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GENERALIZATION OF THE SOBOLEV-LIEB-THIRRING INEQUALITIES AND APPLICATIONS TO THE DIMENSION OF ATTRACTORS

JEAN-MICHEL GHIDAGLIA, MARTINE MARION, AND ROGER TEMAM
Laboratoire d'Analyse Numérique, C.N.R.S. et Université Paris-Sud
Bâtiment 425, Université Paris-Sud, 91405 Orsay Cédex, France

Abstract. In relation to the question of stability of matter, Lieb and Thirring have given a remarkable improvement of Sobolev-Gagliardo- Nirenberg inequalities for finite families of functions which are orthonormal in $L^2(\mathbf{R}^n)$. These inequalities have also found utilization in the study of the dimension of the attractors associated with dissipative parabolic equations, in particular Navier-Stokes equations in a bounded domain with zero boundary conditions, where they have led to sharp estimates on this dimension. The generalizations that we propose here are partly motivated by the study of other equations of fluid mechanics for which the previous inequalities do not apply. Our extensions are made possible, in particular, by the fact that we replace the global condition of orthonormality in $L^2(\mathbf{R}^n)$ by a local one which is called suborthonormality condition (this concept is defined hereafter). We are hence able to consider arbitrary boundary conditions; we can also consider higher order derivatives and vector valued functions. The last Section of the paper contains some applications to viscous incompressible fluid flows which illustrate some of the generalized inequalities.

1. Introduction. In relation to the question of stability of matter, Lieb and Thirring [19] have proved a remarkable improvement of Sobolev- Gagliardo-Nirenberg inequalities for a finite family of functions which are orthonormal in $L^2(\mathbf{R}^n)$.

Let $\varphi_1, \dots, \varphi_N$ be a finite family of functions in $H^1(\mathbf{R}^n)$ which are orthonormal in $L^2(\mathbf{R}^n)$ i.e.

$$\int_{\mathbf{R}^n} \varphi_i \varphi_j \, dx = \delta_{ij}, \qquad 1 \le i, j \le N. \tag{1.1}$$

According to the extension of the Sobolev-Gagliardo-Nirenberg inequalities due to Lieb and Thirring [19] (see also Cwickel [6]), for every p satisfying $\max(1, n/2) < p \le 1 + (n/2)$, there exists a constant $\kappa = \kappa(n,p)$ independent of N and of the φ_j's such that

$$\Big[\int_{\mathbf{R}^n} \Big(\sum_{j=1}^N \varphi_j(x)^2\Big)^{p/(p-1)} dx\Big]^{2(p-1)/n} \le \kappa \sum_{j=1}^N \int_{\mathbf{R}^n} \sum_{i=1}^n \Big(\frac{\partial \varphi_j}{\partial x_i}\Big)^2 dx. \tag{1.2}$$

Besides its initial motivation for the stability of matter, this inequality has played an important role in the estimate of the trace of certain linear operators arising in the study of

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infinite dimensional dynamical systems and has led to sharp bounds on the dimension of the corresponding attractors in terms of the physical data [25, 18, 5, 27, 9]. In such problems, the evolution partial differential equations under consideration involve functions which are defined on limited sets of $\mathbf{R}^n$. In the form given in [19] (i.e. in the form (1.1)–(1.2)), the Sobolev–Lieb–Thirring inequalities are only applicable to problems where the boundary conditions are of Dirichlet type. This is very restrictive in view of many physically relevant applications (for instance thermohydraulics [9] or magnetohydrodynamics, see Section 5.1).

Our aim in this work (announced in [15]) is to generalize these inequalities in several directions. First, we can consider higher order derivatives in (1.2) or vector valued functions (this was already done in [28]). Second, we do not assume any property of the functions on the boundary when they are defined on a limited bounded or unbounded subset Ω of $\mathbf{R}^n$. Finally, we replace the global condition (1.1) by a very less restrictive condition which is of local type: we ask the family $\{\varphi_j\}_{j=1}^N$ to be suborthonormal in the sense that

$$\sum_{i,j=1}^{N} \xi_i \xi_j \int_\Omega \varphi_i \varphi_j \, dx \leq \sum_{k=1}^{N} \xi_k^2 , \quad \forall \xi \in \mathbf{R}^N. \tag{1.3}$$

In contrast with (1.1), this constraint is stable by localization and regularization. In particular this allows us to use implicitly or explicitly localizations in the proofs of the extended inequalities. As a consequence of these generalizations, we also are able to consider new applications which will be reported elsewhere (see also §5).

The paper is organized as follows. In the next Section, we state the main result of this work (Theorem 2.1) and deduce some corollaries. Theorem 2.1 is then proved in Section 3. In Section 4, we derive a few generalizations to the case where the functions are defined on unbounded sets of $\mathbf{R}^n$. Finally Section 5 is devoted to some applications. The Navier–Stokes equations and related equations in which general boundary conditions occur are considered; in particular, the estimates on the dimension of the attractor for Navier–Stokes equations on a sphere-like two-dimensional manifold (geophysical flows) are as sharp as the ones obtained previously for the Dirichlet conditions.

When Ω is an open subset of $\mathbf{R}^n$, we shall denote as usual by $L^2(\Omega)$ the set of classes of real functions which are measurable in Ω and square integrable on Ω; $H^m(\Omega)$ is the Sobolev space of order m constructed on $L^2(\Omega)$. The closure of $\mathcal{D}(\Omega)$ (the set of $\mathcal{C}^\infty$–compactly supported functions in Ω) in $H^m(\Omega)$ is denoted by $\overset{\circ}{H}{}^m(\Omega)$ and, as usual, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ is a multi-index, we set $[\alpha] = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \dfrac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

Finally, we say that a constant which depends on Ω, depends only on the shape of Ω if it is invariant by translation and homothety of Ω.

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2. Generalized Sobolev-Lieb-Thirring Inequalities. In this Section, we give the main result of this work and derive some consequences. Let us first introduce some notations which will be used all along the paper. We consider a bounded open set in $\mathbf{R}^n$ ($n \geq 1$) on which we make the following regularity hypothesis

$$\begin{cases} \text{There exists a linear prolongation operator } \Pi_m \text{ mapping} \\ H^m(\Omega) \text{ into } H^m(\mathbf{R}^n) \text{ such that } \Pi_m \in \mathcal{L}(H^r(\Omega)\,,\, H^r(\mathbf{R}^n)), \\ r = 0, 1, \ldots, m \text{ and } \Pi_m\, u(x) = u(x) \text{ for a.e. } x \in \Omega. \end{cases} \tag{2.1$_m$}$$

Recall that this property is in particular fulfilled when the boundary of Ω is an $(n-1)$–dimensional $\mathcal{C}^m$–manifold (see [23, p. 75]).

