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MULTIVALUED DIFFERENTIAL EQUATIONS ON CLOSED SETS

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Abstract. Let X be a Banach space, $D \subset X$ closed, $J = [0, a] \subset \mathbf{R}$ and $F : J \times D \to 2^X \backslash \{\emptyset\}$ a multivalued map. We consider the initial value problem

$$u' \in F(t, u) \quad \text{a.e. on } J,\ u(0) = x_0 \in D\,. \tag{1}$$

Solutions of (1) are understood to be a.e. differentiable with $u' \in L^1_X(J)$ such that $u(t) = x_0 + \int_0^t u'(s)\,ds$ on J and (1) is satisfied. We first give sufficient conditions for existence of solutions in the autonomous case $F(t, x) = F(x)$ and indicate relations to the fixed problem $x \in F(x)$. We concentrate on upper semicontinuous F with compact convex images and give counter-examples if F is lower semicontinuous or the $F(x)$ are not convex. Then the time-dependent problem (1) is considered by reduction to the autonomous case. Finally we prove existence of solutions which are monotone with respect to a preorder or Lyapaunov-like functions. Since we allow $\dim X = \infty$, we find it necessary to exploit Zorn's Lemma. As a by-product of this approach we achieve considerable simplification of proofs or improvement of some results known for $\dim X < \infty$.

1. Preliminaries. All concepts not discussed in detail in the sequel can be found in several places, e.g. in [5], [6]. First of all, it is easy to see that a necessary condition for $u' \in F(u)$, $u(0) = x_0$ to have a solution for every $x_0 \in D$ is given by $F(x) \cap T_D(x) \neq \emptyset$ on D where

$$T_D(x) = \{y \in X : \varliminf_{\lambda \to 0+} \lambda^{-1} \rho(x + \lambda y\,,\, D) = 0\} \text{ with } \rho(z, D) = \inf_D |z - x|. \tag{2}$$

In case D has nonempty interior $\overset{0}{D}$ this is only a condition at the boundary ∂D, since $T_D(x) = X$ for $x \in \overset{0}{D}$. If D is closed and convex we have $T_D(x) = \overline{\{\lambda(y - x) : \lambda \geq 0\,,\ y \in D\}}$.

In the majority of applications the "multis" F are *upper semicontinuous* (*usc*), i.e., $F^{-1}(A) = \{x \in D : F(x) \cap A \neq \emptyset\}$ is closed whenever $A \subset X$. F is said to be *lower semicontinuous* (*lsc*), if $F^{-1}(V)$ is open whenever $V \subset X$ is, and *continuous* if F is continuous with respect to the Hausdorff metric. Since our main interest is in $\dim X = \infty$

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we also need, say, a condition of compactness type like $\alpha(F(B)) \leq k\alpha(B)$ for some $k \geq 0$ and all bounded $B \subset D$, where $\alpha(B) = \inf\{d > 0 : B = \bigcup_{i=1}^m B_i$ for some m and B_i with $\operatorname{diam}(B_i) \leq d\}$ is Kuratowski's measure of noncompactness and $F(B) = \bigcup_{x \in B} F(x)$; see e.g. §7.3 of [6] for properties of $\alpha(\cdot)$. Much depends also on properties of the set $F(x)$. We assume that $F(x)$ is closed convex; in connection with the α-estimate this then means compact convex, since $\alpha(F(x)) \leq k\alpha(\{x\}) = 0$.

Without convexity of $F(x)$ there may be no solution. A known counter-example is $D = \overline{B}_1(0) \subset \mathbf{R}^2$ (the closed unit ball), $F(x) = \{(-1,0),(1,0)\}$ and $x_0 = (0,1)$, in which case $T_D(x) = \{y \in \mathbf{R}^2 : \langle x, y\rangle \ \leq 0\}$ for $|x| = 1$, hence $F(x) \cap T_D(x) \neq \emptyset$ on D.

