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A RANDOM SCHROEDINGER EQUATION: WHITE NOISE MODEL

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Abstract. This paper is essentially an application of the author's theory of abstract stochastic bilinear equations to the problem of laser beam propagation in a turbulent medium, and the associated random Schroedinger equation. The white noise theory is shown to provide a consistent self-contained model for the Markov approximation of the refractive index field, and in particular avoids invoking ad hoc Stratanovich correction terms.

1. Introduction. The problem of evaluating the phase and amplitude fluctuations of a laser beam in refractive-index turbulence is important to many applications, in particular to adaptive optics correction. There is an extensive literature on the subject; a recent review is presented in [8]. The treatise of Tatarskii [6] is a recognized landmark reference on the subject.

In this paper we study the so-called Markov approximation [6] to the turbulence-field which we model by the finitely additive white noise theory initiated by the author [1]. Previous mathematical treatments–notably [7] –use an Ito model with an ad hoc "Stratanovich correction" term. Although our results for the moments are not new, our technique justifies the interpretation of the results as the limiting case when the bandwidth is allowed to expand in arbitrary fashion. The stationarity of the turbulence field appears to be essential for this, as well as a degree of smoothness of the covariance.

Central to the theory is the random Schroedinger equation – a bilinear stochastic partial differential equation of the kind studied initially in [1]. The current paper can in fact be viewed as an application/extension of that work. Referring to [8] for details, we shall now review briefly the genesis of this equation.

With E denoting the electric field, the usual wave equation in the standard notation takes the form

$$\mu\epsilon\frac{\nabla^2 E}{\partial t^2} = \nabla^2 E + \nabla(E \times \nabla \log \epsilon) \tag{1.1}$$

where

$$E = E(t, x, y, z)$$

$\epsilon = \epsilon(x, y, z)$ dielectric constant of medium

$\mu =$ permittivity, which we take to be a constant.

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The main interest is in the dielectric constant $\epsilon(\cdot)$ which constitutes a "random field." Assuming that ϵ changes rather slowly, the first approximation of (1.1) is to neglect the $\nabla \log \epsilon$ term – this goes back to Tatarskii [6] – leading to:

$$\mu\epsilon \frac{\partial^2 E}{\partial t^2} = \nabla^2 E. \tag{1.2}$$

Next we take care of the time dependence by the sinusidal representation:

$$E(t,x,y,z) = E(x,y,z)e^{-i\omega t}$$

where the angular frequency ω is large (optical). This yields for the spatially dependent field:

$$-\omega^2 \mu\epsilon E = -k^2 n^2 E = \nabla^2 E$$

where k is the wave number and n is the index of refraction:

$$n^2 = c^2 \mu\epsilon$$

c being the velocity of light, and the spatial dependence is now expressed by:

$$\nabla^2 E + k^2 n^2 E = 0 \tag{1.3}$$

where n^2 is a random field. Since (1.3) may be broken down to 3 component equations, it is enough to study any one of them. Hence we may consider the scalar equation:

$$\nabla^2 E + k^2 n^2 E = 0.$$

We assume next that the wave is propagating in the z-axis direction, so that we can write:

$$E(x,y,z) = V(x,y,z)e^{ikz}$$

where $V(x,y,z)$ is slowly varying in z. Then we obtain

$$\frac{\partial^2 V}{\partial z^2} + 2ik\frac{\partial V}{\partial z} + k^2(n^2-1)V + \Delta V = 0$$

where

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}.$$

Since by assumption, V is slowly varying in z and $k \approx 10^6$, for optical frequencies we may use the approximation:

$$\frac{\partial}{\partial z}\left[\frac{\partial V}{\partial z} + 2ikV\right] \approx 2ik\,\frac{\partial V}{\partial z}$$

to obtain finally:

$$2ik\frac{\partial V}{\partial z} + \Delta V + k^2(n^2-1)V = 0. \tag{1.4}$$

(Implicit in our assumption is that there is no back-scatter, so this is really the forward-scatter equation.) The nonhomogeneity of the medium is expressed by

$$(n^2 - 1) = \nu$$

where ν can be modelled as a zero mean Gaussian random field. It is natural to assume that the field is isotropic and homogeneous (i.e., the statistical properties are invariant under translation or rotation). Thus (1.4) is our final form of the stochastic partial differential equation, and taking advantage of the unique role played by the z-coordinate, we may cast it as

$$2ik \frac{\partial V}{\partial z} = -\Delta V - k^2 \nu V , \qquad z > 0 \tag{1.5}$$

and view it as a Cauchy problem. We recognize (1.5) as a Schroedinger-type equation, with initial data given at $z = 0$: a "random wave equation." In the Markov approximation (Tatarskii [6]) it is assumed that $\nu(\cdot)$ is "delta correlated" in the z-direction. We model $\nu(\cdot)$ as "white noise" in the z-direction, so that we can justify it as "high badnwidth noise" in the z-direction. We see that (1.5) is a bilinear stochastic partial differential equation of the kind initiated in [1].

We begin in Section 2 recalling basic definitions and properties of polynomials mapping one Hilbert space into another since we use polynomial expansion as the basic tool. Section 3 reviews essential notions of the finitely additive white noise theory, in particular of white noise polynomials and their orthogonalization leading to Ito polynomials. The main results are in Section 4 which develops an appropriate abstract version of the random Schroedinger equation, leading to an orthogonal polynomial expansion of the solution. Differential equations for the moments (coherence functions) are deduced therefrom.

2. Polynomials. In this section we review the basic definitions and relevant results on polynomials mapping one separable Hilbert Space H_1 into another Hilbert space H_2. See [1] for more details.

Let $K_n(\dots)$ denote a symmetrice n-linear form mapping H_1 into H_2 :

$$y = K_n(x_1, \dots, x_n) , \qquad x_i \in H_1 .$$

We shall say that such an n-linear form for $n \geq 2$ is nuclear, if for every $cos\{\phi_i\}$ in H_1 the following two properties hold:

$$\text{(i)} \qquad \sum_{i_1} \cdots \sum_{i_n} \|k_n(\phi_{i_1}, \dots, \phi_{i_n})\|^2 < \infty \tag{2.1}$$

($K_n(\dots)$ is Hilbert-Schmidt, in other words.)

$$\text{(ii)} \qquad \text{For each} \quad \nu,\ \nu \leq [n/2]$$
$$\sum_{i_1} \cdots \sum_{i_\nu} K_n(\phi_{i_1}, \phi_{i_1}, \dots, \phi_{i_\nu}, \phi_{i_\nu}, x_{2\nu+1}, \dots, x_n) \tag{2.2}$$

converges weakly in H_2 to a limit which is independent of the cos chosen, and defines an $(n - 2\nu)$ linear form (denote it)

$$K_{n,n-2\nu}(x_{2\nu+1}, \dots, x_n)$$

which is Hilbert-Schmidt. Note that if $K_n(\cdot)$ is nuclear, so is $K_{n,n-2\nu}(\cdot)$.

Our technique for proving property (2.2) will be to show that an appropriate linear integral operator is nuclear. Recall that if L is a nuclear operator mapping a separable Hilbert space H into H, then for any cos$\{\phi_i\}$ in H we have

$$\sum_1^\infty |[L\phi_i\,,\,\phi_i]| < \infty$$

and

$$\text{Trace } L = \sum_1^\infty [L\phi_i\,,\,\phi_i]$$

independent of the cos chosen.

