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A MULTIPLICITY RESULT FOR PERIODIC SOLUTIONS OF HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Ambrosetti-Prodi type results are proved for the periodic solutions of some higher-order nonlinear differential equations. The proofs are based upon a priori estimates, degree theory and Lyapunov-Schmidt method.

Introduction. In a recent paper, Fabry, Mawhin and Nkashama [2] have considered periodic problems of the form

$$\begin{aligned} &u'' + f(x,u) = s \\ &u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \end{aligned}$$

and have proved that if

$$f(x,u) \to +\infty$$

as $|u| \to \infty$ uniformly in $x \in [0, 2\pi]$, an Ambrosetti-Prodi type result [1] holds, namely there exists s_1 such that the above problem has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$. A similar result holds for

$$\begin{aligned} &u' + f(x,u) = s \\ &u(0) = u(2\pi) \end{aligned}$$

(see [5]) and the corresponding proofs rely on a combination of techniques of lower and upper solutions and degree theory.

If we consider higher order problems

$$\begin{aligned} &u^{(m)} + f(x,u) = s \\ &u(0) - u(2\pi) = \cdots = u^{(m-1)}(0) - u^{(m-1)}(2\pi) = 0, \end{aligned} \tag{1_s}$$

$(m \geq 3)$, the situation becomes different as the method of upper and lower solutions is only available when $m = 1$ or 2. The aim of this note is to explore the following special case of (1_s)

$$\begin{aligned} &u^{(m)} + g(u) = s + e(x,u) \\ &u(0) - u(2\pi) = \cdots = u^{(m-1)}(0) - u^{(m-1)}(2\pi) = 0 \end{aligned} \tag{2_s}$$

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($m \geq 3$) and prove for it in Theorem 1 a somewhat weakened Ambrosetti-Prodi-like result, namely that if e is bounded on $I \times \mathbf{R}$,

$$g(u) \to +\infty \quad \text{as} \quad |u| \to \infty$$

and, when m is even, if moreover, for all $u \neq v$ and some $\beta < 1$,

$$(-1)^{m/2}\left(g(u) - g(v)\right)(u - v) \geq -\beta(u - v)^2,$$

then there exist numbers $s_0 < s_1$ such that (1_s) has no solution if $s < s_0$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$. The technique of proof relies on the obtainment of suitable a priori bounds and computations of topological degrees given in Sections 2 and 3 respectively.

If we write $u = \overline{u} + \widetilde{u}$, with $\overline{u} = (1/2\pi) \int_0^{2\pi} u(x)\, dx$, a main ingredient of the proof is the obtainment of a priori estimates independent of s for $\widetilde{u}$, when u is a possible solution of (2_s). In the more general equation (1_s), we could only obtain estimates on $\widetilde{u}$ which depend upon s. Therefore, the question remains open if, say for m odd, the above conclusions holds for (1_s) when

$$f(x, u) \to +\infty \quad \text{as} \quad |u| \to \infty$$

uniformly on $[0, 2\pi]$. Another open question is to know if $s_0 = s_1$ for (2_s). We can only prove it under supplementary conditions upon e (Theorem 2).

In what follows, $I = [0, 2\pi]$, $C^k(I)$ is the space of real functions of class C^k on I, $L^P(I)$ the space of measurable functions u on I for which $|u|^P$ is Lebesgue integrable on I and the corresponding usual norms are denoted by $\|u\|_{C^k}$ and $\|u\|_{L^P}$ respectively.

2. A priori estimates. For $k \in \mathbf{N}$, let $C^k_{2\pi}(I) = \{u \in C^k(I) : u^{(j)}(0) = u^{(j)}(2\pi),\ 0 \leq j \leq k\}$. If $m \geq 3$ is an integer, let $D(L) = \{u \in C^{(m-1)}_{2\pi}(I) : u^{(m-1)}$ is absolutely continuous in $I\}$ and $L : D(L) \subset C^0(I) \to L^1(I)$ be a linear Fredholm operator of index zero (see e.g. [4] for terminology) such that $\ker L = \{u \in D(L) : u$ is constant on $I\}$ and $R(L) = \{h \in L^1(I) : \int_I h(x)\, dx = 0\}$. If u and v are real functions on I such that $uv \in L^1(I)$, we shall write

$$(u, v) = \int_I u(x)v(x)\, dx$$

and if $u \in L^2(I)$, we write $\|u\| = (u, u)^{1/2}$.

