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REACTION–DIFFUSION SYSTEMS AND VECTOR LYAPUNOV FUNCTIONS

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Introduction. Employing Lyapunov-like functions to reduce the study of parabolic differential equations to that of ordinary differential equations is considered in [10]. Recently, due to the applications to real world phenomena such as population dynamics, chemical reactor theory etc., this technique has gained more popularity [2-8.13]. Furthermore, it is known that using vector Lyapunov function instead of a single Lyapunov function offers more flexibility [10, 11, 12] and vector Lyapunov function is a natural tool in the investigation of large scale dynamical systems [14]. In this paper, we investigate stability properties of weakly coupled reaction-diffusion systems by means of vector Lyapunov functions and show that this effective approach is also a natural setting for the discussion of such systems. We have utilized an idea of [8] in developing the method of vector Lyapunov functions for the study of reaction-diffusion systems.

1. Comparison results. Let Ω be a bounded domain in R^n and let $H = (t_0, \infty) \times \Omega$, $t_0 \geq 0$. Suppose that the boundary ∂H of H is split into two parts ∂H_0, ∂H_1, such that $\partial H = \partial H_0 \cup \partial H_1$, $\{t_0\} \times \partial\Omega \subset \partial H_0$ and $\partial H_0 \cap \partial H_1$ is empty.

A vector ν is said to be an outer normal at $(t, x) \in \partial H_1$ if $(t, x - h\nu) \in H$ for small $h > 0$. The outer normal derivative is then given by

$$\frac{\partial u(t,x)}{\partial \nu} = \lim_{h \to 0} \frac{u(t,x) - u(t, x - h\nu)}{h}$$

for any $u \in C[\overline{H}, R^N]$. We shall always assume that an outer normal exists on ∂H_1 and the functions in question have outer normal derivatives on ∂H_1. If $u \in C[\overline{H}, R^N]$ is such that the partial derivatives u_t, u_x, u_{xx} exist and are continuous in H, then we shall say that $u \in C^*(J)$, $J = [t_0, \infty)$.

Let $f \in C\big[H \times R^N \times R^n \times R^{n^2}, R^N\big]$ (Here f represents the vector $f_k(t, x, u, u_x^k, u_{xx}^k)$, $k = 1, 2, \ldots, N$; it is important to note that each component f_k contains partial derivatives of kth component of u only. For convenience, we shall use the notation $f(t, x, u, u_x, u_{xx})$ to

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represent such vector functions). Then f is said to be elliptic at $(t_1, x_1) \in H$ if for any u, P, Q, R, the quadratic form

$$\sum_{i,j=1}^{N} (Q_{ij} - R_{ij})\lambda_i\lambda_j \leq 0, \quad \lambda \in R^n$$

implies

$$f(t_1, x_1, u, P, Q) \leq f(t_1, x_1, u, P, R).$$

If this property holds for every $(t, x) \in H$, then f is said to be elliptic in H. Here and in what follows, the inequalities between vectors are understood component wise.

Also, f is said to be quasimonotone nondecreasing in u if

$$u \leq v, \quad u_i = v_i \quad \text{for} \quad 1 \leq i \leq N \quad \text{implies} \quad f_i(t, x, u, P, Q) \leq f_i(t, x, v, P, Q).$$

The following comparison result is crucial to our discussion and it is a special case of a result in [Th. 3.2. 9].