We consider an arbitrary finite family of functions $\varphi_1, \ldots, \varphi_N$ defined on Ω with values on $\mathbf{R}^k$ ($k \geq 1$). It will be assumed that the φ_j's belong to $\mathbf{H}^m(\Omega)$ ($\equiv H^m(\Omega)^k$) for some fixed integer $m \geq 1$ and that

$$\sum_{i,j=1}^{N} \xi_i \xi_j \int_\Omega \varphi_i \cdot \varphi_j \, dx \leq \sum_{i=1}^{N} \xi_i^2 \quad , \quad \text{for all } \xi_1, \ldots, \xi_N \in \mathbf{R}. \tag{2.2}$$

The condition (2.2) is in particular satisfied when the φ_j's are orthonormal in $\mathbf{L}^2(\Omega)$ ($\equiv L^2(\Omega)^k$), i.e. such that

$$\int_\Omega \varphi_i \cdot \varphi_j \, dx = \delta_{i\,j},$$

and a family for which (2.2) holds will be called suborthonormal in $\mathbf{L}^2(\Omega)$. We set for almost every $x \in \Omega$,

$$\rho(x) = \rho_\varphi(x) = \sum_{j=1}^{N} |\varphi_j(x)|^2, \tag{2.3}$$

where $|\cdot|$ denotes the usual euclidian norm on $\mathbf{R}^k$.[1] With these notations, we state

Theorem 2.1. *Let Ω be a bounded open set of $\mathbf{R}^n$ satisfying $(2.1)_m$. For every p with*

$$\max(1, \frac{n}{2m}) < p \leq 1 + \frac{n}{2m}, \tag{2.4}$$

there exist two positive constants κ and χ such that, for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m(\Omega)$ which is suborthonormal in $\mathbf{L}^2(\Omega)$ (i.e. which satisfies (2.2)), we have

$$\Big(\int_\Omega \rho(x)^{p/(p-1)} \, dx \Big)^{2m(p-1)/n} \leq \kappa \sum_{j=1}^{N} \int_\Omega \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2 \, dx + \frac{\chi}{\delta(\Omega)^{2m}} \int_\Omega \rho(x) \, dx, \tag{2.5}$$

where $\delta(\Omega)$ is the diameter of Ω and ρ is given by (2.3). The constants κ and χ depend on m, n, k, p and the shape of Ω.

Before proceeding to the proof of this result, which is given in the next Section, we make a few remarks and derive some corollaries.

[1] We write ρ instead of ρ_φ when no confusion can occur.

Remark 2.2. Relation with the Sobolev-Gagliardo-Nirenberg inequalities. According to these well-known inequalities, under the hypotheses of Theorem 2.1, there exist two constants c_1 and c_2 such that, for every $\varphi \in \mathbf{H}^m(\Omega)$,

$$\Big(\int_\Omega |\varphi(x)|^{2p/(p-1)}\,dx\Big)^{2m(p-1)/n} \leq \Big(\int_\Omega |\varphi(x)|^2\,dx\Big)^{(2mp-n)/n}\Big(c_1 \sum_{[\alpha]=m}\int_\Omega |D^\alpha\varphi(x)|^2\,dx + \frac{c_2}{\delta(\Omega)^{2m}}\int_\Omega |\varphi(x)|^2\,dx\Big). \tag{2.6}$$

Application of (2.6) to each of the φ_j's (note that thanks to (2.2) we have $\int_\Omega |\varphi_j(x)|^2\,dx \leq 1$) and the utilization of Hölder inequalities lead to (2.5) with constants $c_1(N)$ and $c_2(N)$ instead of κ and χ, where $c_1(N)$ and $c_2(N)$ go to infinity with N. The fundamental interest of Theorem 2.1 is that the constants κ and χ do not depend on N.

We observe that, due to the suborthonormality assumption, the inequality (2.5) is stable by localization. Indeed, if θ denotes a $\mathcal{C}^\infty$ function on $\mathbf{R}^n$ with $0 \leq \theta(x) \leq 1$, $\forall x \in \mathbf{R}^n$, and if the family $\{\varphi_j\}_{j=1}^N$ is suborthonormal, we deduce from Theorem 2.1 the

Corollary 2.3. *Under the assumptions of Theorem 2.1, let θ be a $\mathcal{C}^\infty$ function from $\mathbf{R}^n$ into $[0,1]$. Then, there exist two constants $\kappa(\theta)$ and $\chi(\theta)$, with the same dependence as κ and χ, such that*

$$\begin{aligned}
&\Big(\int_\Omega \rho_{\theta\varphi}(x)^{p/(p-1)}\,dx\Big)^{2m(p-1)/n} \\
&\leq \kappa\sum_{j=1}^N\int_\Omega\sum_{[\alpha]=m}|D^\alpha(\theta\varphi_j)(x)|^2\,dx + \frac{\chi}{\delta(\Omega)^{2m}}\int_\Omega \rho_{\theta\varphi}(x)\,dx, \\
&\leq \kappa(\theta)\sum_{j=1}^N\int_\Omega\sum_{[\alpha]=m}|D^\alpha\varphi_j(x)|^2\,dx + \frac{\chi(\theta)}{\delta(\Omega)^{2m}}\int_\Omega \rho_\varphi(x)\,dx.
\end{aligned}$$

This property allows in particular the extension of (2.5) to functions defined on a manifold (see §5.2 below).

When the φ_j's belong to $\mathbf{H}^m_\circ(\Omega)$, we can remove the smoothness assumption on Ω and set $\chi = 0$ in (2.5) (see also the comments after Theorem 4.1 in §4.1).

Corollary 2.4. *Let Ω be an arbitrary open bounded set of $\mathbf{R}^n$. For every p satisfying (2.4), there exists a positive constant $\kappa_\circ$ such that, for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m_\circ(\Omega)$ which is suborthonormal in $\mathbf{L}^2(\Omega)$, we have*

$$\Big(\int_\Omega \rho(x)^{p/(p-1)}\,dx\Big)^{2m(p-1)/n} \leq \kappa_\circ\sum_{j=1}^N\int_\Omega\sum_{[\alpha]=m}|D^\alpha\varphi_j(x)|^2\,dx, \tag{2.7}$$

where $\kappa_\circ$ depends on m, n, k and p but not on Ω.