Let $g : \mathbf{R} \to \mathbf{R}$ be continuous and $e : I \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function with $|e(x, u)| \leq h(x)$ for a.e. $x \in I$, all $u \in \mathbf{R}$ and some $h \in L^1(I)$. We shall obtain a priori estimates for the possible solutions of

$$(Lu)(x) + \lambda g(u(x)) = \lambda s + \lambda e(x, u(x)), \tag{3_s^λ}$$

when $\lambda \in]0, 1]$ and $s \in \mathbf{R}$.

Lemma 1. *Assume that there exist an integer $1 \le p \le m-1$ and $\delta > 0$ such that*

$$|(Lu, u')| \ge \delta \|u^{(p)}\|^2 \tag{4}$$

for all $u \in D(L)$. Then each possible solution u of (3_s^λ) satisfies the inequality

$$\|u^{(p)}\| \le 3^{-1/2}\delta^{-1}\pi\|h\|_{L^1} = M$$

Proof: If u satisfies (3_s^λ), then

$$(Lu, u') + \lambda(g \circ u, u') = \lambda(s, u') + \lambda\big(e(\cdot, u(\cdot)), u'\big),$$

and hence, using the periodicity and (4), we get

$$\delta\|u^{(p)}\|^2 \le |(Lu, u')| = \lambda|(e, u')| \le \|h\|_{L^1}\|u'\|_{L^\infty}.$$

Now, by Sobolev and Wirtinger's inequalities (see e.g. [6]), we have

$$\|u'\|_{L^\infty} \le 3^{-1/2}\pi\|u''\| \le 3^{-1/2}\pi\|u^{(p)}\|$$

and hence

$$\|u^{(p)}\| \le 3^{-1/2}\delta^{-1}\pi\|h\|_{L^1}$$

Example 1. The conditions of Lemma 1 are satisfied for L defined by $Lu = \pm\, u^{(m)}$ where $m \ge 3$ is an odd number as

$$(u^{(m)}, u') = (-1)^{(m-1)/2}\,\|u^{(m+1)/2}\|^2.$$

Conditions implying that (4) holds for more general operators of the form

$$Lu = \sum_{j=0}^{m} a_j u^{(j)}$$

can be found in [3].

When m is an even integer, (4) is never satisfied for L defined by $Lu = u^{(m)}$. Letting, for $u \in L^1(I), u = \overline{u} + \widetilde{u}$, with

$$\overline{u} = (1/2\pi)\int_I u(x)\,dx,$$

we can cover this situation by the following lemma.

Lemma 2. *Assume that there exists an integer $1 \le p \le m-1$, $0 \le \beta < \alpha$ and $\epsilon = \pm 1$ such that*

$$\begin{aligned} \epsilon(Lu, \widetilde{u}) &\ge \alpha\|u^{(p)}\|^2, \quad u \in D(L) \\ \epsilon\big(g(v) - g(w)\big)(v - w) &\ge -\beta(v - w)^2, \quad v, w \in \mathbf{R}. \end{aligned} \tag{5}$$

Then there exists $M > 0$ such that for each $s \in \mathbf{R}$, each $\lambda \in]0, 1]$ and each possible solution u of (3_s^λ) one has

$$\|u^{(p)}\| \le M.$$

Proof: If u satisfies (3_s^λ), then

$$\epsilon(Lu,\widetilde{u}) + \epsilon\lambda(g\circ u,\widetilde{u}) = \epsilon\lambda(s,\widetilde{u}) + \epsilon\lambda\big(e(\cdot,u(\cdot)),\widetilde{u}\big)\,,$$

and hence, by the definition of $\widetilde{u}$ and periodicity,

$$\epsilon(Lu,\widetilde{u}) + \epsilon\lambda\big((g\circ u - g\circ\widetilde{u}),\, u-\widetilde{u}\big) = \epsilon\lambda(e,\widetilde{u})\,. \tag{6}$$