Theorem 1.1. *Assume that*

(i) $v, w \in C^*(J)$, $f \in C\big[H \times R^n \times R^n \times R^{n^2}, R^N\big]$, *$f$ is elliptic in H and quasimonotone nondecreasing in u and*

$$v_t \leq f(t, x, v, v_x, v_{xx}), \quad w_t \geq f(t, x, w, w_x, w_{xx}) \;\; \text{in} \;\; H;$$

(ii) $Z \in C^*(J)$, $Z > 0$ *on* $\overline{H}$, $\partial Z/\partial\nu \geq \beta > 0$ *on* ∂H_1 *and for sufficiently small* $\epsilon > 0$, *either*

a) $\epsilon Z_t > f(t, x, v, v_x, v_{xx}) - f(t, x, v - \epsilon Z, v_x - \epsilon Z_x, v_{xx} - \epsilon Z_{xx})$ *or*

b) $\epsilon Z_t > f(t, x, w + \epsilon Z, w_x + \epsilon Z_x, w_{xx} + \epsilon Z_{xx}) - f(t, x, w, w_x, w_{xx})$ *on* H;

(iii) $v \leq w$ *on* ∂H_0 *and* $\dfrac{\partial v}{\partial \nu} \leq \dfrac{\partial w}{\partial \nu}$ *on* ∂H_1.

Then, $v \leq w$ *on* $\overline{H}$.

As an example, consider the interesting special case

$$f_k(t, x, u, u_x^k, u_{xx}^k) = a^k u_{xx}^k + b^k u_x^k + F_k(t, x, u) \tag{1.1}$$

where

$$a^k u_{xx}^k = \sum_{i,j=1}^{n} a_{ij}^k u_{x_i x_j}, \quad b^k u_x^k = \sum_{j=1}^{n} b_j^k u_{x_j}^k, \quad k = 1, 2, \ldots, N$$

and F is Lipschitzian and quasimonotone nondecreasing in u. That is, F satisfies

$$\Big|F_k(t, x, u) - F_k(t, x, v)\Big| \leq L \sum_{\mu=1}^{n} |u_\mu - v_\mu|, \quad (t, x) \in H.$$

Assume also that the boundary H is smooth enough, that is, there exists a $h \in C^2[\Omega, R_+]$ such that $\partial h/\partial\nu \geq 1$ on ∂H_1 and h_x, h_{xx} are bounded.

Let $M > 1$ and define $H(x) = \exp\big((mLh(x)\big)$, $Z(x) = \big(\exp(N_0 t)\big)H(x)$ and $\widetilde{Z}(x) = \widetilde{e}Z(x)$, where $\widetilde{e} = (1,1,\ldots,1)$, $N_0 = MLN + A$, L is the Lipschitz constant for F and $|a^k H_{xx} + b^k H_x| \leq A_k \leq A$, $k = 1,2,\ldots,N$. Then

$$\frac{\partial Z}{\partial \nu} = ML\big(\frac{h(x)}{\partial \nu}\big)\,\widetilde{e} \geq ML\widetilde{e} > 0 \quad \text{on} \quad \partial H_1$$

and

$$\epsilon\big(\widetilde{Z}^k_t - a^k\,\widetilde{Z}^k_{xx} - b^k\widetilde{Z}^k_x\big) \geq \epsilon(N_0 - A)\widetilde{Z}^k = \epsilon MLN\,\widetilde{Z}^k > \epsilon LNZ.$$

Using the Lipschitz condition of F, we have

$$\big|F^k(t,x,w+\epsilon\widetilde{Z}) - F^k(t,x,w)\big| \leq L\sum_{\mu=1}^{n}\epsilon\widetilde{Z}_\mu = \epsilon LNZ$$

and consequently, we get for $\epsilon > 0$,

$$\epsilon\widetilde{Z}_t > \epsilon\big[a\widetilde{Z}_{xx} + b\widetilde{Z}_{xx}\big] + F(t,x,w+\epsilon\widetilde{Z}) - F(t,x,w)$$

which is exactly condition (b) of (ii) in Theorem 1.1. Similarly (a) of (ii) is also verified.