Let us finally recall that $u : J \to X$ is (Bochner-) integrable iff $|u(\cdot)| \in L^1_{\mathbf{R}}(J)$ and u is strongly measurable, i.e., the a.e. limit of a sequence of step functions. The Banach space of equivalence classes of such u will be denoted by $L^1_X(J)$. The Bochner integral has the usual properties, say $(\int_0^t w(s)\,ds)' = w(t)$ a.e. for $w \in L^1_X(J)$. A sufficient condition for weak compactness (= weak sequential compactness) in $L^1_X(J)$ is given by Theorem 2 in [8], namely

Lemma 1. *Consider $L^1_X(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$ and let $K \subset X$ be weakly compact convex. Then $\widetilde{K} = \{u \in L^1_X(\Omega,\Sigma,\mu) : u(\omega) \in K$ a.e.$\}$ is weakly compact.*

2. The Autonomous Problem. Our basic existence theorem for upper semicontinuous F is:

Theorem 1. *Let X be a Banach space, $D \subset X$ closed, $F : D \to 2^X \backslash \{\emptyset\}$ usc, $F(x)$ closed convex for all $X \in D$, $\alpha(F(B)) \leq k\alpha(B)$ for some $k \geq 0$ and all bounded $B \subset D$, $F(x) \cap T_D(x) \neq \emptyset$ on D and*

$$\|F(x)\| = \sup\{|y| : y \in F(x)\} \leq c(1 + |x|) \text{ on } D \text{ for some } c > 0\,. \tag{3}$$

Then $u' \in F(u)$, $u(0) = x_0 \in D$ has a solution on $\mathbf{R}_+$.

Proof: Evidently it is enough to show that a solution exists on $J = [0,a]$ for some $a > 0$. Given $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 < 1$ sufficiently small, we consider an approximate solution $v : J \to X$ such that

$$v(t) = x_0 + \int_0^t v'(s)\,ds \text{ and } D \cap \overline{B}_\eta(v(t)) \neq \emptyset \text{ on } J, v(a) \in D \tag{4}$$

$$v'(t) \in F(D \cap \overline{B}_\eta(v(t))) + \overline{B}_\epsilon(0) \text{ a.e. on } J \tag{5}$$

where $\eta = \epsilon(1+d)$ with d such that $F(D \cap \overline{B}_\eta(v(t))) \subset \overline{B}_d(0)$ on J. Notice first that if we let $\varphi(t) = |v(t)|$ then $\varphi(0) = |x_0|$ and

$$\varphi'(t) \leq c(1 + \epsilon d + \epsilon) + \epsilon + c\varphi(t) \text{ a.e. on } J\,,$$

hence $\varphi(t) \leq (|x_0| + 2 + 1/c)e^{ca} + \epsilon d e^{ca}$, and therefore we find the desired d by choosing ϵ_0 such that $c\epsilon_0(1 + e^{ca}) < 1$. Now, the existence of a v satisfying (4), (5) can be shown by means of Zorn's Lemma as follows. We find $y \in F(x_0) \cap T_D(x_0)$, $\delta \in (0, \epsilon]$ and $z \in D$ such that $|x_0 + \delta y - z| \leq \delta\, \epsilon$. Let $v(t) = x_0 + (z - x_0)t/\delta$ on $[0, \delta]$. Then

$$v' = (z - x_0)/\delta \in \overline{B}_\epsilon(y) \subset F(x_0) + \overline{B}_\epsilon(0) \text{ and } |v(t) - x| \leq \eta \text{ on } [0, \delta]\,,$$

hence $M = \{(v, \delta) : \delta \in (0, a],\, v : [0, \delta] \to X$ satisfies (4), (5) on $J = [0, \delta]\} \neq \emptyset$. We introduce the partial ordering $(v_1, \delta_1) \leq (v_2, \delta_2)$ iff $\delta_1 \leq \delta_2$ and $v_{2\,|\,[0,\delta_1]} = v_1$. It is clear that every chain has an upper bound, and therefore M has a maximal element (v^*, δ^*). Then necessarily $\delta^* = a$, since otherwise we can extend v^* beyond δ^* by means of a segment constructed as before, with $(v^*(\delta^*), \delta^*)$ instead of $(x_0, 0)$.