Then

$$\begin{aligned}\epsilon\lambda(e,\widetilde{u}) \geq& \alpha\,\|u^{(p)}\|^2 + \lambda\int_I \epsilon\big(g(u(x)) - g(\overline{u})\big)(u(x)-\overline{u})\,dx\\ \geq& \alpha\,\|u^{(p)}\|^2 - \lambda\beta\int_I |u(x)-\overline{u}|^2\,dx\\ \geq& \alpha\,\|u^{(p)}\|^2 - \beta\|\widetilde{u}\|^2\,.\end{aligned}$$

Therefore, by Wirtinger and Sobolev inequalities, we get

$$\alpha\|u^{(p)}\|^2 \leq \beta\|u^{(p)}\|^2 + 3^{-1/2}\pi\|h\|_{L^1}\,\|u^{(p)}\|\,,$$

i.e.

$$\|u^{(p)}\| \leq (\alpha-\beta)^{-1}\,3^{-1/2}\,\pi\|h\|_{L^1} = M\,.$$

Example 2. For $Lu = u^{(m)}$, m even, we have

$$(Lu,\widetilde{u}) = (Lu,u) = \int_I u^{(m)}(x)\,u(x)\,dx = (-1)^{m/2}\,\|u^{m/2}\|^2\,,$$

and hence

$$(-1)^{m/2}\,(Lu,\widetilde{u}) \geq \|u^{(m/2)}\|^2\,.$$

Thus, we can have $p = m/2$, $\alpha = 1$, $\epsilon = (-1)^{m/2}$.

Conditions implying that the first inequality in (5) holds for more general operators of the form

$$Lu = \sum_{j=0}^{m} a_j\,u^{(j)}$$

can be found in [3].

We shall now obtain a priori estimates for the mean value $\overline{u}$ of possible solutions u of (3_s^λ).

Lemma 3. *Assume that condition (4) or condition (5) is satisfied. If*

$$\lim_{|u|\to\infty} g(u) = +\infty \tag{7}$$

then, for each $s^* \in \mathbf{R}$, there exists $r(s^*)$ such that for each $s < s^*$, each $\lambda \in]0,1]$ and each possible solution $u = \overline{u} + \widetilde{u}$ of (3_s^λ), one has

$$|\overline{u}| < r(s^*) + 3^{-1/2}\,\pi M$$

where M is given by Lemma 1 or Lemma 2.

Proof: Let $s^* \in \mathbf{R}$, $s \leq s^*$, $\lambda \in]0,1]$ and u a possible solution of (3_s^λ). Then, by the property of $R(L)$

$$(1/2\pi) \int_0^{2\pi} g(u(x))\, dx \leq s^* + \overline{h}\,,$$

and, by (7), there exists $r(s^*) > 0$ such that

$$g(v) > s^* + \overline{h}$$

whenever $|v| \geq r(s^*)$. Consequently, there must exist $y \in I$ such that

$$|u(y)| < r(s^*)\,,$$

and hence, by Lemmas 1 and 2 and Sobolev inequality,

$$|\overline{u}| = |u(y) - \widetilde{u}(y)| < r(s^*) + \|\widetilde{u}\|_{L^\infty} \leq r(s^*) + 3^{-1/2}\,\pi M\,.$$

3. Degree computations. Let us define $G : \mathbf{R} \times C^0(I) \to L^1(I)$ by

$$G(u,s) = g \circ u - s - e(\cdot, u(\cdot))$$

so that (3_s^λ) can be written

$$F_s^\lambda(u) \equiv Lu + \lambda G(u,s) = 0\,.$$

It is standard to check that the degree $D_L(F_s^\lambda, \Omega)$ in the sense of [4] is defined for each bounded open set $\Omega \subset C^0(I)$ for which $F_s^\lambda(u) \neq 0$ whenever $\lambda \in]0,1]$, $s \in \mathbf{R}$ and $u \in D(L) \cap \partial\Omega$.

Lemma 4. *If either (4) or (5) holds and (7) is satisfied, then, for each $s^* \in \mathbf{R}$ and each open bounded set $\Omega \subset C^0(I)$ such that*

$$\Omega \supset \{u \in C^0(I) : |\overline{u}| < r(s^*) + \widetilde{r}\,,\ \|\widetilde{u}\|_{L^\infty} < \widetilde{r}\}$$

where $\widetilde{r} > 3^{-1/2}\pi\, M$ and M is given by Lemma 1 or Lemma 2, one has

$$D_L(F_s^1, \Omega) = 0$$

whenever $s \leq s^*$.