Proof: Let $R > 0$ denote the radius of a ball in $\mathbf{R}^n$ which contains $\Omega : \Omega \subset B_R$. Let $\{\varphi_j\}$ be a family satisfying the hypotheses of Corollary 2.4. The natural extensions of the φ_j's to B_R by 0 outside of Ω are still denoted by φ_j. They are suborthonormal in $\mathbf{L}^2(B_R)$ and belong to $\mathbf{H}^m_\circ(B_R)$. Hence, according to Theorem 2.1,

$$\Big(\int_{B_R} \rho(x)^{p/(p-1)}\, dx\Big)^{2m(p-1)/n} \leq\leq \kappa \sum_{j=1}^{N} \int_{B_R} \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx + \frac{\chi}{(2R)^{2m}} \int_{B_R} \rho(x)\, dx\,, \tag{2.8}$$

where κ and χ depend on m, n, k and p but are independent of R. On the other hand, according to an appropriate variant of Poincaré inequality, there exists a constant c which only depends on n such that

$$\frac{1}{R^{2m}} \int_{B_R} |\varphi(x)|^2\, dx \leq c \int_{B_R} \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx, \quad \forall \varphi \in \mathbf{H}^m_\circ(B_R). \tag{2.9}$$

Now since

$$\int_{B_R} \rho(x)^{p/(p-1)}\, dx = \int_\Omega \rho(x)^{p/(p-1)}\, dx\,,$$

(2.7) follows from (2.8) and (2.9) with $\kappa_\circ = \kappa + \dfrac{c\chi}{2^{2m}}$.

We now state two corollaries of Theorem 2.1 of a slightly different nature which are very useful in the applications.

Corollary 2.5. *Let Ω be a bounded open set of $\mathbf{R}^n$ satisfying $(2.1)_m$. There exists a constant κ_1 such that for every finite family $\{(\varphi_j, \psi_j)\}_{j=1}^N$ in $\mathbf{H}^m(\Omega) \times \mathbf{L}^2(\Omega)$ which is orthonormal in $\mathbf{L}^2(\Omega)^2$:*

$$\int_\Omega (\varphi_i \cdot \varphi_j + \psi_i \cdot \psi_j)\, dx = \delta_{i\,j} \tag{2.10}$$

we have

$$\delta(\Omega)^{2m} \sum_{j=1}^{N} \int_\Omega \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx + \int_\Omega \rho_\psi(x)\, dx \geq N - \kappa_1 \tag{2.11}$$

where κ_1 depends on m, n, k, p, and on the shape of Ω.

Proof: We first note that, due to (2.10), the family $\{\varphi_j\}_{j=1}^N$ is suborthonormal in $\mathbf{L}^2(\Omega)$. Indeed

$$\begin{aligned}\sum_{i,j=1}^{N} \xi_i \xi_j \int_\Omega \varphi_i \cdot \varphi_j\, dx &= \sum_{i,j=1}^{N} \xi_i \xi_j \big(\delta_{i\,j} - \int_\Omega \psi_i \cdot \psi_j\, dx\big)\,, \\ &= \sum_{i=1}^{N} \xi_i^2 - \Big|\sum_{i=1}^{N} \xi_i \psi_i\Big|^2_{\mathbf{L}^2(\Omega)} \leq \sum_{i=1}^{N} \xi_i^2.\end{aligned}$$

We also infer from (2.10) that

$$\int_\Omega (\rho_\varphi(x) + \rho_\psi(x))\, dx = N.$$

Hence, the left hand side of (2.11) is equal to

$$\begin{aligned} &N + \delta(\Omega)^{2m} \sum_{j=1}^{N} \int_\Omega \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx - \int_\Omega \rho_\varphi(x)\, dx, \\ &\geq \text{ (by using (2.5) with } p = 1 + \frac{n}{2m}) \\ &\geq N + \frac{1}{\kappa}\delta(\Omega)^{2m} \int_\Omega \rho_\varphi(x)^{1+(n/2m)}\, dx - (\frac{\chi}{\kappa} + 1) \int_\Omega \rho_\varphi(x)\, dx\,. \end{aligned} \tag{2.12}$$

Thanks to Hölder inequalities, the last integral in (2.12) is majorized as follows

$$(\frac{\chi}{\kappa} + 1) \int_\Omega \rho_\varphi(x)\, dx \leq (\frac{\chi}{\kappa} + 1)|\Omega|^{2m/(n+2m)} \Big(\int_\Omega \rho_\varphi(x)^{1+(2m/n)}\, dx \Big)^{n/(n+2)}, \tag{2.13}$$

where $|\Omega|$ denotes the n-dimensional volume of Ω. Since $|\Omega| \leq \delta(\Omega)^n$, we infer from (2.13) by using Young inequality

$$(\frac{\chi}{\kappa} + 1) \int_\Omega \rho_\varphi(x)\, dx \leq \frac{1}{2\kappa}\delta(\Omega)^{2m} \int_\Omega \rho_\varphi(x)^{1+(n/2m)}\, dx - \kappa_1\,. \tag{2.14}$$

Thus, combining (2.12) and (2.14), we obtain

$$\begin{aligned} &\delta(\Omega)^{2m} \sum_{j=1}^{N} \int_\Omega \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx + \int_\Omega \rho_\psi(x)\, dx \\ &\qquad \geq N + \frac{1}{2\kappa}\delta(\Omega)^{2m} \int_\Omega \rho_\varphi(x)^{1+(n/2m)}\, dx - \kappa_1\,, \end{aligned}$$

which implies (2.11).

Corollary 2.6. *Let Ω be a bounded open set of $\mathbf{R}^n$ satisfying $(2.1)_m$. There exist two constants κ_2 and χ_2 such that for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m(\Omega)$, orthonormal in $\mathbf{L}^2(\Omega)$,*

$$\delta(\Omega)^{2m} \sum_{j=1}^{N} \int_\Omega \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx \geq \kappa_2\, N^{1+(2m/n)} - \chi_2\,, \tag{2.15}$$

where κ_2 and χ_2 depend on m, n, k, p and on the shape of Ω.

Proof: The proof consists of applying Theorem 2.1 to the orthonormal family $\{\varphi_j\}_{j=1}^N$ and in using Hölder and Young inequalities as in the proof of Corollary 2.5. Details are omitted.

Remark 2.7. The inequality (2.15) could also be derived from the asymptotic expansion of the eigenvalues of the m^{th} laplacian on $\Omega\big((-1) \sum_{[\alpha]=m} D^{2\alpha}\big)$ associated with the Neumann boundary conditions.

3. Proof of Theorem 2.1. We now prove Theorem 2.1. We first state in Section 3.1 an extension of the Sobolev-Lieb-Thirring inequalities to periodic functions proved by applying the methods of [19]. We then give the proof of Theorem 2.1 (§3.2).