Now, consider $\epsilon_n \to 0+$ and v_n satisfying (4), (5) on $J = [0, a]$ with $\epsilon = \epsilon_n$. Then (v_n) is bounded and equicontinuous, since Lipschitz of constant $d + 1$. The function $\varphi(t) = \alpha(\{v_n(t) : n \geq 1\})$ is a.c. with $\varphi(0) = 0$ and

$$\begin{aligned}\varphi'(t) \leq& 2\alpha(\{v_n'(t) : n \geq p\}) \\ \leq& 2\alpha(F(\{y_n(t) : n \geq p\})) + 4\epsilon_p \text{ (with } |y_n - v_n|_\infty \leq \eta_n \text{)} \\ \leq& 2k\varphi(t) + 4\epsilon_p(k(d+1) + 1) \text{ a.e.}\,,\end{aligned} \tag{6}$$

hence $\varphi(t) = 0$ on J, since $p \geq 1$ was arbitrary. Thus, (v_n) is relatively compact in $C_X(J)$ with max-norm $|\cdot|_0$, and therefore we may assume $|v_n - v|_0 \to 0$ for some $v \in C_D(J)$. Consequently

$$v_n'(t) \in F(D \cap \overline{B}_\delta(v(t))) + \overline{B}_\delta(0) \text{ a.e. on } J \text{ for all } n \geq n_\delta\,, \tag{7}$$

hence $v_n' \in C_\delta = \{u \in L^1_X(J) : u(t) \in K_\delta(t) \text{ a.e. }\}$ for $n \geq n_\delta$, where $K_\delta(t)$ is $\overline{\text{conv}}$ of the right hand side in (7). Since C_δ is closed convex, hence weakly closed, we only have to show that (v_n') is relatively weakly compact. Indeed, we may then assume $v_n' \rightharpoonup w \in L^1_X(J)$, hence $v_n(t) = x_0 + \int_J \chi_{[0,t]} v_n'\, ds$ yields $v(t) = x_0 + \int_0^t w(s)\, ds$ on J and therefore $v' = w$ a.e., consequently $v' \in C_\delta$ for all $\delta > 0$ and therefore $v'(t) \in F(v(t))$ a.e., since F is usc and $F(y)$ is compact convex for all $y \in D$.

To see that (v_n') has a weakly convergent subsequence, consider $K = \overline{\text{conv}} F(v(J))$ which is compact convex. Let $R : X \to K$ be a retraction and $w_n = R \circ v_n'$. By Lemma 1 we may assume $w_n \rightharpoonup w$ and, given $\epsilon > 0$, we find $\eta = \eta(\epsilon)$ such that $|R(x) - x| \leq \epsilon$ on $K + \overline{B}_\eta(0)$. For $\delta \leq \delta(\epsilon)$ we have K_δ, introduced behind (7), in $K + \overline{B}_\eta(0)$, hence $|w_n(t) - v_n'(t)| \leq \epsilon$ a.e. on J for all $n \geq n(\epsilon)$, and therefore $v_n' \rightharpoonup w$.

Remarks. 1. Theorem 1 for closed bounded convex D and reflexive X is Lemma 1 in [11], the proof of which was based on Alaoglu's theorem instead of Lemma 1.

2. Theorem 1 can be used to prove the following general fixed point theorem, Theorem 1 in [7]: "If D is closed bounded convex, $G : D \to 2^X \backslash \{\emptyset\}$ is usc and α-condensing (i.e., $\alpha(G(B)) < \alpha(B)$ for $B \subset D$ with $\alpha(B) > 0$), $G(x)$ is closed convex and $G(x) \cap (x + T_D(x)) \neq \emptyset$ on D (i.e., G is weakly inward) then G has a fixed point". The proof given in [7] was possible without knowing Theorem 1 since approximate solutions for $u' \in G(u) - u$, $u(0) = x \in D$ are sufficient for this purpose.

3. A case much simpler than Theorem 1 was considered in [12]. The main Theorem 3.1 of which says that a solution exists if D and $F(x)$ are weakly compact convex, $F(x) \cap T_D(x) \neq \emptyset$

on D and F is usc in the weak topology on D and X, very restrictive conditions. The proof is very easy, since application of Lemma 1 is obvious and v_n can be chosen as polygons

$$v_n(t) = x_i + (x_{i+1} - x_i)(t - ih)/h \text{ on } [ih\,, ih + h] \text{ with } h = a/n\,,$$

due to the fact that $hF(x) + x_i - x$ has a zero x_{i+1} if $x_i \in D$. Clearly, knowing the fixed point theorem from Remark 2, one can also use these polygons to prove Theorem 1 for closed bounded convex D, since $G = hF + x_i$ is α-condensing for small h and $(G(x)-x) \cap T_D(x) \neq \emptyset$ on D is obvious. However, this is at most interesting if one can prove the fixed point result without relying on differential equations, which is not the case so far, even for singlevalued maps F.