We now state and prove that under sufficient smoothness conditions on the kernel – of the kind we can verify for our needs – a linear integral operator will be nuclear.

Theorem 2.1. *Let H be a separable Hilbert space and let $W(T)$ denote the L_2-space:*

$$W(T) = L_2\big((0,T)\,;\,H\big)\,, \qquad 0 < T < \infty\,.$$

Fix T. Let $M_1(t)$, $M_2(t,s)$ be Hilbert-Schmidt operators on H into H, strongly continuous on $0 \le s \le t \le T$ and suppose that

$$\sup_{0\le s\le t\le T} \|M_2(t,s)\|^2_{H-S} + \|M_1(s)\|^2_{H-S} < \infty\,.$$

Define the operator L on $W(T)$ into $W(T)$, by

$$Lf = g\,; \qquad g(t) = \int_0^t M_1(t)\Big(\int_\sigma^t M_2(s,\sigma)\,f(\sigma)\,ds\Big)\,d\sigma\,.$$

Then L is nuclear and its trace is zero.

Proof: Define the operators L_1 and L_2 on $W(T)$ into $W(T)$ by:

$$L_1 f = g\,; \qquad g(t) = M_1(t)\int_0^t f(s)\,ds \qquad 0 \le t \le T$$

$$L_2 f = g\,; \qquad g(t) = \int_0^t M_2(t,s)\,f(s)\,ds \qquad 0 \le t \le T\,.$$

Then both L_1 and L_2 are $H-S$ and we can verify that L is their product:

$$L = L_1\,L_2$$

and hence is nuclear. Further being Volterra, its trace has to be zero.

3. White noise polynomials and Ito polynomials. We begin by recalling briefly some essential notions from white noise theory [2]. See [3] for recent mathematical extensions. Let H denote a separable Hilbert space (we shall use $[\,,]$ generically to denote inner products in a Hilbert space) and for each t, $0 < T < \infty$, let $W(T)$ denote the L_2-space over H :

$$W(T) = L_2\big((0,T)\,;\,H\big)\,.$$

By (finitely additive) white noise in $W(T)$, we mean the process with sample paths $\omega(\cdot)$ in $W(T)$, with Gauss measure on the cylinder sets (with Borel bases) with the characteristic function defined by

$$C(h) = E\big[exp(i\int_0^T [N(t), h(t)]\,dt)\big] = exp\big(-\frac{1}{2}\int_0^T [h(t), h(t)]\,dt\big)\,.$$

This measure cannot be extended to be countably additive on (the Borel sets of) $W(T)$. Let $f(\cdot)$ denote any Borel measurable function mapping $W(T)$ into another (separable) Hilbert space H_r. In general it need not define a distribution over H_r. We shall need to consider only a special class of such functions which do in an appropriate limiting sense. Thus let P_N be any sequence of finite-dimensional projections on $W(T)$ into $W(T)$ such that P_N converges strongly to the identity. Then of course for each N, $f(P_N\omega)$ is a random variable. Suppose now that

$$\{f(P_N\omega)\}$$

is Cauchy in probability. Then

$$C(h) = \operatorname*{limit}_{n}\, C_N(h)\,, \qquad h \in H_r \tag{3.1}$$

where

$$C_N(h) = E\big[exp\big(i[f(P_N\omega), h]\big)\big]$$

defines a countably additive measure on (the Borel sets of) H_r. If the limiting characteristic function $C(h)$ is independent of the particular sequence of projections P_N, we call $f(\cdot)$ a physical random variable (Prv) and the corresponding limiting measure μ_f is defined as the distribution of $f(\cdot)$. We recall in this connection the Prokhrov theorem (see [4]): for every Borel set B in H_r :

$$\mu_n(B) \to \mu_f(B) \qquad \text{if} \qquad \mu_f(\partial B) = 0 \tag{3.2}$$

where $\mu_N(\cdot)$ is the distribution of $f(P_N\omega)$.

Let us next specialize the functions $f(\cdot)$ to polynomials. Let $K_n(\dots)$ denote a continuous symmetric n-linear form with range in H_r as in section 2 and define

$$p_n(\omega) = K_n(\omega, \omega, \dots, \omega) \tag{3.3}$$

This is a homogeneous polynomial of degree n and every such polynomial can be so expressed. We shall call $p_n(\cdot)$ nuclear if $K_n(\cdot)$ is. If $p_n(\cdot)$ defines a Prv, we shall call it a white-noise (W-N) polynomial. Our basic result is:

Theorem 3.1. *A homogeneous polynomial $p_n(\cdot)$ defines a physical random variable if the corresponding symmetric n-linear form $K_n(\cdot)$ is nuclear.*

Proof: The statement needs no proof for degree zero. For degree 1,

$$p_1(\omega) = L\omega\,,$$

where L is a linear bounded operator, it is necessary and sufficient for $p_1(\cdot)$ to be a prv that L is Hilbert-Schmidt as shown in [2]. Note that for any sequence P_N of projections converging strongly to the identity:

$$p_1(P_N\omega) = L(P_N\omega)$$

is Cauchy in the mean of order two. In particular

$$E([L\omega, h]) = \operatorname*{limit}_n E([LP_N\omega, h]) = 0$$
$$E|[L\omega, h]|^2 = \operatorname*{limit}_n E|[LP_N\omega, h]|^2 = [LL^*h, h] .$$

Note that any W-N polynomial of degree one is orthogonal to any W-N polynomial of degree zero, in the sense that

$$E\big([p_1(\omega), h][p_0(\omega), g]\big) = 0$$

for g, h in H_r. Let us next consider a polynomial of degree n, $n \geq 2$. Fix a strongly convergent sequence P_N of projections approximating the identity operator. Let

$$p_n^N(\omega) = p_n(P_N\omega) .$$

Let P_N be spanned by the orthonormal basis $\phi_1, \ldots, \phi_N$. Then $K_n(\cdot)$ denoting the symmetric n-linear form corresponding to $p_n(\cdot)$, we have

$$p_n(P_N\omega) = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} K_n(\phi_{i_1}, \ldots, \phi_{i_n})[\omega, \phi_{i_1}] \ldots [\omega, \phi_{i_n}] . \tag{3.4}$$

Let us consider more generally the cylinder-set measurable function

$$p_n^N(\omega) = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} a_{i_1, i_2, \ldots, i_n}[\omega, \phi_{i_1}] \ldots [\omega, \phi_{i_n}]$$

where the (H_r-valued) coefficients are symmetric in the indices. For any ψ in H_r,

$$\big[p_n^N(\omega), \psi\big] = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} b_{i_1, i_2, \ldots, i_n}[\omega, \phi_{i_1}] \ldots [\omega, \phi_{i_n}]$$

where

$$b_{i_1, i_2, \ldots, i_n} = \big[a_{i_1, i_2, \ldots, i_n}, \psi\big]$$

defines a scalar polynomial in the independent (0,1) Gaussian random variable

$$\varsigma_i = [\omega, \phi_i] , \qquad i = 1, \ldots, N$$

and hence by the usual theory of Hermite polynomials (or otherwise) we know that

$$E\big[\big|\big(p_n^N(\omega), \psi\big)\big|^2\big] = n! \sum_{i_1}^{N} \cdots \sum_{i_n}^{N} \big|b_{i_1, i_2, \ldots, i_n}\big|^2 +$$