Proof: By (7), $s_0 = \min_{\mathbf{R}} g$ exists and if (3_s^λ) has a solution u for some $s \in R$ and $\lambda \in]0,1]$, then

$$\overline{h} + s \geq (1/2\pi) \int_I g(u(x))\, dx \geq s_0\,.$$

Thus, (3_s^λ) has no solution for $s < s_0 - \overline{h} = s_0^*$, so that

$$D_L(F_s^1, \Omega) = 0, \quad s < s_0^*\,.$$

Now, by Lemmas 1 or 2 and Lemma 3, and the homotopy invariance of degree [4], we have

$$D_L(F_s^1, \Omega) = D_L(F_{s^*}^1, \Omega)$$

for all $s \leq s^*$, and the proof is complete.

Lemma 5. *if either (4) or (5) holds and if (7) is satisfied, then there exists $s_1 \geq s_0$ such that, for each $s > s_1$, one can find an open bounded set $\Delta(s)$ in $C^0(I)$ for which*

$$|D_L(F_s^1, \Delta(s))| = 1\,.$$

Proof: Let $u_0 \in \mathbf{R}$ be such that

$$g(u_0) = \min_{\mathbf{R}} g$$

and let

$$s_1 = \max_{[u_0 - \tilde{r}, u_0 + \tilde{r}]} g$$

and, for $s > s_1$, let $\Delta(s) = \{u \in C^0(I) : u_0 < \overline{u} < r(s) + r\,, \|\tilde{u}\|_{L^\infty} < \tilde{r}\}$, where $\tilde{r} > 3^{-1/2}\pi M$, M given by Lemma 1 or 2, and $r(s)$ is given by Lemma 3. If $s > s_1$, $\lambda \in]0,1]$ and u is a possible solution of (3_s^λ) such that $u \in \partial\Delta(s)$, then by Lemmas 1 or 2 and 3, one has necessarily $\overline{u} = u_0$ and

$$u_0 - \tilde{r} < u(x) = \overline{u} + \tilde{u}(x) < u_0 + \tilde{r}$$

for all $x \in I$. But then,

$$s \leq \overline{h} + (1/2\pi) \int_I g(u(x))\, dx \leq s_1\,,$$

a contradiction. Thus $D_L(F_s^\lambda, \Delta(s))$ is well-defined for $s > s_1$ and $\lambda \in]0,1]$. On the other hand, for each $s > s_1$, the mapping

$$\overline{G}_s : \mathbf{R} \to \mathbf{R}, \quad a \longmapsto g(a) - s - (1/2\pi) \int_I e(x,a)\, dx$$

is such that

$$\overline{G}_s(u_0) \leq s_1 - s < 0$$

and

$$\overline{G}_s\big(r(s) + \tilde{r}\big) > 0$$

by definition of s_1 and $r(s)$. Therefore, $\overline{G}_s$ has opposite signs at the boundary points of

$$]u_0, r(s) + \tilde{r}[\approx \Delta(s) \cap \ker L.$$

By Proposition II.12 of [4], this implies that

$$\big|D_L\big(F_s^1, \Delta(s)\big)\big| = \big|D_B\big(\overline{G}_s,]u_0\,;\ r(s) + \tilde{r}[\big)\big| = 1,$$

where D_B denotes the Brouwer degree, and the proof is complete.

4. Existence and multiplicity results. The results of Section 3 allow us to state and prove an existence and multiplicity result of Ambrosetti-Prodi type for equation

$$(Lu)(x) + g(u(x)) = s + e(x, u(x))\,. \tag{8_s}$$

Theorem 1. *Assume that the following conditions are satisfied.*

1) $L : D(L) \subset C^0(I) \to L^1(I)$ *is a linear Fredholm operator of index zero such that, for some integer* $m \geq 3$,

$$D(L) = \{u \in C_{2\pi}^{m-1}(I) : u^{(m-1)} \text{ is absolutely continuous on } I\}$$

$$\ker L = \{u \in D(L) : u \text{ is constant on } I\}$$

$$R(L) = \{v \in L^1(I) : \int_I v(x)\,dx = 0\}$$

2) $e : I \times \mathbf{R} \to \mathbf{R}$ *is Caratheodory and* $|e(x,u)| \leq h(x)$ *with* $h \in L^1(I)$.