4. Of course "$\alpha(F(B)) \leq k\alpha(B)$" can be replaced by "$\alpha(F(B)) \leq \varphi(\alpha(B))$," where $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, $\varphi(0) = 0$ and $\rho' = \varphi(\rho)$, $\rho(0) = 0$ has only the trivial solution; see e.g. Theorem 2.2 in [5].

5. In case F is lsc we also have to assume that the $F(x)$ are closed convex; see the counter-example in section 2. However, the following counter-example shows that Theorem 1 is wrong if we replace usc by lsc, even if D is closed bounded convex. Another example where neither D nor the $F(x)$ are convex was given in [3].

Counter-Example. As a slight modification of the last example in [11], let $D = \{x \in \mathbf{R}^2 : |x| \leq 1,\, x_2 \geq 0\}$, consider the triangle Δ with vertices $(-1,1)$, $(1,1)$ and $(0,2)$, let $x_0 = (0,1)$, define $G(x_0) = \{(1,1)\}$ and

$$G(x) = \Delta \cap \overline{B}_{r(x_2)}((0,\, 1 + r(x_2))) \text{ with } r(x_2) = (1 - x_2)^{-1} \text{ for } x_0 \neq x \in D\,,$$

and let $F(x) = G(x) - x$. It is easy to see that F is lsc, $F(x)$ is compact convex and $F(x) \cap T_D(x) \neq \emptyset$ on D. But $u' \in F(u)$, $u(0) = x_0$ has no solution, since otherwise $u_2' \geq 1 - u_2$ a.e., and $u_2(0) = 1$, hence $u_2(t) \geq 1$ and therefore $u(t) \equiv x_0$, contradicting $x_0 \notin G(x_0)$. This example shows also that the fixed point theorem in Remark 2 becomes wrong if we replace usc by lsc.

6. In case D is compact, appropriate finite open coverings of D allow the construction of approximate solutions as polygons; see e.g. the finite-dimensional reference [1]. In the situation of Theorem 1 the polygons may break down, i.e., it is not clear whether they exist on a fixed interval for all sufficiently small $\epsilon > 0$. Therefore we used Zorn's Lemma, which is more elegant anyway.

3. The Time–Dependent Problem. Given $F : J \times D \to 2^X \backslash \{\emptyset\}$ with $J = [0, a] \subset \mathbf{R}$ and $D \subset X$ closed, we try to find a solution of (1) by reduction to the autonomous case. For this purpose it is convenient to extend F by means of $F(t,x) = F(a,x)$ for $t \geq a$. So, consider $X_0 = \mathbf{R} \times X$ with $|(\tau, x)| = |\tau| + |x|$, let $D_0 = \mathbf{R}_+ \times D$ and $F_0 : D_0 \to 2^{X_0} \backslash \{\emptyset\}$ be defined by $F_0(\tau, x) = \{1\} \times F(\tau, x)$. Obviously

$$(1, y) \in F_0(\tau, x) \cap T_{D_0}(\tau, x) \text{ iff } y \in F(\tau, x) \cap T_D(x)\,.$$

Therefore we assume

$$F(t,x) \cap T_D(x) \neq \emptyset \text{ on } J \times D \tag{8}$$

and consider $u' \in F_0(u)$ a.e., on $\mathbf{R}_+$, $u(0) = (0, x_0)$. Evidently

$$\|F(t,x)\| \le c(1+|x|) \text{ on } J \times D \tag{9}$$

is good enough to get reduction to the case of bounded F_0 as in the proof to Theorem 1. Finally, let $\alpha_0(\cdot)$ be the $\alpha(\cdot)$ for X_0. Since the corresponding approximate solutions can now be chosen of the form $\overline{v}_n(t) = (\tau_n(t), v_n(t))$ with $|\tau_n(t) - t| \le \epsilon_n t$ on J and since $\alpha_0(\{\overline{v}_n(t) : n \ge 1\}) \ge \alpha(\{v_n(t) : n \ge 1\})$, it is also easy to see that

$$\lim_{\tau \to 0+} \alpha\big(F(J_{t,\tau} \times B)\big) \le k(t)\alpha(B) \text{ a.e. on } J, \text{ with } J_{t,\tau} = [t-\tau, t+\tau] \cap J \tag{10}$$

for all bounded $B \subset D$ and some $k \in L^1_{\mathbf{R}}(J)$ is sufficient to get $\alpha_0\big(\{\overline{v}_n(t) : n \ge 1\}\big) = 0$ on J. Therefore Theorem 1 implies