$$\sum_{i=\nu}^{[n/2]} \frac{1}{(n-2\nu)!}\Big(\frac{n!}{2^\nu \nu!}\Big)^2 \sum_{i_{2\nu+1}}^{N} \cdots \sum_{i_n}^{N} \Big| \sum_{i_1}^{N} \cdots \sum_{i_\nu}^{N} b_{i_1, i_1, i_2, i_2, \ldots, i_\nu, i_\nu, i_{2\nu+1}, \ldots, i_n} \Big|^2$$

and hence

$$
\begin{aligned}
&E\big(\|p_n^N(\omega)\|^2\big) = n!\sum_{i_1}^{N}\cdots\sum_{i_n}^{N}\|a_{i_1,i_2,\dots,i_n}\|^2 + \\
&\sum_{\nu=1}^{[n/2]}\frac{1}{(n-2\nu)!}\Big(\frac{n!}{2^\nu\nu!}\Big)^2\sum_{i_{2\nu+1}}^{N}\cdots\sum_{i_n}^{N}\Big\|\sum_{i_1}^{N}\cdots\sum_{i_\nu}^{N}a_{i_1,i_1,\dots,i_\nu,i_\nu,i_{2\nu+1},\dots,i_n}\Big\|^2 .
\end{aligned}
\tag{3.5}
$$

Associating the sequence of projections P_N with a $\cos\{\phi_i\}$ such that P_N is spanned by the $\phi_i,\ i = 1,\dots,N$, we see that if $K_n(\cdot)$ is nuclear, then

$$p_n(P_N\omega)$$

is Cauchy in the mean of order two and, further,

$$
\begin{aligned}
E\big[\|p_n(\omega)\|^2\big] &= \operatorname*{limit}_N\ E\|p_n(P_N\omega)\|^2 \\
&= n!\|K_n(\cdot)\|_{HS}^2 + \sum_{\nu+1}^{[n/2]}\frac{1}{(n-2\nu)!}\Big(\frac{n!}{2^\nu\nu!}\Big)^2\|K_{n,n-2\nu}(\cdot)\|_{H-S}^2 .
\end{aligned}
\tag{3.6}
$$

It is convenient to use $p_{n,n-2\nu}(\cdot)$ for the polynomial of degree $(n-2\nu)$ defined by the $(n-2\nu)$ linear form $K_{n,n-2\nu}(\cdot)$.

From now on we shall consider only W-N polynomials. Given a sequence of W-N polynomials $\{p_n(\omega)\},\ n \geq 0$, let us proceed to orthogonalize them. Thus let

$$
\begin{aligned}
\widetilde{p}_0(\omega) &= p_0(\omega)\\
\widetilde{p}_1(\omega) &= p_1(\omega)\\
\widetilde{p}_2(\omega) &= p_2(\omega) - p_{2,0}(\omega)
\end{aligned}
$$

and by induction:

$$\widetilde{p}_n(\omega) = p_n(\omega) - \sum_{1}^{[n/2]}\frac{n!}{(n-2\nu)!2^\nu\nu!}\widetilde{p}_{n,n-2\nu}(\omega)\,. \tag{3.7}$$

The polynomials $\widetilde{p}_n(\omega)$ so defined are nonhomogeneous but orthogonal polynomials:

$$E\big([\widetilde{p}_n(\omega),\psi][\widetilde{p}_m(\omega),\psi]\big) = 0\,.$$

Moreover

$$E[\|\widetilde{p}_n(\omega)\|^2] = \operatorname*{limit}_N\ E\big[\|\widetilde{p}_n(P_N\omega)\|^2\big] = n!\,\|K_n(\cdot)\|_{HS}^2\,.$$

These are recognized as the "Ito multiple-integrals" [5], and we shall call them Ito polynomials. In this connection see also [9] for the use of Hermite polynomials.

4. The Schroedinger equation. In this section we present our main results on the abstract version of the propagation equations replacing $z > 0$ by $t > 0$ and using r to denote the position vector in $\mathbf{R}^2$. Let H denote $L_2(R^2)$ and let A denote the closed linear operator with dense domain (H^2) in H, defined by

$$Af = g\,; \qquad g(r) = \frac{i}{2k}\Delta f(r)\,. \tag{4.1}$$

Let $B(\phi, \psi)$ denote the bilinear form over H defined by:

$$h = B(\phi, \psi)\,; \qquad h(r) = ik\phi(r)g(r)\,, \qquad \text{a.e. } r \in R^2 \tag{4.2}$$

where

$$g(r) = \int W(r, r')\psi(r')\, dr' \tag{4.3}$$

where $W(r.r')$ is real-valued, Lebesgue measurable in $R^2 \times R^2$, and

$$\sup_r \int_{R^2} |W(r, r')|^2\, d|r'| \leq Q < \infty\,. \tag{4.4}$$

Note that

$$|g(r)| \leq \sqrt{Q}\, \|\psi\|\,, \tag{4.5}$$

and

$$\|B(\phi, \psi)\| \leq k\|\phi\|\sqrt{Q}\, \|\psi\|\,. \tag{4.6}$$

Let $\omega(\cdot)$ denote white noise in $W(T)$ in the notation of Section 3. Our abstract equation takes the form:

$$\frac{dv(t)}{dt} = Av(t) + B\big(v(t), \omega(t)\big)\,, \qquad 0 < t < T \tag{4.7}$$

where we shall consider the equation for each fixed $T < \infty$. It is well known that A generates a C_0-semigroup $S(t)$ (actually a unitary group) over H such that:

$$\|S(t)\| = 1$$
$$S(t)^* = S(t)^{-1} = S(-t)\,.$$

For each $\omega(\cdot)$ in $W(T)$, we shall seek a "mild" solution of (4.7). In other words, we consider the integral equation:

$$v(t) = S(t)x + \int_0^t S(t-\sigma)B\big(v(\sigma), \omega(\sigma)\big)\, d\sigma \qquad 0 < t < T \tag{4.8}$$

where we require that $v(\cdot) \in W(T)$, for given x in H. The existence and uniqueness of such "sample-path" solutions to (4.6) follows as in [1] by noting the fact that for each $\omega(\cdot)$ in $W(T)$, the operator $L(\omega)$ defined on $W(T)$ by:

$$L(\omega)\, f = g$$

$$g(t) = \int_0^t S(t-\sigma)\, B\big(f(\sigma), \omega(\sigma)\big)\, d\sigma \qquad 0 < t < T \tag{4.9}$$

maps $W(T)$ into itself. As in [1], using (4.7) we can see that L is quasi-nilpotent and writing (4.8) in the form

$$v = g + L(\omega)v$$

where we use g to denote the function

$$S(t)x \qquad 0 < t < T \tag{4.10}$$

as an element of $W(T)$, we see that the solution can be expressed

$$v = \big(I - L(\omega)\big)^{-1} g = \sum_0^\infty L(\omega)^n\, g\,. \tag{4.11}$$

Hence (4.11) defines the pathwise solution, and the solution $v(t)$ is defined for each t. $0 \le t \le T$ as well by

$$v(t,\omega) = \sum_0^\infty p_n(t;\omega) \tag{4.12}$$

where

$$\begin{aligned} p_0(t,\omega) &= S(t)x \\ p_{n+1}(t;\omega) &= \int_0^t S(t-\sigma)\, B\big(p_n(\sigma;\omega), \omega(\sigma)\big)\, d\sigma\,. \end{aligned} \tag{4.13}$$

For each x fixed, let

$$v = f(\omega) \tag{4.14}$$

denote the solution (4.11), defined further for each t by (4.12). We want to show that for each t, $0 \le t \le T$ (4.12) defines a prv with range in H; and also that (4.11) defines a prv with range in $W(T)$. Let us begin with the latter. For each n,

$$p_n(\omega) = L(\omega)^n\, g \tag{4.15}$$

defines a (homogeneous) polynomial of degree n mapping $W(T)$ into $W(T)$. We shall show that $p_n(\omega)$ is a W-N polynomial for each n. For this purpose we show that the symmetric n-linear form

$$\widetilde{K}_n(\omega_1, \ldots, \omega_n) = \frac{1}{n!} \sum_p L(\omega_n) \ldots L(\omega_1)\, g \tag{4.16}$$

where the summation is over the $n!$ possible permutations over the distinct indices $1, 2, \ldots, n$, is nuclear.

