)$ it follows from Landau's inequality (see [H]) that $u_0'(r)$ is also bounded on $[0,\infty)$. Further, (2.5) implies that $\lim_{r\to\infty} u_0'(r)/a(r) = 0$. Hence we conclude that

$$\lim_{r\to\infty} y'(r) = \lim_{r\to\infty} \frac{a(r)u_0'(r) - u_0(r)a'(r)}{a^2(r)} = 0.$$

Now, since $\lim_{r\to\infty} u_0(r)/a(r) = 0$ and $a'(r)/a(r)$ is bounded for large r, the result follows.

For case (2), we need to apply l'Hôpital's rule one more time in (2.22). Since $a'(0) = 0$ and $0 \le u_0(r) \le a(r)$, we have that $u_0'(0) = 0$. Hence, in (2.22), we have that

$$\lim_{r\to 0+} y(r) = \lim_{r\to 0+} \frac{u_0''(r)}{a''(r)} = \frac{u_0''(0)}{a''(0)} = y_0 \ge 0. \tag{2.24}$$

Note that in (2.23) the limit is again 0 since $a/a' \to 0$ as $r \to 0$. This completes the proof of part (2) since condition (2.3) holds as in part (1) above.

Suppose finally that $a'(0) = a''(0) = 0$. Since $0 \le u_0(r) \le a(r)$ we have that $u_0(0) = 0 = u_0''(0)$. We need to show first that $\lim_{r\to 0+} y(r) = \lim_{r\to 0+} u_0(r)/a(r)$ exists. If not, then let $m \equiv \limsup_{r\to 0+} u_0(r)/a(r)$ and l the liminf of the same quantity, with $m > l \ge 0$. We may then find sequences $\{r_n\}$, $\{\tau_n\}$ with $r_n \downarrow 0$, $\tau_n \downarrow 0$, $r_{n+1} < \tau_n < r_n$ for all n and such that

$$\begin{cases} y'(r_n) = 0, \ y''(r_n) \le 0, \ \text{(maxima)} \\ y'(\tau_n) = 0, \ y''(\tau_n) \ge 0, \ \text{(minima)}. \end{cases} \tag{2.25}$$

Notice that $y' = 0 \Leftrightarrow au_0' = u_0 a'$ and we easily find that

$$a^2 y'' = au'' - ua'', \text{when } y' = 0. \tag{2.26}$$

Since $y(\tau_n) < y(r_n)$ we have that $f(y(r_n)) \geq f(y(\tau_n))$ and from (2.25), (2.26) and (2.9) it follows that

$$\begin{cases} y''(r_n) \leq 0 \implies f(y(r_n)) \leq g(r_n) \\ y''(\tau_n) \geq 0 \implies f(y(\tau_n)) \geq g(\tau_n). \end{cases} \tag{2.27}$$

Since $g(r_n) < g(\tau_n)$ by (2.7), (2.27) implies $f(y(r_n)) < f(y(\tau_n))$, a contradiction. Therefore, $\lim_{r\to 0+} y(r) = \lim_{r\to 0+}(u_0(r)/a(r)) \equiv y_0 \geq 0$ exists. Again as in part (2), the limit in (2.23) is 0 since $a/a' \to 0$ as $r \to 0$ and the condition (2.3) as in parts (1) and (2). This completes the proof.

A straightforward application of Gronwall's inequality yields that, if $a(r)$ is linear near $r = 0$ (so that $a''(0) = 0$) (or more generally, if $a''(r)/a(r)$ is bounded near $r = 0$), the boundary value problem

$$\begin{cases} u'' = F(r,u) = a(r)f(\dfrac{u}{a(r)}) - a(r)g(r) + \dfrac{a''(r)}{a(r)}u \\ u(0) = 0, \quad u'(0) = u_0 \geq 0 \end{cases} \tag{2.28}$$

has at most one solution. The general situation requires a somewhat more detailed analysis, which we shall not pursue here.

We next wish to establish a comparison result for the case that $a'(0) > 0$.

Theorem 2.4. *Let u, w be solutions of (2.8) satisfying*

$$u(0) = w(0) = 0, \qquad 0 < u'(0) < w'(0), \tag{2.29}$$

which exist on $[0, b)$, $\quad b \leq \infty$. Further assume that

$$a(r)f'(\frac{u}{a(r)}) + a''(r) \geq 0, \qquad r > 0, \qquad 0 \leq u \leq \eta(r). \tag{2.30}$$

Then $u(r) < w(r)$ on $(0, b)$.