3.1. A result on periodic functions. When Ω is an n-dimensional hypercube $\Omega =]0, L[^n$, we denote by $H^m_{per}(\Omega)$ the closure in $H^m(\Omega)$ of trigonometric polynoms which are L-periodic in each direction. We set also $\mathbf{H}^m_{per}(\Omega) \equiv (\mathbf{H}^m_{per}(\Omega))^k$ and, when $\Omega =]0, 2\pi[^n$, $\mathbf{H}^m_{per}(\Omega) \equiv \mathbf{H}^m(\pi^n)$. With these notations we state the

Proposition 3.1. *For every p with*

$$\max(1, \frac{n}{2m}) < p \leq 1 + \frac{n}{2m} \tag{3.1}$$

there exist two constants κ_t and χ_t such that for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m(\pi^n)$ which is orthonormal in $\mathbf{L}^2(\pi^n)$, we have

$$\Big(\int_{\pi^n} \rho(x)^{p/(p-1)}\, dx\Big)^{2m(p-1)/n} \leq \kappa_t \sum_{j=1}^N \int_{\pi^n} \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx + \chi_t \int_{\pi^n} \rho(x)\, dx\,. \tag{3.2}$$

The constants κ_t and χ_t depend on m, n, k and p.

Remark 3.2. 1. In the case where the φ_j's are defined on the hypercube $\Omega =]0, L[^n$ and belong to $\mathbf{H}^m_{per}(\Omega)$, it is easy to infer from (3.2) that

$$\Big(\int_{\Omega} \rho(x)^{p/(p-1)}\, dx\Big)^{2m(p-1)/n} \leq \kappa_t \sum_{j=1}^N \int_{\Omega} \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx + (\frac{2\pi}{L})^{2m} \chi_t \int_{\Omega} \rho(x)\, dx. \tag{3.3}$$

2. When the φ_j's belong to $\mathbf{H}^m_\circ(\Omega) \subset \mathbf{H}^m_{per}(\Omega)$, $\Omega = x_\circ +]0, L[^n$, thanks to Poincaré inequality, the inequality (3.3) can be strenghtened to

$$\Big(\int_{\Omega} \rho(x)^{p/(p-1)}\, dx\Big)^{2m(p-1)/n} \leq \kappa'_t \sum_{j=1}^N \int_{\Omega} \sum_{[\alpha]=m} |D^\alpha \varphi_j(x)|^2\, dx\,, \tag{3.3$'$}$$

where κ'_t depends on m, n, k and p.

We refer the reader to R. Temam [28] for the details of the proof of Proposition 3.1, which uses techniques similar to those in [19]. Let us simply point out some properties which are essential in the proof of (3.2). They are associated to the m^{th} laplacian on $\mathbf{H}^m_{per}(\Omega)$, $\Omega = x_\circ +]0, L[^n$:

$$\Delta^{(m)} \varphi = (-1)^m \sum_{[\alpha]=m} D^{2\alpha} \varphi\,. \tag{3.4}$$

This self-adjoint operator possesses a sequence of eigenfunctions w_j , $j \in \mathbf{N}$, which form an *orthonormal* basis of $\mathbf{L}^2(\Omega)$

$$\begin{aligned} &\Delta^{(m)} w_j = \lambda_j w_j\,, \\ &0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty\,, \quad \text{as} \quad j \to +\infty\,. \end{aligned}$$

In fact, the sequence of λ_j's is the sequence of numbers $(\Lambda_{q,\ell})_{\substack{1 \leq q \leq k \\ \ell \in \mathbf{N} \times \mathbf{Z}^{n-1}}}$,

$$\Lambda_{q,\ell} = (\frac{2\pi}{L})^{2m} \sum_{[\alpha]=m} \ell^{2\alpha} = (\frac{2\pi}{L})^{2m} \sum_{\alpha_1 + \cdots + \alpha_n = m} \ell_1^{2\alpha_1} \ldots \ell_n^{2\alpha_n}\,, \tag{3.5}$$

the eigenvectors associated to $\lambda_j = \Lambda_{q,\ell}$ being

$$\begin{cases} \text{the real and imaginary parts of } \dfrac{\sqrt{2}}{L^{n/2}} e^{i(2\pi/L)\ell.(x-x_\circ)}\, e_q \quad \text{for} \quad \ell \neq 0 \\ \dfrac{1}{L^{n/2}} e_q \quad \text{for} \quad \ell = 0\,, \end{cases} \tag{3.6}$$

where $(e_1, \ldots, e_k)$ is the canonical basis of $\mathbf{R}^k$.

The three following properties are essential in the proof of Proposition 3.1.

$$\begin{cases} \text{There exist two constants } c_1 \text{ and } c_2 \text{ which depend on} \\ m,\, n \text{ and } k \text{ such that } \ \lambda_j \geq \dfrac{1}{L^{2m}}(c_1\, j^{2m/n} - c_2), \end{cases} \tag{3.7}$$

$$\sup_{x\in\Omega} |w_j(x)|^2 \leq \frac{2}{L^n}\,, \quad \forall j \in \mathbf{N}, \tag{3.8}$$

$$\begin{cases} \text{For every } r > 0, \text{ the operator } (\Delta^{(m)} + r)^{-1} \text{ with periodic} \\ \text{boundary conditions can be extended as a linear continuous} \\ \text{operator from } L^s(\Omega) \text{ into } W^{2m,s}(\Omega), \text{ for every } s \in (1, +\infty). \end{cases} \tag{3.9}$$

The property (3.8) follows immediately from (3.6), while (3.9) is a classical regularity result on elliptic equations (see e.g. [1]). Finally, the lower bound (3.7) follows from (3.5) as we briefly show now. We suppose first that $L = 2\pi$ and set for $p \in \mathbf{N}$,

$$\xi_p = \Big\{(q\,,\ell) \in \mathbf{N} \times \mathbf{Z}^n\,;\, 1 \leq q \leq k,\, \sum_{[\alpha]=m} \ell^{2\alpha} \leq p\Big\}$$

and denote by N_p its cardinal.

When $\sum_{[\alpha]=m} \ell^{2\alpha} \leq p$, we have $\ell_j^{2m} \leq p$, for $j = 1, \ldots, n$, hence $|\ell_j| \leq p^{1/(2m)}$ and then $\xi_p \subset [1,k] \times [-p^{1/(2m)}, p^{1/(2m)}]^n$. This shows that

$$N_p \leq k\Big[1 + 2I(p^{1/(2m)})\Big]^n,$$

where $I(x)$, $x \in \mathbf{R}$, denotes the integer part of x. Also, by the definition of the λ_j's and the w_j's, N_p is the cardinal of the set $\{j \in \mathbf{N}\,;\, \lambda_j \leq p\}$. Therefore, we get

$$\lambda_{N_p} \leq p\,,\, \lambda_{N_p+1} \geq p + 1,$$

and since the sequence $\{\lambda_j\}$ is nondecreasing, we obtain

$$\lambda_{k[1+2I(p^{1/(2m)})]^n+1} \geq \lambda_{N_p+1} \geq p + 1. \tag{3.10}$$

For every $j \geq k + 1$, there exists a unique $q = q(j) \in \mathbf{N}$ such that

$$k\Big[1 + 2I(q^{1/(2m)})\Big]^n + 1 \leq j < k\Big[1 + 2I((q+1)^{1/(2m)})\Big]^n + 1.$$

Then

$$\lambda_j \geq \lambda_{k[1+2I(q(j)^{1/(2m)})]^n+1} \geq \text{ (due to (3.10)) } \geq q(j)+1.$$

By definition of $q(j)$, we have

$$\Big(q(j)+1\Big)^{1/(2m)} \geq I\Big((q(j)+1)^{1/(2m)}\Big) \geq \frac{1}{2}\left[(\frac{j}{k})^{1/n}-1\right],$$

hence

$$\lambda_j \geq \left[\frac{1}{2}(\frac{j}{k})^{1/n}-1\right]^{2m}, \; j \geq 1+k,$$

which clearly implies (3.7) when $L=2\pi$. The general case ($L \neq 2\pi$) reduces to the previous one by homothety and this yields (3.7).