We note that if H_1 is empty so that $\partial H = \partial H_0$, then assumption (ii) can be replaced by a one-sided Lipschitz condition of the form

$$f^k(t,x,u,P,Q) - f^k(t,x,\overline{u},P,Q) \leq \sum_{\mu=1}^{n}(u_\mu - \overline{u}_\mu) \quad u \geq \overline{u} \tag{1.2}$$

where L is a positive constant. Even when ∂H_1 is non-empty assumption (1.2) is enough if (iii) is strengthened to

$$\frac{\partial v}{\partial \nu} + \psi(t,x,v) \leq \frac{\partial w}{\partial \nu} + \psi(t,x,w) \quad \text{on} \quad \partial H_1$$

where $\psi \in C\big[\partial H_1 \times R^N, R^N\big]$ and $\psi(t,x,u)$ is strictly increasing in u. Of course, if ψ is not strictly increasing in u or $\psi \equiv 0$, then we need condition (ii) which implies that we require smooth boundary information when considering such reaction-diffusion systems.

2. Method of vector Lyapunov functions. We consider the system of reaction-diffusion equations

$$\begin{cases} u_t = Lu + f(t,x,u) \quad \text{in} \quad J \times \Omega, \quad J = (t_0,\infty),\ t_0 \geq 0 \\ u(t_0,x) = \phi_0(x) \quad \text{in } \overline{\Omega}, \quad \dfrac{\partial u(t,x)}{\partial \nu} = 0 \quad \text{on } J \times \partial\Omega, \end{cases} \tag{2.1}$$

where the elliptic operator L is such that

$$L_k u^k = \sum_{i,j=1}^{n} a^k_{ij}\,u^k_{x_i x_j} + \sum_{j=1}^{n} b^k_j\,u^k_{x_j}, \quad k = 1,2,\ldots,N,$$

$$\sum_{i,j=1}^{n} a^k_{ij}\,\lambda_i\lambda_j \geq \beta\,|\lambda|^2, \quad \lambda \in R^n;$$

and $f \in C\big[R_+ \times \overline{\Omega} \times R^N, R^N\big]$. Here $\Omega \subset R^n$ is assumed to be bounded, open connected region equipped with a smooth boundary. We assume existence and uniqueness of solutions of (2.1) in $C^*(J)$. For existence results, see [1].

On the basis of Theorem1.1, we can now extend the method of vector Lyapunov functions (see [10-12]) to study the stability properties of solutions of (2.1).

Theorem 2.1. *Assume that*

(i) $V \in C^1[R_+ \times S(\rho)\,,\, R_+^N]$, $V_u(t,u)Lu \le LV(t,u)$, *and* $V_t(t,u) + V_u(t,u)f(t,x,u) \le g(t,V(t,u))$ *on* $R_+ \times \Omega \times S(\rho)$, *where* $g \in C[R_+ \times R_+^N\,,\, R_+^N]$, $g(t,u)$ *is quasimonotone nondecreasing and locally Lipschitzian in* u;

(ii) $f(t,x,0) \equiv 0$ *and* $g(t,0) \equiv 0$;

(iii) *On* $R_+ \times S(\rho)$, $b(\|u\|) \le \sum_{i=1}^n V_i(t,u) \le a(\|u\|)$, *where* $a,\, b \in K$, $K = \{\phi \in C[R_+\,,\, R_+] \,:\, \phi(0) = 0$ *and* $\phi(s)$ *is strictly increasing in* $s\}$.

Then, the stability properties of the trivial solution of either (a)

$$y' = g(t,y), \quad y(t_0) = y_0 \ge 0, \tag{2.2}$$

or (b)

$$\begin{cases} v_t = Lv + g(t,v) & \text{in} \quad J \times \Omega \\ v(t_0,x) = \psi_0(x) \ge 0 \quad \text{in } \overline{\Omega}, & \dfrac{\partial v(t,x)}{\partial \nu} = 0 \quad \text{on } J \times \partial\Omega, \end{cases} \tag{2.3}$$

imply the corresponding stability properties of the trivial solution of (2.1).