Theorem 2. *Let X be a Banach space, $D \subset X$ closed, $J = [0,a] \subset \mathbf{R}$, $F : J \times D \to 2^X \backslash \{\emptyset\}$ usc, $F(t,x)$ compact convex for all $(t,x) \in J \times D$. Suppose also that F satisfies (8),(9) and (10). Then (1) has a solution on J.*

Remarks. 7. Even if F is nice, the α-estimates for the sections $F(t,\cdot)$ may be bad, also for singlevalued F. An estimate more careful than (6) shows that it is enough to have $\lim_{\tau \to 0+} \alpha(F(J_{t,\tau} \times B)) \le \varphi(t, \alpha(B))$ for appropriate φ such that $\varphi(t,\cdot)$ is continuous and increasing and $\rho' = \varphi(t,\rho)$, $\rho(0) = 0$ has only the trivial solution; see e.g. the proof of Theorem 2.1 in [5].

8. By the method of proof we had to assume that F is jointly usc. In case we have no constraints, i.e., $D = X$, it is easy to weaken this condition. In fact, Theorem 2 with $D = X$ remains true if we replace "F usc" by "$F(t,\cdot)$ is usc and $F(\cdot,x)$ has a strongly measurable selection", c in (9) is a function from $L^1_{\mathbf{R}_+}(J)$ and (10) (or the estimate mentioned in Remark 7) is true with $J_{t,\tau} = [t-\tau, t] \cap J$. This is Theorem 4.4 in [14] proved by means of tedious arguments. Therefore it is worth to show that the result can be obtained by the usual fixed point approach as follows.

First, if $v \in C_X(J)$, then approximation by step functions shows that $F(\cdot, v(\cdot))$ also admits a strongly measurable selection w (we write $w \in F(\cdot, v(\cdot))$ for short); see e.g. Lemma 1.2 in [13]. Since (9) yields a priori bounds, let

$$K = \left\{ v \in C_X(J) : v(0) = x_0; |v(t)| \le \psi_1(t) \text{ and } |v(t) - v(s)| \le \left| \int_s^t \psi_2(s)\, ds \right| \right\}$$

with $\psi_1 \in C_{\mathbf{R}}(J)$ and $\psi_2(t) = c(t)(1 + \psi_1(t))$ such that the multi G, defined by

$$G(v) = \left\{ u \in C_X(J) : u(t) = x_0 + \int_0^t w(s)\, ds \text{ on } J,\ w \in F(\cdot, v(\cdot)) \right\},$$

maps K into $2^K \backslash \{\emptyset\}$. Since K is equicontinuous we have $\alpha_0(B) = \max_J \alpha(B(t))$ for $B \subset K$, where $\alpha_0(\cdot)$ is the $\alpha(\cdot)$ for $C_X(J)$ and $B(t) = \{v(t) : v \in B\}$. Now, consider the usual sequence $K_0 = K$, $K_{n+1} = \overline{\operatorname{conv}}\, G(K_n)$ and $K_\infty = \bigcup_{n \ge 1} K_n$. Let $\rho_n(t) = \alpha(K_n(t))$ and $\gamma(t) = \alpha(G(K_n)(t))$. Then it is easy to see that $\gamma'(t) \le \varphi(t, \rho_n(t))$ a.e., for φ from Remark 7, hence $\rho_{n+1}(t) \le \int_0^t \varphi(s, \rho_n(s))\, ds$ on J. Since ρ_n converge uniformly to ρ this means

$\rho(t) \leq \int_0^t \varphi(s, \rho(s))\, ds$, hence $\rho = 0$ and therefore K_∞ is compact convex. Application of Lemma 1 shows that $G : K_\infty \to 2^{K_\infty} \backslash \{\emptyset\}$ is usc and the $G(v)$ are compact convex, hence the fixed point theorem from Remark 2 applies trivially.

It is, however, not clear whether this kind of reasoning can be used to generalize Theorem 2.

9. Since we have used α-estimates to get relative compactness of the sequence of approximate solutions, $F(t, x)$ is not allowed to be unbounded, say a subspace or a cone. In case $\dim X < \infty$, results for such F have been obtained recently in [10].