3.2. Proof of Theorem 2.1. We are given an open bounded set Ω of $\mathbf{R}^n$ satisfying $(2.1)_m$ and a finite family $\{\varphi_j\}_{j=1}^N$ of $\mathbf{H}^m(\Omega)$, which is suborthonormal in $\mathbf{L}^2(\Omega)$ and our aim is to derive (2.5).
(i) We set $L = 2\delta(\Omega) = 2\max_{(x,y)\in\overline{\Omega}^2}|x-y|$. The closure of Ω is included in a hypercube Q_1 of edge $2L$, of the form $x_\circ +]-L,+L[^n$ which is in its turn included in a larger hypercube $Q = Q_M = x_\circ +]-ML\,,\, ML[^n,\;\; M>1$.

The hypotheses ensure the existence of two linear continuous operators R_1, R mapping respectively $\mathbf{H}^m(\Omega)$ into $\mathbf{H}^m_\circ(Q_1)$ and $\mathbf{H}^m_\circ(Q)$ with

$$(R_1\varphi)(x) = (R\varphi)(x) = \varphi(x)\,,\;\; \forall\varphi\in\mathbf{H}^m(\Omega)\,,\;\; a.e.\, x\in\Omega, \tag{3.11}$$

$$(R\varphi)(x) = 0,\;\; \forall\varphi\in\mathbf{H}^m(\Omega),\;\; a.e.\; x\in Q\backslash Q_1. \tag{3.12}$$

Indeed, $R_1\varphi$ is the product of $\pi_m\varphi$ (see $(2.1)_m$) with a $\mathcal{C}^\infty$-truncation function equal to 1 in a neighborhood of Ω and whose compact support is included in Q_1. Then, $R\varphi$ is the trivial extension of $R_1\varphi$ by 0 outside of Q_1.

Moreover, since $\pi_m \in \mathcal{L}(H^\ell(\Omega)\,,\, H^\ell(\mathbf{R}^n))$, $\ell = 0,\ldots,m$, there exist two positive constants $r_\circ$ and r_m such that for every $\varphi\in\mathbf{H}^m(\Omega)$

$$\int_Q |R\varphi|^2\,dx \leq r_\circ^2\int_\Omega |\varphi|^2\,dx, \tag{3.13}$$

$$\int_Q \sum_{[\alpha]=m} |D^\alpha R\varphi|^2\,dx \leq r_m^2\int_\Omega\{\sum_{[\alpha]=m}|D^\alpha\varphi|^2 + \frac{|\varphi|^2}{L^{2m}}\}\,dx. \tag{3.14}$$

It is easy to check that $r_\circ, r_m$ are invariant by homothety, i.e., they are the same for Ω, Q and $\lambda\Omega$, λQ, $\forall\lambda>0$. They depend therefore on m, n and the shape of Ω.
(ii) We consider then an open ball $B_\eta = B_\eta(a_\eta)$ included in $Q\backslash\overline{Q}_1$ centered at a_η of radius $\eta>0$, and we denote by v_j , $j\in\mathbf{N}$, the infinite sequence of eigenfunctions of the operator $\Delta^{(m)}$ (see (3.4)) on $\mathbf{H}^m_\circ(B_\eta)$ (i.e. associated to the Dirichlet boundary conditions). Denoting by μ_j the eigenvalue corresponding to v_j, we have

$$v_j\in\mathbf{H}^m_\circ(B_\eta),\;\; \Delta^{(m)}v_j = \mu_j v_j\,,\;\; j\in\mathbf{N}. \tag{3.15}$$

We assume, this is licit, that the sequence v_j is orthonormal in $\mathbf{L}^2(B_\eta)$ and we denote again by v_j the function equal to v_j on B_η and to 0 on $Q\backslash B_\eta$; we observe that the so-defined function v_j belongs to $\mathbf{H}_\circ^m(Q)$.

(iii) The proof consists now in applying Proposition 3.1 to a well chosen family of functions $\psi_1, \dots, \psi_N$ of $\mathbf{H}_\circ^m(Q)$. These functions are searched of the form

$$\psi_j = R\varphi_j + \alpha_{j\,1}v_1 + \cdots + \alpha_{j\,N}v_N\,, \quad \text{for } 1 \le j \le N, \tag{3.16}$$

and we will require that

$$\int_Q \psi_i(x).\psi_j(x)\,dx = r_\circ^2\,\delta_{i\,j}\,, \quad 1 \le i\,, j \le N\,. \tag{3.17}$$

We recall that the supports of the functions $R\varphi_i$ and v_j do not intersect and that the family $\{v_j\}_{j\in\mathbf{N}}$ is orthonormal. It is then easy to express (3.17) in terms of the φ_i's and $\alpha_{i\,j}$'s; we find

$$(\psi_i\,,\,\psi_j)_{\mathbf{L}^2(Q)} = (R\varphi_i\,,\,R\varphi_j)_{\mathbf{L}^2(Q)} + \sum_{\ell=1}^N \alpha_{i\,\ell}\,\alpha_{j\,\ell} = r_\circ^2\,\delta_{i\,j}\,. \tag{3.18}$$

Considering the matrices $(b_{i\,j})$, $(\alpha_{i\,j})$,

$$b_{i\,j} = r_\circ^2\,\delta_{i\,j} - (R\varphi_i\,,\,R\varphi_j)_{\mathbf{L}^2(Q)}\,, \tag{3.19}$$

we see that (3.18) is equivalent to $b = \alpha\cdot{}^t\alpha$ and, b being given, we have to find a $N\times N$ matrix α such that $b = \alpha\cdot{}^t\alpha$. It is well known that this problem has a solution α if and only if b is a symmetric positive matrix. It is clear that b is symmetric, and in order to check that b is positive, we consider $\xi \in \mathbf{R}^N$ and form

$$q(\xi) = \sum_{i\,,j=1}^N b_{i\,j}\xi_i\,\xi_j = r_\circ^2\sum_{i=1}^N \xi_i^2 - \sum_{i\,,j=1}^N \xi_i\,\xi_j(R\varphi_i\,,\,R\varphi_j)_{\mathbf{L}^2(Q)}\,.$$