Proof: Let $u(t,x)$ be any solution of (2.1) and $u \in C^*(J)$. Setting $m(t,x) = V(t,u(t,x))$ and using assumption (i), we get

$$\begin{cases} m_t \le Lm + g(t,m) & \text{in} \quad J \times \Omega \\ m(t_0,x) = V(t_0,\phi_0(x)) \quad \text{in } \overline{\Omega}, & \dfrac{\partial m(t,x)}{\partial \nu} = 0 \quad \text{in } J \times \partial\Omega\,. \end{cases} \tag{2.4}$$

Let $y(t)$ and $r(t,x)$ be the solutions of (2.2) and (2.3) respectively existing for $t \ge t_0$ and $x \in \overline{\Omega}$. Then we have

$$y' = g(t,y), \quad y(t_0) = y_0 \ge m(t_0,x) \quad \text{in } \overline{\Omega} \tag{2.5}$$

and

$$\begin{cases} r_t = Lr + g(t,r) & \text{in } J \times \Omega, \\ r(t_0,x) \ge m(t_0,x) \quad \text{in } \overline{\Omega}, & \dfrac{\partial m(t,x)}{\partial \nu} = 0 \quad \text{in } J \times \partial\Omega\,. \end{cases} \tag{2.6}$$

Consequently, applying Theorem 1.1 yields with $v = m$ and $w = y$ or $w = r$ the estimates $V(t,u(t,x)) \le y(t)$, or $V(t,u(t,x)) \le r(t,x)$ in $J \times \overline{\Omega}$.

From these estimates and assumptions (ii) and (iii), it is now easy to prove, using standard arguments, the conclusion of the theorem.

If we have the same operator L, that is, the same diffusion law for all components of u in (2.1), then one can use a single Lyapunov function. On the other hand, if the system (2.1) does not enjoy this luxury, then it is not possible to employ a single Lyapunov function. We note also that if each component of V is convex then clearly $V_u(t,u)\,Lu \le LV$ holds. Thus, it is clear that for general reaction-diffusion systems, utilizing vector Lyapunov functions appears to be natural and advantageous.

Let $\psi(x)$ be the solution of the steady state problem

$$\begin{cases} L\psi + f(t,x,\psi) = 0 \;\text{ in }\; \Omega. \\ \dfrac{\partial \psi(x)}{\partial \nu} = 0 \;\text{ on }\; \partial\Omega. \end{cases} \tag{2.7}$$

Then, setting $w = u - \psi$, we see that w satisfies

$$\begin{cases} w_t = Lw + F(t,x,w) \;\text{ in }\; J \times \Omega, \\ w(t_0,x) = \phi_0(x) - \psi(x) = \overline{\psi}_0(x) \;\text{ in }\; \overline{\Omega}, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0 \;\text{ in }\; J \times \partial\Omega, \end{cases} \tag{2.8}$$

where $F(t,x,w) = f(t,x,w+\psi) - f(t,x,\psi)$. Noting that $F(t,x,0) = 0$, we observe that the stability properties of the trivial solution of (2.8) imply the corresponding stability properties of solutions of (2.1) relative to the steady state solution $\psi(x)$.

For the purpose of illustration, let us consider a typical comparison system

$$\begin{cases} v_t = Av_{xx} - bv_x + g(t,v), \quad 0 < x < 1, \quad t \geq 0 \\ v(0,x) = \psi_0(x) \geq 0, \; 0 \leq x \leq 1, \\ v_x(t,0) = v_x(t,1) = 0, \; t \geq 0, \end{cases} \tag{2.9}$$

where $A > 0$ is a diagonal matrix, $b > 0$ and $g(t,v)$ satisfies

$$g(t,u_1) - g(t,u_2) \leq L(u_1 - u_2), \quad \text{whenever } \; 0 \leq u_2 \leq u_1 \leq Q, \tag{2.10}$$

for some $Q > 0$, where L is a N by N matrix. We shall consider two cases.
(i) L is a positive matrix with

$$\max_i \sum_{k=1}^{n} L_{ik} = \widetilde{L}, \quad 1 \leq i \leq N;$$

(ii) L is a Metzeler matrix satisfying dominant diagonal condition, that is,

$$L_{ij} \geq 0 \quad \text{for } \; i \neq j \; \text{ and } \; -L_{ii} > \sum_{\substack{j=1 \\ i \neq j}}^{n} L_{ij}.$$