The basic observation in this argument is that by (4.2), for any ϕ, ψ in H we can express the bilinear form $B(\cdot,\cdot)$ as

$$B(\psi, \phi) = M(\psi)\phi \tag{4.17}$$

where $M(\psi)$ for fixed ψ in H, is a linear bounded transformation on H into H defined by

$$M(\psi)x = y\,; \qquad y(r) = \int_{R^2} ik\, \psi(r) W(r,r') x(r')\, d|r'| \tag{4.18}$$

and since

$$\int_{R^2}\int_{R^2} \left|\psi(r)W(r,r')\right|^2 d|r|\,d|r'| < \|\psi\|^2 Q < \infty$$

it follows that $M(\cdot)$ is Hilbert-Schmidt and

$$\|M(\psi)\|_{HS}^2 \leq \|\psi\|^2\, Qk^2\,. \tag{4.19}$$

Now the unsymmetrized n-linear form:

$$K_n(\omega_1,\ldots,\omega_n) = L(\omega_n)\ldots L(\omega_1)g$$

is defined by

$$h = K_n(\omega_1,\ldots,\omega_n)\,;$$

$$h(t) = \int_0^t \int_0^t K\Big(t;\, \sigma_n,\ldots,\sigma_1; \omega_1(\sigma_1),\ldots,\omega_n(\sigma_n)\Big)d\sigma_1\ldots d\sigma_n \quad 0 \leq t \leq T \tag{4.20}$$

$$= K_n(t;\, \omega_1,\ldots,\omega_n) \qquad 0 \leq t \leq T \tag{4.21}$$

where the kernel

$$\begin{aligned} &K\big(t;\, \sigma_n,\ldots,\sigma_1;\, x_1,x_2,\ldots,x_n\big)\\ &= S(t-\sigma_n)B\Big(S(\sigma_n-\sigma_{n-1})B(S(\sigma_{n-1}-\sigma_{n-2})\ldots\\ &\qquad B(S(\sigma_1)x,x_1),x_2)\ldots),x_n\Big) \quad \text{for } t \geq \sigma_n \geq \sigma_{n-1} \geq \cdots \geq \sigma_1\\ &= 0 \qquad \text{otherwise}\,. \end{aligned} \tag{4.22}$$

By (4.19) the n-linear form defining the kernel (4.17) is Hilbert-Schmidt for fixed σ_i, $0 \leq \sigma_i \leq \sigma_{i+1} \leq t$; and further

$$\|K(t;\, \sigma_n,\ldots,\sigma_1;\, \ldots)\|_{HS}^2 \leq (Qk^2)^n \|x\|^2\,.$$

Hence the n-linear form $K_n(\ldots)$ defined by (4.20) is also Hilbert-Schmidt and in fact (by virtue of (4.22)):

$$\begin{aligned} \|\overline{K}_n(\ldots)\|_{HS}^2 &= \int_0^T \Big(\int_0^t\int_0^t\cdots\int_0^t \|\overline{K}(t;\,\sigma_n,\ldots,\sigma_1;\ldots)\|^2\, d\sigma_1\ldots d\sigma_n\Big)dt\\ &\leq \frac{T^{n+1}}{(n!)^2}(Qk^2)^n\|x\|^2 \end{aligned} \tag{4.23}$$

where $\overline{K}_n(\ldots)$ is the symmetrized version of the kernel corresponding to (4.16). In particular we have that for each $0 < t$:

$$E\big[\|\widetilde{p}_n(t,\omega)\|^2\big] \leq \frac{t^n}{(n!)}(Qk^2)^n\|x\|^2 \tag{4.23a}$$

Next we come to the crucial part – of proving property (ii), the nuclearity of $K_n(\omega_1, \ldots, \omega_n)$. We shall prove it first for $K_n(t;\, \omega_1, \ldots, \omega_n)$ for each $t > 0$. Since the random field $\nu(\cdot)$ is real-valued, we may confine ourselves to the real L_2-space

$$W_R(t) = L_2[(0,t);\, H_R]$$

where H_R is the real L_2space of real-valued functions over $\mathbf{R}^2$. Specializing to $n = 2$ for specificity, we wish to show that for any $\cos\{\phi_1\}$ in $W_R(t)$,

$$\sum_i \left|[K_2(t;\, \phi_i, \phi_i), \psi]\right| < \infty$$

for any ψ in H. For this purpose it is convenient to work with the real parts and imaginary parts separately. Thus we may take ψ in H_R. Now

$$[K_2(t;\, \phi_i, \phi_i), \psi] = \int_0^t \int_0^{\sigma_2} [S(t-\sigma_2)B\Big(S(\sigma_2-\sigma_1)B(S(\sigma_1)x, \phi_i(\sigma_1)), \phi_i(\sigma_2)\Big), \psi]\, d\sigma_1\, d\sigma_2$$

and we shall split up the integrand to real and imaginary parts and it is enough to show the absolute summability for the real-valued case. Thus let $x \in H_R$ and use

$$S_R(\sigma) = \frac{S(\sigma) + S(-\sigma)}{2}$$
$$S_I(\sigma) = \frac{S(\sigma) - S(-\sigma)}{2i}$$

and we shall use

$$S_k(\sigma) = S_R(\sigma) \qquad or \qquad S_I(\sigma)\,.$$

Hence the integrand can be taken in the form

$$\big[S_3(t-\sigma_2)B\Big(S_2(\sigma_2-\sigma_1)B(S_1(\sigma_1)x, \phi_i(\sigma_1)), \phi_i(\sigma_2)\Big), \psi\big]\,.$$

The main point is that for ϕ_1, ϕ_2 in H_R, we can write (fixing t and ψ!)

$$\big[S_3(t-\sigma_2)B\Big(S_2(\sigma_2-\sigma_1)B(S_1(\sigma_1)x, \sigma_1), \sigma_2\Big), \psi\big] = \big[L_2(\sigma_2, \sigma_1)\phi_1, L_1(\sigma_2)\phi_2\big]$$

where

$$L_2(\sigma_2, \sigma_1) = S_2(\sigma_2 - \sigma_1)M(S_1(\sigma_1)x)$$
$$L_1(\sigma_2) = M^*(S_3(t-\sigma_2)^*\psi)$$

are H-S operators on H into H, strongly continuous in $\sigma_1 \leq \sigma_2 \leq t$, and

$$\sup_{0 \leq \sigma_1 \leq \sigma_2 \leq t} \|L_2(\sigma_2, \sigma_1)\|_{HS}^2 + \|L_1(\sigma_2)\|_{HS}^2 < \infty\,.$$