But

$$\begin{aligned}\sum_{i\,,j=1}^N \xi_i\,\xi_j(R\varphi_i\,,\,R\varphi_j)_{\mathbf{L}^2(Q)} &= \Big|R\Big(\sum_{i=1}^N \xi_i\,\varphi_i\Big)\Big|^2_{\mathbf{L}^2(Q)}\\ &\le \text{(by (3.13))} \ \le r_\circ^2\Big|\sum_{i=1}^N \xi_i\,\varphi_i\Big|^2_{\mathbf{L}^2(\Omega)}\\ &\le \text{(since the } \varphi_i\text{'s are suborthonormal in } \mathbf{L}^2(\Omega)\text{, see (2.2)),}\\ &\le r_\circ^2\sum_{i=1}^N \xi_i^2\,.\end{aligned}$$

We conclude that $q(\xi) \ge 0$ and the matrix b is positive.

(iv) Now, we observe that the functions $\psi_j/r_\circ$, $j = 1, \dots, N$, are orthonormal in $\mathbf{L}^2(Q)$ and belong to $\mathbf{H}_\circ^m(Q)$. Proposition 3.1 is then applicable (more precisely we apply (3.3)$'$). This gives

$$\Big(\int_Q \rho_\psi(x)^{p/(p-1)}dx\Big)^{2m(p-1)/n} \le r_\circ^{2(2mp-n)/n}\kappa_t'\sum_{j=1}^N\int_Q\sum_{[\alpha]=m}|D^\alpha\psi_j(x)|^2dx. \tag{3.20}$$

Since $\psi_j = \varphi_j$ on Ω, the left hand side of (3.20) is larger than

$$\left(\int_\Omega \rho_\varphi(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n}.$$

In order to majorize the right hand side of (3.20) we recall that the supports of the functions $R\varphi_i$ and v_j do not intersect and that

$$\sum_{[\alpha]=m} \int_Q D^\alpha v_i . D^\alpha v_j\,dx = \int_{B_\eta} \Delta^{(m)} v_i . v_j\,dx = \mu_i\,\delta_{ij}\,.$$

Using (3.16), we then write

$$\int_Q \sum_{[\alpha]=m} |D^\alpha \psi_j|^2\,dx = \int_Q \sum_{[\alpha]=m} |D^\alpha R\varphi_j|^2\,dx + \sum_{i=1}^N \alpha_{j\,i}^2 \int_Q \sum_{[\alpha]=m} |D^\alpha v_i|^2\,dx.$$

But

$$\alpha_{j\,i} = \int_Q \psi_j(x).v_i(x)\,dx\,,$$
$$\alpha_{j\,i}^2 \le \left|\psi_j\right|^2_{\mathbf{L}^2(Q)} \left|v_i\right|^2_{\mathbf{L}^2(Q)} = r_\circ^2\,.$$

Finally

$$\left(\int_\Omega \rho_\varphi(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \le r_\circ^{2(2mp-n)/n}\,\kappa_t' \sum_{j=1}^N \left(\int_Q \sum_{[\alpha]=m} |D^\alpha R\varphi_j|^2\,dx + r_\circ^2 \sum_{i=1}^N \mu_i\right). \tag{3.21}$$

In (3.21), η does not appear except in μ_i that we now write $\mu_i(B_\eta)$ to emphasize the dependence in η. But we know that $\mu_i(B_\eta) = \eta^{-2m}\mu_i(B_1)$. Hence, choosing $M = 1 + 4\eta$ and letting $\eta \to \infty$ in (3.21), we obtain exactly the same inequality without the μ_i's. We then conclude by using (3.14) (recall that $L = 2\delta(\Omega)$). This gives (2.5) and the proof of Theorem 2.1 is therefore complete.

4. Other generalizations of the Sobolev-Lieb-Thirring inequalities. The purpose of this Section is to give some extensions of the results of Section 2 to families of functions defined on unbounded open sets Ω of $\mathbf{R}^n$. We first consider in Section 4.1 the case $\Omega = \mathbf{R}^n$ (which is the case considered in [19]). Then we give an analog of Theorem 2.1 for unbounded open sets (Section 4.2).

4.1. The case $\Omega = \mathbf{R}^n$. In the case of families of functions belonging to $\mathbf{H}^m(\mathbf{R}^n)$, we can state the following result.

Theorem 4.1. *Let $m \geq 1,\ n \geq 1,\ k \geq 1$ and p with*

$$\max(1, \frac{n}{2m}) < p \leq 1 + \frac{n}{2m}\,. \tag{4.1}$$

There exists a constant $\widetilde{\kappa} = \widetilde{\kappa}(m,n,p,k)$ such that for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m(\mathbf{R}^n)$ which is suborthonormal in $\mathbf{L}^2(\mathbf{R}^n)$, we have

$$\left(\int_{\mathbf{R}^n} \rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \leq \widetilde{\kappa}\sum_{j=1}^{N}\int_{\mathbf{R}^n}\sum_{[\alpha]=m}|D^\alpha\varphi_j(x)|^2\,dx. \tag{4.2}$$

Remark 4.2 When $m=1$ and the family $\{\varphi_j\}_{j=1}^N$ of $\mathbf{H}^1(\mathbf{R}^n)$ is assumed to be orthonormal in $\mathbf{L}^2(\mathbf{R}^n)$, (4.2) is the initial Sobolev-Lieb-Thirring inequality proved in [19].

Proof: It is parallel to the proof of Theorem 2.1 (Section 3) and some details will be omitted.