3) *Either*

$$|(Lu, u')| \geq \delta \|u^{(p)}\|^2$$

for some $\delta > 0$, *some integer* $1 \leq p \leq m-1$ *and all* $u \in D(L)$ *or there exist* $0 \leq \beta < \alpha$, *an integer* $1 \leq p \leq m-1$ *and* $\epsilon = \pm 1$ *such that*

$$\epsilon(Lu, \tilde{u}) \geq \alpha \|u^{(p)}\|^2 , \quad u \in D(L)$$

$$\epsilon\big(g(v) - g(w)\big)(v - w) \geq -\beta(v - w)^2 , \quad v, w \in \mathbf{R}.$$

4) $\lim_{|v| \to \infty} g(v) = +\infty$.

Then there exist real numbers $s_0 \leq s_1$ *such that*

(i) (8_s) *has no solution for* $s < s_0$,
(ii) (8_s) *has at least one solution for* $s = s_1$,
(iii) (8_s) *has at least two solutions for* $s > s_1$.

Proof: Part (i) has been proved in the beginning of the proof of Lemma 4. For part (iii), if $s > s_1$ with s_1 given by Lemma 5, we can always choose Ω in Lemma 4 in such a way that $\Omega \underset{\neq}{\supset} \Delta(s)$, where $\Delta(s)$ is defined in Lemma 5. By the additivity of degree we have

$$0 = D_L(F_s^1, \Omega) = D_L\big(F_s^1, \Delta(s)\big) + D_L(F_s^1, \Omega \backslash \overline{\Delta(s)})$$

and hence, by the results of Lemmas 4 and 5,

$$|D_L(F_s^1, \Omega \backslash \overline{\Delta(s)})| = 1 ,$$

and (8_s) has one solution in $\Delta(s)$ and one in $\Omega \backslash \overline{\Delta(s)}$. Finally, taking a sequence (σ_k) in $]s_1, \infty[$ which converges to s_1 and a corresponding sequence (u_k) of solutions of (8_{σ_k}), we deduce from the a priori bounds of Lemmas 1 or 2 and 3 and from Ascoli-Arzela's theorem that (u_k) has a subsequence (u_{j_k}) wich converges in $C^0(I)$ to some $u \in C^0(I)$. It then follows easily from the closedness of the graph of L that $u \in D(L)$ and is a solution of (8_{s_1}), which completes the proof.

Remark. We do not know if $s_0 = s_1$ under the conditions of Theorem 1. However, the following partial result is true.

Theorem 2. *Assume that the assumptions of Theorem 1 hold with $e(x,u) = e(x) \in R(L)$ and (3) replaced by*

3′) *There exist $0 \le \beta < \alpha$, an integer $1 \le p \le m-1$ and $\epsilon = \pm 1$ such that*

$$\begin{aligned} \epsilon(Lu,u) &\ge \alpha \|u^{(p)}\|^2\,, \quad (u \in D(L)) \\ \epsilon\big(g(v) - g(w)\big)(v-w) &\ge -\beta |v-w|^2\,, \ (v, w \in \mathbf{R})\,. \end{aligned} \tag{9}$$

Then there exists $s_1 \in \mathbf{R}$ such that

(i) (8_s) *has no solution for $s < s_1$,*
(ii) (8_s) *has at least one solution for $s = s_1$,*
(iii) (8_s) *has at least two solutions for $s > s_1$.*