Here $M(\cdot)$ has been defined in (4.17) and $M^*(\cdot)$ is defined by (for x, ϕ, ψ all in H_R)

$$[B(x, \phi), \psi] = [x, M^*(\psi)\phi] \tag{4.24}$$

and

$$\|M^*(\psi)\|_{HS}^2 = \|M(\psi)\|_{HS}^2 .$$

To proceed further we need to first state the property:

$$B(B(x,\phi),\psi) = B(B(x,\psi),\phi) . \tag{4.25}$$

Next we need to make what is no more than a smoothness assumption on the function $W(r,r')$.
Assumption: For x in $D(A)$ and ϕ in H

$$B(S(\sigma)x,\psi) \in D(A)$$

and

$$AB(S(\sigma)x,\phi) = T(\sigma)\phi$$

define a H-S operator on H into H, such that $T(\sigma)$ is strongly continuous in σ, $\sigma < \infty$, and further

$$\sup_{0\leq\sigma} \|T(\sigma)\|_{HS}^2 < \infty .$$

This assumption holds if $W(r,r')$ is smooth enough – in the stationary case (which is the one we are really interested in) where

$$W(r,r') = W(r-r')$$

it would be enough if, for instance, $W(\cdot) \in D(A)$. Let us note that under this Assumption which is assumed from now on,

$$AB(S_k(\sigma)x,\phi)$$

defines a H-S operator on H into H and

$$\sup_{\sigma} \|AB(S_k(\sigma)x,\cdot)\|_{HS}^2 < \infty .$$

We assume also from now on that $x \in D(A)$ and $x \in H_R$.

Hence we can write

$$L_2(\sigma_2,\sigma_1)\phi = \int_{\sigma_1}^{\sigma_2} M_2(s,\sigma)\phi\, ds + L_2(\sigma_1,\sigma_1)\phi$$

where $M_2(s,\sigma)$ is H-S and

$$\sup_{\sigma\leq s} \|M_2(s,\sigma)\|_{HS}^2 < \infty .$$

Hence if we define the operator on $W_R(t)$ into $W_R(t)$ by:

$$L\phi = h; \qquad h(\sigma_2) = \int_0^{\sigma_2} L_1(\sigma_2)^* L_2(\sigma_2,\sigma_1)\phi(\sigma_1)\, d\sigma_1, \quad 0 < \sigma_2 < t$$

we have:

$$h(\sigma_2) = \int_0^{\sigma_2}\int_{\sigma_1}^{\sigma_2} L_1(\sigma_2)^* M_2(s,\sigma_1)\phi(\sigma_1)\, ds\, d\sigma_1 + \int_0^{\sigma_2} L_1(\sigma_2)^* L_2(\sigma_1,\sigma_1)\phi(\sigma_1)\, d\sigma_1 .$$

Here the first term defines a nuclear operator with zero trace, by Theorem 2.1. So we need only to work with the second term; denote it again by L :

$$L\phi = g; \qquad g(\sigma_2) = \int_0^{\sigma_2} L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)\phi(\sigma_1)\, d\sigma_1\,, \quad 0 < \sigma_2 < t\,.$$

Now, for ϕ_1, ϕ_2 in H_R :

$$\big[L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)\phi_1, \phi_2\big] = \big[S_3(t - \sigma_2)B\Big(S_2(0)B(S_1(\sigma_1)x, \phi_1), \phi_2\Big), \psi\big]$$

and

$$S_2(0) = \text{ zero } \qquad \text{or} \qquad \text{Identity}$$

and by (4.25):

$$B\Big(S_2(0)B(S_1(\sigma_1)x, \phi_1), \phi_2\Big) = B\Big(S_2(0)B(S_1(\sigma_1)x, \phi_2), \phi_1\Big)$$

hence

$$\big[L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)\phi_1, \phi_2\big] = \big[L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)\phi_2, \phi_1\big]$$

or, $L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)$ is self-adjoint. To show that $(L + L^*)$ is nuclear also, we exploit our Assumption and note that we can express

$$L_1(\sigma_2)^*\phi - L_1(\sigma_1)^*\phi = \int_{\sigma_1}^{\sigma_2} M(s)\phi\, ds$$

and hence for $\phi_i(\cdot)$ in $W(t)$:

$$\int_0^{\sigma_2} L_1(\sigma_2)^* L_2(\sigma_1, \sigma_1)\phi_i(\sigma_1)\, d\sigma_1 = \int_0^{\sigma_2} d\sigma_1 \int_{\sigma_1}^{\sigma_2} M(s) L_2(\sigma_1, \sigma_1)\phi_i(\sigma_1)\, ds$$
$$+ \int_0^{\sigma_2} L_1(\sigma_1)^* L_2(\sigma_1, \sigma_1)\phi_i(\sigma_1)\, d\sigma_1 \quad 0 \le \sigma_2 \le t\,.$$

Because of the self-adjointness of the kernel, the second term defines an operator, denote it L_3, such that $(L_3 + L_3^*)$ is nuclear, and

$$\text{Tr. } (L_3 + L_3^*) = \int_0^t \text{Tr. } L_1(\sigma)^* L_2(\sigma, \sigma)\, d\sigma\,.$$

Let us denote the operator defined by the first term as L_4. Then L_4 can be expressed as the product:

$$L_4 = L_5 L_6$$

where L_5, L_6 are defined by:

$$L_5 f = g; \qquad g(\sigma_2) = \int_0^{\sigma_2} M(s) f(s)\, ds\,, \qquad 0 \le \sigma_2 \le t$$
$$L_6 f = g; \qquad g(\sigma_2) = \int_0^{\sigma_2} L_2(\sigma_1, \sigma_1) f(\sigma_1)\, d\sigma_1\,, \quad 0 \le \sigma_2 \le t\,.$$

Because L_5 and L_6 are both Hilbert-Schmidt, it follows that L_4 is nuclear, and because it is Volterra, its trace is zero.

Hence it follows that $p_2(t, \omega_1, \omega_2)$ is nuclear and

$$\sum_i [p_2(t, \phi_i, \phi_i), \psi] = \frac{1}{2} \int_0^t \sum_i \left[S(t-\sigma) B\big(B(S(\sigma)x,\, e_i),\, e_i\big),\, \psi\right] d\phi$$
$$= \frac{1}{2} \Big[\int_0^t \Big(\sum_i S(t-\sigma) B\big(B(S(\sigma)x,\, e_i),\, e_i\big)\Big) d\sigma,\, \psi \Big] .$$

Let for x in H :

$$\sum_i B\big(B(x, e_i),\, e_i\big) = \Lambda x$$

where the convergence of the sum is in the weak sense. Then Λ is a linear bounded operator mapping H into H, and

$$\sum_i p_2(t, \phi_i, \phi_i) = \frac{1}{2} \int_0^t S(t-\sigma)\, \Lambda S(\sigma) x \, d\sigma \tag{4.26}$$

Let us next prove property (ii) for $K_2(\omega_1, \omega_2)$. The proof will be similar. for fixed g in $W(T)$, introduce the operator $L(\cdot)g$ by

$$L(\phi) g = \psi; \qquad \psi(t) = \int_0^t S(t-\sigma) B(g(\sigma)\,,\, \phi(\sigma))\, d\sigma \,. \tag{4.27}$$

Then $L(\cdot)g$ is H-S and

$$\|L(\cdot)g\|^2 \le T \|g\|^2 Q k^2 \,.$$

Introduce next the operator $L^*(g)\phi$ by

$$L^*(g)\phi = \psi; \qquad \psi(t) = \int_t^T M^*(S^*(\sigma - t) g(\sigma)) \phi(t)\, d\sigma \tag{4.28}$$