Let θ be a $\mathcal{C}^\infty$-truncation function from $\mathbf{R}_+$ into [0,1] which is equal to 1 on [0,1] and to 0 on $[2,+\infty[$. Let $M>2$ and define a linear continuous operator $\widetilde{R}$ mapping $\mathbf{H}^m(\mathbf{R}^n)$ into $\mathbf{H}^m(B_{Mr})$ (where $r>0$ and $B_{Mr} = B_{Mr}(0)$) by setting

$$(\widetilde{R}\varphi)(x) = \theta\Big(\frac{|x|^2}{r^2}\Big)\varphi(x)\,, \quad \text{for } |x| \leq Mr\,.$$

This operator is such that $\widetilde{R}\varphi \equiv 0$ on $B_{Mr}\backslash\overline{B}_{2r}$, for all $\varphi \in \mathbf{H}^m(\mathbf{R}^n)$. Also we consider a ball $B_\eta(a_\eta)$ included in $B_{Mr}\backslash\overline{B}_{2r}$ and the corresponding sequences of eigenvectors and eigenvalues of $\Delta^{(m)}$, again denoted by $\{v_j\}$ and $\{\mu_j\}$. We set

$$\psi_j = \widetilde{R}\varphi_j + \sum_{i=1}^{N}\alpha_{j\,i}v_i\,.$$

Then, as in the proof of Theorem 2.1, it can be shown that there exists a matrix α such that the corresponding ψ_j's satisfy

$$\int_{B_{Mr}}\psi_j(x).\psi_i(x)\,dx = \delta_{i\,j}\,, \quad 1 \leq i\,,\,j \leq N\,,$$

and with computations similar to those in the proof of Theorem 2.1, we obtain

$$\left(\int_{B_r}\rho_\varphi(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \leq \kappa_t'\sum_{j=1}^{N}\int_{\mathbf{R}^n}\sum_{[\alpha]=m}|D^\alpha\widetilde{R}\varphi_j|^2\,dx\,. \tag{4.3}$$

Recalling that $(\widetilde{R}\varphi)(x) = \theta\big(|x|^2/r^2\big)\varphi(x)$ and using Leibniz rule, we get

$$D^\alpha\widetilde{R}\varphi = \sum_{\beta+\gamma=\alpha} C_\alpha^\beta\, D^\beta\Big(\theta\big(\frac{|x|^2}{r^2}\big)\Big)D^\gamma\varphi\,. \tag{4.4}$$

But, for $1 \le [\beta] \le m$, $D^\beta(\theta(|x|^2/r^2))$ is the sum of terms which are the product of a negative power of r with a monom in x/r and a derivative of θ of order $\le m$ evaluated at $|x|^2/r^2$. Since θ is compactly supported and $\mathcal{C}^\infty$, all these terms are uniformly bounded for $x \in \mathbf{R}^n$ and there exists a constant $\widetilde{c}_1$ depending on θ, n and m such that

$$\sup_{\substack{x\in\mathbf{R}^n \\ 1\le[\beta]\le m}} \left|D^\beta\left(\theta\left(\frac{|x|^2}{r^2}\right)\right)\right| \le \frac{\widetilde{c}_1}{r}, \quad \text{for } r \ge 1.$$

Hence, we infer from (4.4) that there exists a constant $\widetilde{c}_2 = \widetilde{c}_2(\theta, m, n)$ such that for all $\varphi \in \mathbf{H}^m(\mathbf{R}^n)$ and α with $[\alpha] = m$

$$\left|D^\alpha \widetilde{R}\varphi\right|^2_{\mathbf{L}^2(\mathbf{R}^n)} \le 2\left|D^\alpha\varphi\right|^2_{\mathbf{L}^2(\mathbf{R}^n)} + \frac{\widetilde{c}_2}{r^2}\sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{\alpha}\left|D^\gamma\varphi\right|^2_{\mathbf{L}^2(\mathbf{R}^n)}.$$

Returning to (4.3), we obtain

$$\begin{aligned}\left(\int_{B_r}\rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} &\le 2\kappa_t'\sum_{j=1}^N\sum_{[\alpha]=m}\left|D^\alpha\varphi_j\right|^2_{\mathbf{L}^2(\mathbf{R}^n)} + \\ &+\frac{\kappa_t'\widetilde{c}_2}{r^2}\sum_{j=1}^N\sum_{[\alpha]=m}\sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{\alpha}\left|D^\gamma\varphi_j\right|^2_{\mathbf{L}^2(\mathbf{R}^n)}.\end{aligned}$$

Now making $r \to +\infty$, the left hand side of this inequality tends to

$$\left(\int_{\mathbf{R}^n}\rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n},$$

while the right hand side tends to

$$2\kappa_t'\sum_{j=1}^N\sum_{[\alpha]=m}\left|D^\alpha\varphi_j\right|_{\mathbf{L}^2(\mathbf{R}^n)}.$$

This proves (4.1).

As an immediate consequence of Theorem 4.1, we obtain an extension of Corollary 2.4 to arbitrary unbounded open sets.

Corollary 4.3. *For every m, n, k and p as in Theorem 4.1, the constant $\widetilde{\kappa}$ is such that for every open subset Ω of $\mathbf{R}^n$ and every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}_\circ^m(\Omega)$ which is suborthonormal in $\mathbf{L}^2(\Omega)$ we have*

$$\left(\int_{\Omega}\rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \le \widetilde{\kappa}\sum_{j=1}^N\int_\Omega\sum_{[\alpha]=m}|D^\alpha\varphi_j(x)|^2\,dx.$$

4.2. The general case. We consider now an unbounded open set Ω satisfying the regularity assumption $(2.1)_m$ and such that $\mathbf{R}^n\backslash\overline{\Omega}$ contains a semi-cone. The aim of this Section is to establish the following result which does not assume any property of the functions on the boundary of Ω :

Theorem 4.4. *Let Ω be an unbounded open set of $\mathbf{R}^n$ as above. For every p satisfying (4.1), there exists a constant κ_u such that for every finite family $\{\varphi_j\}_{j=1}^N$ in $\mathbf{H}^m(\Omega)$ which is suborthonormal in $\mathbf{L}^2(\Omega)$, we have*

$$\left(\int_\Omega \rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \leq \kappa_u \sum_{j=1}^N \|\varphi_j\|_m^2\,,$$

where $\|\cdot\|_m$ denotes the usual norm on $\mathbf{H}^m(\Omega)$ or any equivalent one.The constant κ_u depends on m, n, k, p and Ω.

Proof: This result follows from an application of Theorem 4.1 to a well chosen family $\{\psi_j\}$ of $\mathbf{H}^m(\mathbf{R}^n)$. Its proof is very similar to the one of Theorem 2.1 and we sketch it briefly.

After a translation and a rotation, we can assume that the semi-cone $\mathcal{C}$ has vertex 0 and semi-axis Ox_1, $x_1 \geq 0$. We denote by ς a $\mathcal{C}^\infty$ truncation function from $\mathbf{R}^n$ into $[0,1]$ which is equal to 1 on Ω and vanishes on the translated semi-cone $\mathcal{C}_1 = \mathcal{C} + (1,0,\ldots,0)$. Let $B_\eta(a_\eta)$ be an open ball included in $\mathcal{C}_1$ and denote by $\{v_i\}_{i\in\mathbf{N}}$ the eigenfunctions of $\Delta^{(m)}$ on $\mathbf{H}_\circ^m(B_\eta(a_\eta))$. Then, the family $\{\psi_j\}$ is chosen of the form

$$\psi_j(x) = \varsigma(x)\pi_m\varphi_j(x) + \sum_{i=1}^N \alpha_{j\,i}\,v_i(x)\,,$$

and one shows the existence of α such that

$$\int_{\mathbf{R}^n} \psi_j(x).\psi_i(x)\,dx = \omega_\circ^2\,\delta_{i\,j}\,,$$

where $\omega_\circ$ is the norm of π_m in $\mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\mathbf{R}^n))$ (see $(2.1)_m$).