4. Monotone Solutions. In the study of stability and some economical models, one is naturally lead to consider so-called "monotone" solutions of (1); see e.g. [1] and the references given there. By the idea of proof given in the previous sections, it is easy to extend the existence theorem given in [1] for such solutions, from $\mathbf{R}^n$ to an arbitrary X and to F satisfying growth condition (9). Since reduction to the autonomous case is as easy as in section 3, we consider only

$$u' \in F(u), \;\; u(0) = x_0 \in D\,. \tag{11}$$

Given $x \in D$, $y \in T_D(x)$, $V : D \to \mathbf{R}_+$ and W : graph $(F) \to \mathbf{R}_+$, we let

$$D_+V(x)(y) = \liminf_{\lambda \to 0+,\, z \to y} \lambda^{-1}(V(x + \lambda z) - V(x))$$

and try to find a solution of (11), called monotone w.r. to V and W in [1], such that

$$V(u(t)) - V(u(s)) + \int_s^t W(u(\tau), u'(\tau))\, d\tau \leq 0 \text{ for all } 0 \leq s \leq t \leq a\,. \tag{12}$$

Theorem 3. *Let X, D and F be as in Theorem 1 with "$F(x) \cap T_D(x) \neq \emptyset$ on D" replaced by*

$$F(x) \cap \{y \in T_D(x) : D_+V(x)(y) + W(x, y) \leq 0\} \neq \emptyset \text{ on } D\,, \tag{13}$$

where $V : D \to \mathbf{R}_+$ and $W : D \times \overline{\mathrm{conv}}\, F(D) \to \mathbf{R}_+$ are continuous and $W(x, \cdot)$ is convex for every $x \in D$. Then (11) has a solution on $\mathbf{R}_+$ satisfying (12).

Proof: It is again enough to find such a solution on $J = [0, a]$ for some $a > 0$. Given $\epsilon > 0$ sufficiently small as in the proof of Theorem 1, we find $v : J \to X$ satisfying (4), (5) and

$$V(y_1(t)) - V(y_2(s)) + \int_s^t W(w(\tau))\, d\tau \leq \epsilon\, a \;\text{ for }\; 0 \leq s \leq t \leq a \tag{14}$$

with functions $y_1 : J \to D$, $y_2 : [0, a) \to D$ and $w : J \to$ graph (F) such that

$$y_1(t) \in \overline{B}_\eta(v(t)) \text{ on } J \text{ and } y_1(a) = v(a),\; y_2(t) \in \overline{B}_\eta(v(t)) \text{ on } [0, a) \tag{15}$$

$$w(t) \in \overline{B}_\eta((v(t), v'(t))) \text{ a.e. on } J. \tag{16}$$

To see this, notice that by (13) we find $y \in F(x_0)$, $\delta \in (0,\epsilon]$ and z with $x+\delta z \in D$ and $|z-y|$ sufficiently small such that

$$V(x_0+\delta z) - V(x_0) + \delta W(x_0,y) \le \epsilon\,\delta\,.$$

Hence (14)–(16) hold for $a=\delta$, $v(t)=x_0+zt$, $y_1(t)=x_0+\delta z$, $y_2(t)=x_0$ and $w(t)=(x_0,y)$. Therefore we consider the partial ordering defined by $(y_1,y_2,w,v,\delta) \le (\overline{y}_1,\overline{y}_2,\overline{w},\overline{v},\overline{\delta})$ iff $\delta \le \overline{\delta}$ and the restriction of $(\overline{y}_1,\overline{y}_2,\overline{w},\overline{v})$ to $[0,\delta]$ coincides with (y_1,y_2,w,v). Zorn's Lemma applies and yields a maximal element $(y_1^*,y_2^*,w^*,v^*,\delta^*)$. If $\delta^* < a$, we can extend, choosing in particular $y_1^*(t)=v(\delta^*)+\overline{\delta}\overline{z}$ on $(\delta^*,\delta^*+\overline{\delta}]$ and $y_2^*(t)=v(\delta^*)$ on $[\delta^*,\delta^*+\overline{\delta})$. This yields a contradiction, since this way all conditions are satisfied on $[0,\delta^*+\overline{\delta}]$; notice, for example, that for $s<\delta^*<t$

$$\begin{aligned} V(y_1^*(t)) - V(y_2^*(s)) + \int_s^t W(w^*(\tau))\,d\tau \le \\ &\le \epsilon\overline{\delta} + V(y_2^*(\delta^*)) - V(y_1^*(\delta^*)) + \epsilon\delta^* \\ &= \epsilon(\delta^*+\overline{\delta}) \text{ since } y_2^*(\delta^*) = y_1^*(\delta^*) = v(\delta^*). \end{aligned}$$