Define $R(t,x) = \left(Ke^{-\alpha t + \beta(1-x)}\right)\widetilde{e}$, α, $K > 0$ and $\beta \in R$ to be chosen and $\widetilde{e} = (1,1,\ldots,1)$. Substituting R in (2.9), we get

$$R_t - AR_{xx} + bR_x - g(t,R) \geq R(-\alpha - a\beta^2 - b\beta) - g(t,R), \tag{2.11}$$

where $a = \max A_{ii}$. Let $Q = \max_i \left(\max_{0 \leq x \leq 1} \psi_{0i}(x)\right)$ and $\alpha = L_0 - \widetilde{L} > 0$. In case (i), we now have

$$R_t - AR_{xx} + bR_x - g(t,R) \geq -R\left[L_0 - \widetilde{L} + a\beta^2 + b\beta + \widetilde{L}\right] = 0,$$

if β is a root of $a\beta^2 + b\beta + L_0 = 0$. But since $\beta = \big(- b \pm \sqrt{b^2 - 4aL_0}\,\big)/2a$, if we suppose that $0 < a < (b^2)/(4L_0)$, then β is negative. It is easy to check that R satisfies initial and boundary conditions by choosing $K = Qe^{-\beta}$. As a result, it follows from Theorem 1.1 with $v = r$ and $w = r$ that

$$0 \le r(t,x) \le R(t,x), \;\; t \ge 0, \;\; 0 \le x \le 1 ,$$

which implies exponential asymptotic stability of the trivial solution of (2.9).

In case (ii), set

$$-\gamma_i = L_{ii} + \sum_{\substack{j=1 \\ i \neq j}}^{n} L_{ij} \quad \text{and} \quad \widetilde{L} = \min_i \gamma_i .$$

Then, we get from (2.11),

$$R_t - AR_{xx} + bR_x - g(t,R) \ge -R[a\beta^2 + b\beta + L_0] = 0 ,$$

with $L_0 = \alpha - \widetilde{L}$. We again have two cases: $\alpha > L$ and $0 < \alpha \le \widetilde{L}$. If $\alpha > L$, then as in the previous situation, β is negative if we assume $0 < a < b^2/4L_0$ and we obtain the same conclusion as before. If, on the other hand $0 < \alpha \le \widetilde{L}$ so that $L_0 \le 0$, then $b^2 - 4aL_0$ is always nonnegative and hence we have one negative root β and consequently, the conclusion remains the same as in the previous case.

It is clear that in case (i), diffusion and convection terms are contributing to stability and in case (ii) reaction terms are also playing a role. Form these two cases, one can obtain several possibilities for the coefficients.

Instead of weakly coupled systems (2.1), one can also investigate certain strongly coupled systems with the help of vector Lyapunov functions. As an illustration, consider the following simple example.

Example. Consider

$$\begin{cases} u_{1t} = a_1 u_{1xx} + a_2 u_{2xx} + b_1 u_{1x} + b_2 u_{2x} + e^{-t} u_1 + u_2 \sin t - (u_1^3 + u_1 u_2^2) \sin^2 t, \\ u_{2t} = a_2 u_{1xx} + a_1 u_{2xx} + b_2 u_{1x} + b_1 u_{2x} + u_1 \sin t + e^{-t} u_2 - (u_1^2 u_2 + u_2^3) \sin^2 t, \;\text{ in } J \times \Omega \\ u_1(0,x) = \phi_{01}(x), \;\; u_2(0,x) = \phi_{02}(x) \;\text{ in } \Omega, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 \;\text{ in } J \times \partial\Omega, \end{cases} \tag{2.12}$$