Proof: Because of (2.29) we have that $u(r) < w(r)$ for r sufficiently small. If there exists $r_0 > 0$ with $u(r_0) = w(r_0)$ and $0 < u(r) < w(r)$ on $(0, r_0)$, then using the representation

$$\begin{cases} u(r) = \displaystyle\int_0^r (r-s)F(s,u(s))ds + u'(0)r \\ w(r) = \displaystyle\int_0^r (r-s)F(s,w(s))ds + w'(0)r \end{cases} \tag{2.31}$$

we obtain

$$w(r) - u(r) = \int_0^r (r-s)\frac{\partial F}{\partial u}(s,\eta(s))(w(s) - u(s))ds + (w'(s) - u'(s))r, \tag{2.32}$$

and since $\frac{\partial F}{\partial u} \geq 0$ by (2.30) (cf. (2.10)), we obtain a contradiction. This proves the result.

We next consider the question of uniqueness of solutions of the boundary value problem

$$\begin{cases} u'' = F(r,u), \quad u(0) = 0, \quad u'(\infty) = 0 \\ 0 \leq u(r) \leq \eta(r), \quad r > 0. \end{cases} \tag{2.33}$$

Theorem 2.5. *Let all hypotheses of theorem 2.1 hold and let (2.30) hold. Further, assume that*

$$\lim_{r\to\infty} a(r)g(r) = 0 = \lim_{r\to\infty} \frac{a''(r)}{a(r)} \tag{2.34}$$

and

$$\lim_{r\to\infty} a(r)f(\frac{k}{a(r)}) = 0, \qquad f'(y) \geq 0, \quad y \geq 0, \tag{2.35}$$

(where $k > 0$ is as in (2.12)). Then problem (2.33) has a unique solution.

Proof: Using theorem 2.1 we obtain a solution $u(r)$ of $u'' = F(r,u)$, $u(0) = 0$, with $0 \leq u(r) \leq \eta(r)$, on $[0,\infty)$. It follows from (2.9), (2.34), and (2.35) that $u''(r) \to 0$ as $r \to \infty$. Hence, using once more Landau's inequality (which is valid on any half-line (see [H])) we may conclude that $u'(r) \to 0$ as $r \to \infty$. Therefore the boundary value problem (2.33) has a solution. To see that this solution is unique, suppose that $u(r)$, $w(r)$ are two such solutions. Then, as in [G], using the Green's function

$$G(r,s) = \begin{cases} -r, & 0 \leq r \leq s \\ -s, & s \leq r \leq \infty \end{cases} \tag{2.36}$$

for the boundary value problem

$$u'' = 0, \qquad u(0) = 0, \quad u'(r) \to 0, \quad r \to \infty, \tag{2.37}$$

We may write the difference of the two solutions as

$$u(r) - w(r) = \int_0^r s^2(F(s,w(s)) - F(s,u(s)))ds + \int_r^\infty rs(F(s,w(s)) - F(s,u(s)))ds. \tag{2.38}$$

If now $u(r) > w(r)$ for $r > 0$ in a right neighborhood of 0, it follows from theorem 2.4 that $u(r) > w(r)$ for all $r > 0$. But then both integrals in (2.38) are nonpositive by (2.30). This contradiction completes the proof.

Corollary 2.6. *Let the hypotheses of theorem 2.5 hold and let $u = u(r)$ be the solution of the boundary value problem (2.33) with $u'(0) = u_0 > 0$. Assume also that $\int_0^\infty a(s)g(s)ds < \infty$. Then*

$$u_0 \leq \min\{ka'(0), \int_0^\infty a(s)g(s)ds\}. \tag{2.39}$$

Proof: From the differential equation (2.8) we obtain

$$u'(r) = u'(0) + \int_0^r a(s)f(\frac{u(s)}{a(s)})ds + \int_0^r \frac{a''(s)}{a(s)}ds - \int_0^r a(s)g(s)ds \tag{2.40}$$

so that

$$u'(r) > u'(0) - \int_0^r a(s)g(s)ds, \qquad r > 0. \tag{2.41}$$

Since $u'(r) \to 0$ as $r \to \infty$, we have

$$u'(0) \le \int_0^\infty a(s)g(s)ds, \tag{2.42}$$

which gives (2.39) (since $u'(0) < ka'(0)$ must also hold).

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