Then for ϕ, ψ in $W(T)$:

$$[K_2(\phi, \phi)\,,\, \psi] = [L_1\phi\,,\, L_2\phi]$$

where L_1, L_2 are defined by:

$$L_1\phi = h; \qquad h(t) = \int_0^t S(t-\sigma) B\big(S(\sigma)x\,,\, \phi(\sigma)\big)\, d\sigma$$
$$L_2\phi = h; \qquad h(t) = \int_t^T M^*\big(S^*(\sigma - t)\psi(\sigma)\big)\phi(t)\, d\sigma \,.$$

As before we next work with the corresponding real valued versions – ϕ, $\psi \in W_R(T)$, $x \in H_R$. Then

$$[K_2(\phi, \phi), \psi] = [L_2^* L_1 \phi\,,\, \phi]$$

where we can write:

$$L_2^* L_1 \phi = h; \qquad h(t) = M_1(t) \int_0^t M_2(t,\sigma)\,\phi(\sigma)\,d\sigma$$

$$M_1(t)^* = \int_t^T M^*\big(S_k^*(\sigma - t)\psi(\sigma)\big)\,d\sigma\,, \qquad y \in H_R$$

$$M_2(t,\sigma) = S_k(t-\sigma)M(S_k(\sigma)x)\,, \qquad y \in H_R\,.$$

Writing

$$M_2(t,\sigma)y = \int_\sigma^t M_3(s,\sigma)y\,ds + M_2(\sigma,\sigma)y$$

we can invoke Theorem 2.1, and need only work with operator L defined by

$$L\phi = h; \qquad h(t) = M_1(t) \int_0^t M_2(\sigma,\sigma)\phi(\sigma)\,d\sigma\,.$$

Now for y, z in H_R

$$\big[M_1(t)M_2(\sigma,\sigma)y,\, z\big] = \big[M_1(t)M_2(\sigma,\sigma)z,\, y\big]$$

again using (4.25); and for any cos$\{\phi_i\}$ in $W_R(T)$, the arguments for

$$\sum [L\phi_i\,,\,\phi_i] = \frac{1}{2}\big|[(L+L^*)\phi_i\,,\,\phi_i]\big|$$

to be finite go through as before. Moreover we may calculate

$$\sum_i [K_2(\phi_i\,,\,\phi_i)\,,\,\psi]$$

using a cos of the form

$$\phi_i \sim e_i f_j(\cdot)$$

where $\{e_i\}$ is a cos in H_R and $\{f_j(\cdot)\}$ a cos in $L_2\big((0,T);\,R^1\big)$. This yields

$$\sum_i \big[K_2(\phi_i,\phi_i),\psi\big] = \int_0^T \int_0^T \sum_j \sum_i \big[M_2(\sigma,\sigma)e_i, M_1(t)^* e_i\big] f_j(\sigma) f_j(t)\,d\sigma\,dt.$$

Going back now to the complex form we have

$$\sum_i \big[M_2(\sigma,\sigma)e_i, M_1(t)^* e_i\big] = \sum_i \int_t^T \big[S(\tau - t)B\big(B(S(\sigma)x, e_i), e_i\big),\, \psi(\tau)\big]\,d\tau$$

$$= \int_t^T \big[S(\tau - t)\Lambda S(\sigma)x,\, \psi(\tau)\big]\,d\tau\,.$$

Hence

$$\sum_i K_2(\phi_i,\phi_i) = h\,;$$

$$h(t) = \frac{1}{2}\int_0^t S(t-\sigma)\Lambda S(\sigma)x\, d\sigma\,, \qquad 0 < t < T \tag{4.29}$$

as we expect, consistent with our calculations for $K_2(t, \omega_1, \omega_2)$.

To proceed to the higher order polynomials we observe that the calculations are made much easier if we exploit the stationarity of the random field $\nu(t, r)$ in the variable r. Thus we assume

$$W(r, r') = W(r - r') \tag{4.30}$$

and that

$$W(\cdot) \in D(A)\,.$$

In that case Λ becomes:

$$\Lambda f = g\,; \qquad g(r) = -k^2\gamma f(r), \quad r \in R^2 \tag{4.31}$$

where

$$\gamma = \int_{R^2} W(r)^2\, d|r| \tag{4.32}$$

and of course

$$\gamma \leq Q\,.$$

Hence in particular

$$\begin{aligned} \Lambda S(t) &= S(t)\Lambda\,, \\ B(S(t)\Lambda z\,, y) &= \Lambda B(S(t)z\,, y) \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \sum_i p_2(t, \phi_i, \phi_i) &= \frac{1}{2}\int_0^t S(t-\sigma)\Lambda S(\sigma)x\, d\sigma \\ &= \frac{\Lambda t S(t)x}{2} = \frac{-k^2\gamma t}{2} S(t)x\,. \end{aligned}$$

It is useful to summarize our result for $n = 2$ using the notation of (4.22) as:

$$\sum_i K_2(t, \phi_i, \phi_i) = \frac{1}{2}\int_0^t \text{Trace } K(t;\, \sigma, \sigma;\, \cdot, \cdot)\, d\sigma \tag{4.34}$$

where

$$\text{Trace } K(t;\, \sigma, \sigma;\, \cdot, \cdot) = \sum_i K(t;\, \sigma, \sigma;\, e_i, e_i)\,, \quad \{e_i\} \text{ } cos \text{ in } H_R\,,$$

where the limit is taken in the weak sense.

Let $\{\phi_i\}$ be a cos in $W_R(T)$ again. Now

$$K_n(\phi_{i_1}, \ldots, \phi_{i_n})$$

we can write as

$$L(\phi_{i_n})L(\phi_{i_{n-1}} \ldots L(\phi_{i_1})g$$

where g is defined by:

$$g(t) = S(t)x\,, \qquad 0 \le t \le T; \quad x \in D(A)$$

and $L(\phi)$ defines a H-S operator $W(T)$ into $W(T)$ for each ϕ in $W(T)$. Hence for ψ in $W_R(T)$ and fixed m :

$$\left[L(\phi_{i_n}) \ldots L(\phi_{i_{\nu+m+1}}) L(\phi_{i_\nu}) L(\phi_{i_{\nu+m-1}}) \ldots L(\phi_{i_1}) L(\phi_{i_{\nu-1}}) \ldots L(\phi_{i_1}) g,\, \psi\right]$$

can be written as

$$\left[L(\phi_{i_\nu})h\,,\, TL(\phi_{i_\nu})^*\,\phi\right] \tag{4.35}$$

where

$$\begin{aligned} h &= L(\phi_{i_{\nu-1}}) \ldots L(\phi_{i_1})g \\ T &= L(\phi_{i_{\nu+1}})^* \ldots L(\phi_{i_{\nu+m-1}})^* \\ \phi &= L(\phi_{i_{\nu+m+1}})^* \ldots L(\phi_{i_n})^*\,\psi \end{aligned}$$

We fix m and all indices except i_ν. Then h and ϕ are fixed. If $m = 1$, then T is the identity. Let us consider $m = 1$ first, and work with the real valued version

$$L(\phi_{i_\nu})h = q\,;$$

where

$$\begin{aligned} q(t) &= \int_0^t S_1(t-\sigma)B\big(h(\sigma)\,,\,\phi_{i_\nu}(\sigma)\big)\,d\sigma = \int_0^t S_1(t-\sigma)M(h(\sigma))\phi_{i_\nu}(\sigma)\,d\sigma\,. \\ L(\phi_{i_\nu})^*\,\phi &= \lambda\,; \\ \lambda(t) &= \int_t^T M^*\big(S_2(\sigma-t)^*\,\phi(\sigma)\big)\,\phi_{i_\nu}(t)\,d\sigma\,. \end{aligned}$$