with $a_1 > a_2$. Then, choosing the vector Lyapunov function $V = (V_1, V_2)$ with

$$V_1(u) = \frac{1}{2}(u_1 + u_2)^2 , \;\; V_2(u) = \frac{1}{2}(u_1 - u_2)^2 ,$$

it is easy to obtain, after some computations, the weakly coupled comparison system

$$\begin{cases} V_{1t} = A_1 V_{1xx} + B_1 V_{1x} + g_1(t,v) \\ V_{2t} = A_2 V_{2xx} + B_2 V_{2x} + g_2(t,v), \;\text{ in } J \times \Omega , \\ V(0,x) = \psi_{10}(x) \ge 0 , \;\; V_2(0,x) = \psi_{20}\, x \ge 0 , \;\text{ in } \overline{\Omega} , \\ \dfrac{\partial V_1}{\partial \nu} = \dfrac{\partial V_2}{\partial \nu} = 0 \;\text{ on } J \times \partial\Omega , \end{cases} \tag{2.13}$$

where $\psi_{10}(x) = V_1\big(u(0,x)\big)$, $\psi_{20}(x) = V_2\big(u(0,x)\big)$, $A_1 = a_1 + a_2$, $A_2 = a_1 - a_2$, $B_1 = b_1 + b_2$, $B_2 = b_1 - b_2$, $g_1(t,v) = 2[e^{-t} + \sin t]v_1$ and $g_2(t,v) = 2[e^{-t} - \sin t]v_2$. It is now clear that the stability properties of the comparison system (2.13) imply the corresponding stability properties of the strongly coupled system (2.12).

REFERENCES

[1] H. Amann, *Invariant sets and existence theorems for semilinear parabolic and elliptic systems*, J. Math. Anal. Appl., 65 (1978), 432-467.

[2] N.R. Amundson, *Nonlinear problems in chemical reactor theory*, In SIAM–AMS Proc., Vol III, Amer. Math. Soc. Providence, (1974), 59-84.

[3] R. Aries, *The mathematical theory of diffusion and reaction in permeable catalysts*, Clarendon Press, Oxford, 1975.

[4] R.G. Casten & C.J. Holland, *Stability properties of solutions to systems of reaction-diffusion equations*, SIAM J. Appl. Math., Vol 33 (1977), 353-364.

[5] P. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomath., Vol 28, Springer-Verlag, New York, 1979.

[6] J.K. Hale, *Large diffusivity and asymptotic behavior in parabolic systems*, J. Math. Anal. Appl., 118 (1986), 455-466.

[7] F.A. Howes, *The asymptotic stability of steady solutions of reaction-diffusion-convection equations*, J. Reine. Angeward. Math., (to appear).

[8] F.A. Howes & S. Whitaker, *Asymptotic stability in the presence of convection*, J. Nonlinear Anal. (To appear).

[9] V. Lakshmikantham, *Comparison results for reaction-diffusion equations in a Banach space*, Proc. Conf. Seminar in "A Survey of the theoretical and numerical trends in nonlinear analysis" at the Univ. of Bari, Italy, (1979), 121-156.

[10] V. Lakshmikantham & S. Leela, *Differential and Integral Inequalities*, Vol I and II, Academic Press, New York, 1969.

[11] V.M. Matrosov, *Comparison method in system's dynamics*, Diff. Eq. 10 No. 9 (1974), 408-445.

[12] V.M. Matrosov, *On the theory of stability of motion*, Prikl. mat. Meh. 26 (1962), 992-1002.

[13] R. Redheffer & W. Walter, *On parabolic systems of the Volterra predator-prey type*, J. Nonlinear Anal. TMA, Vol 7 (1983), 333-347.

[14] D.D. Siljak, *Large scale dynamical systems*, North Holland, 1978.