Now, we consider again $\epsilon_n \to 0+$ and the corresponding v_n. We may assume $|v_n-u|_0 \to 0$ and $v_n' \rightharpoonup u'$ in $L^1_X(J)$ as $n\to\infty$, for some solution of (11). By (16) we also have $w_{n2} \rightharpoonup u'$, hence, by Mazur's Theorem (see e.g. Theorem II, 5.2 in [4]), a sequence of convex combinations $\sum_{k=n}^{m_n} \alpha_k w_{k2}$ converges strongly and therefore a.e. (without loss of generality) to u'. Since $w_k(t) \in \operatorname{graph}(F)$, we have $W(u(t),w_{k2}(t)) \le W(w_k(t))+\epsilon$ on J for all $k \ge k_\epsilon$ and, since the $W(u(t),\cdot)$ are convex, it is clear that (14) for each k implies (12).

Remark 10. Theorem 3 with $V=W=0$ is Theorem 1. Other interesting V and W can be found in [1]. Condition (13) is again necessary in the sense that if a solution satisfying (12) exists for every $x_0 \in D$, then (13) holds.

In case $W=0$, condition (12) is the simplest example for a monotonicity concept based on a preorder $\le$ on $D\times D$; i.e. $\le$ is reflexive and transitive. By consideration of $P(x)=\{y\in D : y\le x\}$, it is clear that such a preorder can be described equivalently by a multi $P: D\to 2^D\backslash\{\emptyset\}$ such that $x\in P(x)$ on D and $y\in P(x)$ implies $P(y)\subset P(x)$. For example, $P(x)=\{y\in D : V_i(y)\le V_i(x) \text{ for } i=1,\dots,m\}$ with given functions $V_i : D\to \mathbf{R}$ plays a role in some economical models. In this respect one is interested in solutions of (11) satisfying $u(t)\in P(u(s))$ for $0\le s\le t$. For this problem we have:

Theorem 4. *Let X, D and F be as in Theorem 1 with "$F(x)\cap T_D(x)\neq\emptyset$ on D" replaced by*

$$F(x)\cap T_{P(x)}(x) \neq \emptyset \quad \text{on} \quad D\,, \tag{17}$$

where $P: D\to 2^D\backslash\{\emptyset\}$ has closed graph, $x\in P(x)$ on D and $y\in P(x)$ implies $P(y)\subset P(x)$. Then (11) has a solution on $\mathbf{R}_+$ satisfying $u(t)\in P(u(s))$ for $0\le s\le t$.

Proof: This time we find an approximate solution $v: J\to X$ satisfying (4), (5) and $y_1(t)\in P(y_2(s))$ for $0\le s<t\le a$, with $y_1: J\to D$ such that $y_1(t)\in \overline{B}_\eta(v(t))$ on J and $y_1(a)=v(a)$, and $y_2: [0,a)\to D$ such that $y_2(t)\in\overline{B}_\eta(v(t))$ for $t<a$; notice that the $P(x)$ are closed, and that we can start with $v(t)=x_0+(z-x_0)t/\delta$ with $z\in P(x_0)$, $y_1(t)=z$ and $y_2(s)=x_0$. As $\epsilon_n\to 0+$, v_n and therefore y_{in}, converge to a solution u. Since

$(y_{2n}(s), y_{1n}(t)) \in \operatorname{graph}(P)$ and the latter is closed, we have $(u(s), u(t)) \in \operatorname{graph}(P)$, i.e., $u(t) \in P(u(s))$ for $0 \le s \le t \le a$.

Remarks. 11. For $\dim X < \infty$ a typical finite-dimensional proof for the existence of a local solution was given in [9] under the additional restrictive assumption that P be lsc. It is clear that (17) is again necessary in the sense of Remark 10.

12. Among others the papers [1], [9] have also been reproduced in [2].

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