Hence we only need to make sure

$$B(h(\sigma)\,,\,x) \in D(A)$$

and that

$$\sup_{0\le\sigma\le T} \|AB(h(\sigma),\,\cdot)\|_{HS}^2 < \infty$$

which is readily deduced as a consequence of our smoothness assumption on $W(\cdot)$. Hence it follows that

$$\sum \left|\left[L(\phi_{i_\nu})h\,,\,L(\phi_{i_\nu})\phi\right]\right| < \infty$$

and that

$$\begin{aligned} \sum_i \left[L(\phi_{i_\nu})h\,,\,L(\phi_{i_\nu})\phi\right] &= \frac{1}{2}\int_0^T \Big[\int_0^t S(t-\sigma)\Lambda h(\sigma)\,d\sigma\,,\,\phi(t)\Big]\,dt \\ &= \frac{\Lambda}{2}\int_0^T \Big[\int_0^t S(t-\sigma)h(\sigma)\,d\sigma\,,\,\phi(t)\Big]\,dt. \end{aligned} \tag{4.36}$$

Now

$$h = L(\phi_{i_{\nu-1}})z\,, \qquad z = L(\phi_{i_{\nu-2}})\dots L(\phi_{i_1})g$$

Hence

$$\begin{aligned}\int_0^t S(t-\sigma)h(\sigma)\,d\sigma &= \int_0^t S(t-\sigma)\int_0^\sigma S(\sigma-\tau)B\big(z(\tau)\,,\,\phi_{i_{\nu-1}}(\tau)\big)\,d\tau\\ &= \int_0^t (t-\tau)S(t-\tau)B\big(z(\tau)\,,\,\phi_{i_{\nu-1}}(\tau)\big)\,d\tau\\ &= \widetilde{L}(\phi_{i_{\nu-1}})z\end{aligned}$$

where $\widetilde{L}(\cdot)$ is the operator on $W(T)$ into $W(T)$ defined by

$$\widetilde{L}[\phi]z = y; \quad y(t) = \int_0^t (t-\sigma)S(t-\sigma)B(z(\sigma)\,,\,\phi(\sigma))\,d\sigma\,, \quad 0 < t < T\,. \tag{4.37}$$

hence

$$\sum_i \big[L(\phi_{i_\nu})h\,,\,L(\phi_{i_\nu})\phi\big] = \frac{\Lambda}{2}\big[L(\phi_{i_n})\dots L(\phi_{i_{\nu+2}})\widetilde{L}(\phi_{i_{\nu-1}})\dots L(\phi_{i_1})g\,,\,\psi\big].$$

$$\sum_{i_\nu} K_n\big(\phi_{i_1},\dots,\phi_{i_\nu},\phi_{i_\nu},\phi_{i_{\nu+2}},\dots,\phi_{i_n}\big) \tag{4.38}$$

has the weak limit

$$\frac{\Lambda}{2}L(\phi_{i_n})\dots L(\phi_{i_{\nu+2}})\widetilde{L}(\phi_{i_{\nu-1}})L(\phi_{i_{\nu-2}})\dots L(\phi_{i_1}g)\,. \tag{4.39}$$

For $m \geq 2$, in our notation, T^* is Volterra and Hilbert-Schmidt and hence

$$\big[T^*L(\phi_{i_\nu})h\,,\,L(\phi_{i_\nu})^*\phi\big]$$

which we can write as

$$\big[J\phi_{i_\nu}\,,\,\phi_{i_\nu}\big]$$

where J is Volterra and nuclear being the product of a bounded operator and at least two H-S operators. Hence J has zero trace. Hence we only need to consider the case $m = 1$.

Let us now take account of the fact that we have to work with symmetrized polynomials. Let us denote (4.39) by

$$\frac{\Lambda}{2}J_{n-2}\big(\phi_{i_n},\dots,\phi_{i_{\nu+2}},\phi_{i_{\nu-1}},\phi_{i_{\nu-2}},\dots,\phi_{i_1}\big)$$

being a (non-symmetric) $(n-2)$ linear form with kernel

$$\begin{aligned}&\frac{\Lambda}{2}J_{n-2}(t;\,\sigma_{n-2},\dots,\sigma_1;\,x_{n-2},\dots,x_1)\\ &\quad= \frac{\Lambda}{2}S(t-\sigma_{n-2})B\Big(\dots B(S(\sigma_\nu-\sigma_{\nu-1})B(\dots B(S(\sigma_1)x,\,x_1),\dots,\\ &\qquad\dots,x_{n-2}\Big)(\sigma_\nu-\sigma_{\nu-1}) \qquad \text{for} \quad 0<\sigma_1<\sigma_2<\dots<\sigma_n<t\\ &\quad= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.}\end{aligned} \tag{4.40}$$

Summing over the "location" index ν from $\nu = 1$ to $\nu = n-1$, with $\sigma_{n-1} = t$, and $\sigma_0 = 0$, and noting that

$$\sum_1^{n-1} (\sigma_\nu - \sigma_{\nu-1}) = t$$

we see that the sum yields the kernel

$$\frac{\Lambda t}{2} K_{n-2}\big(t;\, \sigma_{n-2}, \ldots, \sigma_1\,,\, x_{n-2}, \ldots, x_1\big)\,.$$

Since we distinguish by order – $K(\phi_{i_1}, \phi_{i_2}, \ldots)$ is distinct from $K(\phi_{i_2}, \phi_{i_1}, \ldots)$ – we see that for the symmetrized kernel $\overline{K}_n(\cdot)$, we obtain

$$\frac{(n-2)!(\Lambda t)\overline{K}_{n-2}(\cdot)}{n!}$$

and more generally for

$$\sum_{i_1} \sum_{i_\nu} \overline{K}_n\big(\phi_{i_1}, \phi_{i_1}, \ldots, \phi_{i_\nu}, \phi_{i_\nu}, \phi_{i_{2\nu+1}}, \ldots, \phi_{i_n}\big), \qquad \nu \le [\tfrac{n}{2}]$$

we get

$$\frac{(n-2\nu)!(\Lambda t)^\nu \overline{K}_{n-2}(\phi_{i_{2\nu+1}}, \ldots, \phi_{i_n})}{n!}\,. \tag{4.41}$$

Note that in particular for $n = 2m$,

$$\overline{K}_{n,n-2m} = \frac{(\Lambda t)^m}{n!} \overline{K}_0 \tag{4.42}$$

where

$$\overline{K}_0 = h; \qquad h(t) = S(t)x, \qquad 0 < t < T\,.$$

Also using (3.7) we obtain

$$\widetilde{p}_n(t,\omega) = p_n(t,\omega) + \sum_1^{[n/2]} \frac{n!}{(n-2\nu)!} \frac{1}{2^\nu \nu!} \frac{(n-2\nu)!(\Lambda t)}{n!}\, \widetilde{p}_{n,n-2\nu}(t,\omega) \tag{4.43}$$

with

$$\widetilde{p}_0(t,\omega) = S(t)x\,.$$

Hence finally we can write (4.12) as

$$v(t,\omega) = \sum_0^\infty p_n(t,\omega) = \sum_0 a_m \widetilde{p}_m(t,\omega) \tag{4.44}$$

where

$$\begin{aligned} a_m &= \sum_0^\infty \frac{(m+2\nu)!}{2^\nu \nu!} \frac{(\Lambda t)^\nu}{(m+2\nu)!} \\ &= e^{\Lambda t/2}\,. \end{aligned}$$

This is our main result.