Proof: By a usual Lyapunov-Schmidt argument [4], (8_s) is equivalent to the pair of equations

$$L\widetilde{u} + (I-P)g(\overline{u} + \widetilde{u}) = e\,, \tag{10}$$

$$Pg(\overline{u} + \widetilde{u}) = s\,, \tag{11_s}$$

where P is the projector defined on $L^1(I)$ by $Pv = (1/2\pi)\int_I v(x)\,dx$. Let us fix $\overline{u} \in \mathbf{R}$ and consider (10) as an equation in $\widetilde{u} \in \widetilde{C}^0(I) = \{u \in C^0(I) : Pu = 0\}$. By the argument used in the proof of Lemma 2, we see that, for each $\lambda \in [0,1]$ and each possible solution $\widetilde{u}$ of $L\widetilde{u} + \lambda(I-P)g(\overline{u}+\widetilde{u}) = \lambda e$, one has

$$\|\widetilde{u}^{(p)}\| \le (\alpha - \beta)^{-1}\left(3^{-1/2}\pi \|e\|_{L^1}\right) \tag{12}$$

and hence, as L is invertible in $\widetilde{C}^0(I)$, degree theory implies the existence of at least one solution $\widetilde{u}$ of (10). Such a solution is unique, because if $\widetilde{u}_1$ and $\widetilde{u}_2$ solve (10) with $\overline{u}$, then

$$L(\widetilde{u}_1 - \widetilde{u}_2) + (I-P)\big(g(\overline{u} + \widetilde{u}_1) - g(\overline{u} + \widetilde{u}_2)\big) = 0$$

and hence, letting $\widetilde{v} = \widetilde{u}_1 - \widetilde{u}_2$, we get

$$\epsilon(L(\widetilde{v}), \widetilde{v}) + \epsilon\big(g(\overline{u} + \widetilde{u}_1) - g(\overline{u} + \widetilde{u}_2), \widetilde{v}\big) = 0\,.$$

But, then

$$\alpha \|\widetilde{v}^{(p)}\|^2 - \beta \|\widetilde{v}\|^2 \le 0\,,$$

wich implies that $\widetilde{v} = 0$ by Wirtinger's inequality. Denoting this unique solution of (10) by $v(\overline{u})$, we have, for $\overline{u}$, $\overline{u}_0 \in \mathbf{R}$,

$$L\Big(v(\overline{u}) - v(\overline{u}_0)\Big) + (I-P)\Big(g(\overline{u} + v(\overline{u})) - g(\overline{u}_0 + v(\overline{u}_0))\Big) = 0$$

and hence

$$\begin{aligned} L\Big(v(\overline{u}) - v(\overline{u}_0)\,,\, v(\overline{u}) - v(\overline{u}_0)\Big) &+ \Big(g(\overline{u}_0 + v(\overline{u})) - g(\overline{u}_0 + v(\overline{u}_0))\,,\, v(\overline{u}) - v(\overline{u}_0)\Big) \\ &+ \Big((I-P)(g(\overline{u} + v(\overline{u})) - g(\overline{u}_0 + v(\overline{u}))\,,\, v(\overline{u}) - v(\overline{u}_0)\Big) = 0. \end{aligned}$$

Consequently,

$$3^{-1/2}\pi(\alpha-\beta)\|v(\overline{u})-v(\overline{u}_0)\|_{L^\infty} \leq \|g(\overline{u}+v(\overline{u}))-g(\overline{u}_0+v(\overline{u}))\|\,\|v(\overline{u})-v(\overline{u}_0)\|\,,$$

so that

$$\|v(\overline{u})-v(\overline{u}_0)\|_{L^\infty} \leq (\alpha-\beta)^{-1}\,\|g(\overline{u}-v(\overline{u}))-g(\overline{u}_0+v(\overline{u}))\|\,.$$

The continuity of v follows then from that of g and the a priori bound $\|v(\overline{u})\|_{L^\infty} \leq (\alpha-\beta)^{-1}\|e\|_{L^1}$ deduced from (12).

Our equation (8_s) is then reduced to the scalar equation in $\overline{u}$

$$G(\overline{u}) \equiv Pg(\overline{u}+v(\overline{u})) = s \tag{13}$$

and, by the a priori bound on $v(\overline{u})$ and condition 4), we see that $G(\overline{u}) \to +\infty$ as $|\overline{u}| \to \infty$. Consequently, the range of the real continuous function G is a closed interval of the form $[s_1,+\infty[$, where $s_1 = \min_{\mathbf{R}} G$, and, if $G(u_0) = s_1$, on easy argument based upon the intermediate value theorem shows that, for each $s > s_1$, (13) has one solution in $]-\infty, u_0[$ and one in $]u_0,+\infty[$, which completes the proof.

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