The required mean-square convergence for each t follows readily from (4.44) since the $\widetilde{p}_m(t,\omega)$ are orthogonal and we have the estimate (4.23a). Note also that the unsymmetrized kernel corresponding to $e^{(\Lambda t/2)}\widetilde{p}_m(t,\omega)$ can be expressed

$$\begin{aligned} &K(t;\,\sigma_m,\dots,\sigma_1;\,e_{i_1},\dots,e_{i_m}) \\ &\qquad = \widetilde{S}(t-\sigma_m)B\Big(\widetilde{S}(\sigma_m-\sigma_{m-1})B(\dots B(\widetilde{S}(\sigma_1)x,e_{i_1}),\dots),e_{i_m}\Big), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 0\le\sigma_1\le\cdots\le\sigma_m\le t \\ &\qquad = 0 \qquad\qquad\qquad\qquad\quad \text{otherwise}, \end{aligned}$$

where

$$\widetilde{S}(t) = e^{\Lambda t/2}S(t)\,.$$

In other words, use $\widetilde{S}(t)$ in place of $S(t)$. This may be recognized as the genesis of the Stratonovich correction term in [7].

Next let us calculate moments using (4.44). We have

$$m(t) = E[v(t,\omega)] = e^{-\gamma k^2 t/2}\,S(t)x$$

which clearly goes to zero as $t\to\infty$. Of course for x in $D(A^*)$:

$$\frac{d}{dt}[m(t),x] = [m(t),(A^* - \frac{\gamma k^2}{2})x]$$

Next for any z in H, let

$$[R(t)z,z] = E\big[|[v(t,\omega),z]|^2\big]\,.$$

The right side, using (4.44) is

$$= \sum_0^\infty e^{-\gamma t k^2}\,E|[\widetilde{p}_m(t,\omega),z]|^2\,, \tag{4.45}$$

and hence can be calculated term-by-term. Alternately, we can also derive a differential equation characterizing $R(t)$. For this purpose let us note that we can write:

$$v(t+\Delta,\omega) = S(\Delta)v(t,\omega) + \int_0^\Delta S(\Delta-\sigma)B\big(v(t+\sigma,\omega),\omega(t+\sigma)\big)\,d\sigma \tag{4.46}$$

and analogous to (4.44) can be expanded as:

$$\begin{aligned} v(t+\Delta,\omega) &= \sum_0^\infty e^{-\gamma^2k^2\Delta/2}\widetilde{p}_m(\Delta,\omega) \\ &= e^{-\gamma^2k^2\Delta/2}S(\Delta)v(t,\omega) + \sum_1^\infty e^{-\gamma^2k^2\Delta/2}\widetilde{p}_m(\Delta,\omega) \end{aligned} \tag{4.47}$$

where

$$p_n(\Delta,\omega) = \int_0^\Delta \int_0^{\sigma_n} \cdots \int_0^{\sigma_2} S(\Delta-\sigma_n) B\Big(S(\sigma_1-\sigma_{n-1})\ldots$$
$$B\big(S(\sigma_1)v(t,\omega), \omega(t+\sigma_1)\big),\ldots), w(t+\sigma_n)\Big)\, d\sigma_n \ldots d\sigma_1$$

and $\widetilde{p}_n(\Delta,\omega)$ are defined as in (4.43). It follows from (4.46) exploiting the "independence" of white noise over non-overlapping intervals that

$$[R(t+\Delta)z,z] = [R(t)\widetilde{S}(\Delta)^* z, \widetilde{S}(\Delta)^* z] + \sum_1^\infty e^{-\gamma^2 k^2 \Delta} E\big[|[p_m(\Delta,\omega),z]|^2\big]. \tag{4.48}$$

Using the estimate (4.23a), we see that in calculating

$$\Big[\frac{(R(t+\Delta)-R(t))}{\Delta} z, z\Big]$$

we need only retain the first term in the expansion in (4.48), yielding

$$\begin{aligned} &\lim_{\Delta\to 0} \Big[\frac{(R(t+\Delta)-R(t))z}{\Delta}, z\Big] = \\ &\lim_{\Delta\to 0} \frac{1}{\Delta} e^{-\gamma^2 k^2 \Delta} E\Big(|[\widetilde{p}_1(\Delta,\omega),z]|^2\Big) + \lim_{\Delta\to 0} \Big[\frac{[R(t)S(\Delta)z, S(\Delta)z] - [R(t)z,z]}{\Delta}\Big]. \end{aligned} \tag{4.49}$$

Now

$$\widetilde{p}_1(\Delta,\omega) = \int_0^\Delta S(\Delta-\sigma) B\Big(S(\sigma)v(t,\omega), \omega(t+\sigma)\Big)\, d\sigma$$

and hence

$$E\big[|[\widetilde{p}_1(\Delta,\omega),z]|^2\big] = \int_0^\Delta E\big[\|M^*(\widetilde{S}(\sigma)v(t,\omega))\widetilde{S}(\Delta-\sigma)^* z\|^2\big]\, d\sigma$$

Hence the first term in (4.49) is

$$\begin{aligned} &= E\big[[M^*(v(t,\omega))z, M^*(v(t,\omega))z]\big] \\ &= \int_{R^2}\int_{R^2} k^2 \gamma(r,r_1) R(t;r,r_1) z(r)\overline{z}(r_1)\, d|r|\, d|r_1| \end{aligned} \tag{4.50}$$

where

$$\gamma(r,r_1) = \int_{R^2} w(r-r')w(r_1-r')\, d|r'|$$

$$[R(t)z,z] = \int_{R^2}\int_{R^2} R(t;r,r_1) z(r)\overline{z}(r_1)\, d|r|\, d|r_1|.$$

We can express (4.50) as

$$[\Gamma(R(t))z,z]$$

where Γ is a linear bounded transformation defined on the Hilbert space of Hilbert-Schmidt operators on H into H by

$$\Gamma(L)x = y; \qquad y(r_1) = \int_{R^2} \gamma(r, r_1) L(r, r_1) x(r)\, dr\,, \quad r_1 \in R^2\,. \tag{4.51}$$

where $L(\cdot,\cdot)$ is the kernel corresponding to L. Hence our equation is:

$$\frac{d}{dt}[R(t)z, z] = [R(t)z, \widetilde{A}^* z] + [\widetilde{A}^* z, R(t)z] + [\Gamma(R(t))z, z] \quad \text{for} \quad z \in D(A^*) \tag{4.52}$$

where

$$\widetilde{A} = A - \gamma k^2 I/2\,. \tag{4.53}$$

From (4.45) we see that $[R(t)z, z] \to 0$ as $t \to \infty$; as can also be deduced from (4.52). Also from (4.47) we have that for z, y in H :

$$\begin{aligned}[R(t+\sigma, t)z, y] &= E\big[[v(t+\sigma,\omega), z]\overline{[v(t,\omega), y]}\big] \\ &= \big[R(t)e^{-\gamma k^2 \sigma/2}\, S(\sigma)^* z, y\big] \qquad \text{for } \sigma \geq 0.\end{aligned}$$

Differential equations for higher-order moments can be derived using (4.47).

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