The proof then follows the same steps as in Theorem 2.1 (with Q replaced by $\mathbf{R}^n$ and $(3.3)'$ by (4.2)). The details are left to the reader.

5. Applications. As mentioned in the introduction, this work is motivated in part by the study of the dimension of attractors associated with partial differential equations in the case of general boundary conditions. We give, in this Section, two applications of the generalized Sobolev-Lieb-Thirring inequalities to this question. In §5.1, we first consider the magnetohydrodynamic equations. We then give in §5.2 an analogue of Theorem 2.1 for functions defined on a Riemannian manifold and we derive an estimate of the attractor related to the Navier-Stokes equations on a two-dimensional manifold. We refer to [9, 14, 21, 16, 22] for other examples.

5.1. Magnetohydrodynamic equations. The equations are written

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla)u - (B.\nabla)B + \nabla(p + \frac{1}{2}B^2) = f\,, \tag{5.1}$$

$$\frac{\partial B}{\partial t} - \nu_m\Delta B + (u.\nabla)B - (B.\nabla)u = 0\,, \tag{5.2}$$

$$\nabla.u = 0\,, \quad \nabla.B = 0\,, \tag{5.3}$$

$$u = 0\,, \quad B.n = 0 \quad \text{on} \quad \Gamma\,, \tag{5.4}$$

$$\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} = 0 \quad \text{on} \quad \Gamma\,, \tag{5.5}$$

where Γ is the boundary of Ω, a smooth connected and bounded open region of $\mathbf{R}^2$, and n denotes the exterior unit normal on Γ. As usual $u = (u_1, u_2)$ is the velocity, $B = (B_1, B_2)$ the magnetic field and p the pressure. The two positive constants ν and ν_m denote the kinetic viscosity and the magnetic viscosity of the resistive fluid under consideration (see L. Landau and E. Lifchitz [17]); the function f, which is time independent, represents a permanent external force.

It is known (M. Sermange and R. Temam [26], J.M. Ghidaglia [13]) that the long time behavior of the solutions of these equations is characterized by a universal attractor which has finite dimension. An estimate of this finite dimension is an upper bound for the number of degrees of freedom that actually describes the long time behavior of the flow ([4], [5]). However the method of [11], which is used in [26, 13] in order to prove the finite dimensionality, leads to very pessimistic bounds ([12]). On the contrary, that of [5], which is used in [27] for the $2D$ -Navier-Stokes equations (i.e. (5.1)-(5.5) with $B \equiv 0$), leads to bounds which are sharp and perhaps optimal. It relies on estimates of global Lyapunov exponents on the attractor which are obtained using Sobolev-Lieb-Thirring inequalities. These Lyapunov exponents are directly related to the linearized flow

$$\frac{\partial v}{\partial t} - \nu \Delta v + (u.\nabla)v + (v.\nabla)u - (B.\nabla)C - (C.\nabla)B + \nabla(q + B.C) = 0\,. \tag{5.6}$$

$$\frac{\partial C}{\partial t} - \nu_m \Delta C + (u.\nabla)C + (v.\nabla)B - (B.\nabla)v - (C.\nabla)u = 0\,, \tag{5.7}$$

$$\nabla.v = 0\,, \quad \nabla.C = 0\,, \tag{5.8}$$

$$v = 0\,, \quad C.n = 0 \quad \text{on} \quad \Gamma\,, \tag{5.9}$$

$$\frac{\partial C_2}{\partial x_1} - \frac{\partial C_1}{\partial x_2} = 0 \quad \text{on} \quad \Gamma, \tag{5.10}$$

where $\{u(t), B(t)\}$ is a trajectory on the universal attractor $\mathcal{A}$ (see [5, 4]). Let $\{u_\circ, B_\circ\}$ be a point on this set and $\{u(t), B(t)\}$ the solution to (5.1)-(5.5) with initial condition $\{u(x,0), B(x,0)\} = \{u_\circ(x), B_\circ(x)\}$, $x \in \Omega$. We form the quantity[2]

$$\begin{aligned}\sigma_N\Big(u_\circ, B_\circ, \{v^j\,,\, C^j\}_{j=1}^N\,,\, t\Big) = \sum_{j=1}^N \int_\Omega \Big(\nu|\nabla v^j|^2 + \nu_m|\nabla C^j|^2 + \\ + (v^j.\nabla)u.v^j - (C^j.\nabla)B.v^j + (v^j.\nabla)B.C^j - (C^j.\nabla)u.C^j\Big)\, dx\,,\end{aligned} \tag{5.11}$$

[2]This quantity occurs naturally when one considers the N^{th} trace of the linear operator that generates the linearized flow (5.6)-(5.10).

where $\{v^j\,,\,C^j\}_{j=1}^N$ is a family of elements of $H^1(\Omega)^4$ satisfying (5.9), and which is orthonormal in $L^2(\Omega)^4$. We set

$$q_N = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \operatorname{Inf} \sigma_N\Big(u_\circ, B_\circ, \{v^j\,,\,C^j\}, s\Big)\, ds \tag{5.12}$$

where the infimum is taken on all the pairs $\{u_\circ\,,\,B_\circ\} \in \mathcal{A}$, and on all the orthonormalized families of $L^2(\Omega)^4$ which belong to $H^1(\Omega)^4$ and satisfy (5.9). According to [5], if there exists $N_\circ \geq 1$ such that

$$q_{N_\circ} > 0 \tag{5.13}$$

then $\mathcal{A}$ has finite fractal (and Hausdorff) dimension and

$$\text{Fractal dimension of } \mathcal{A} \leq N_\circ \max_{1\leq \ell \leq N_\circ - 1} \Big(1 + \frac{(-q_\ell)_+}{q_{N_\circ}}\Big). \tag{5.14}$$

We recall that the Hausdorff dimension of a set is less than or equal to its fractal dimension and refer to [8] and [20] for the definition of these dimensions.

Our aim now is to estimate (5.12) in order to obtain (5.13) for some $N_\circ \geq 1$. We first notice that

$$\operatorname{Inf} \int_\Omega |\nabla w|^2\, dx = \lambda_1 > 0\,, \tag{5.15}$$

where the infimum is taken on the subset of functions in $H^1(\Omega)^2$ that satisfy $w\cdot n = 0$ on Γ and $\int_\Omega |w|^2\, dx = 1$. Moreover the product

$$\lambda_1 \delta(\Omega)^2 \text{ depends only on the shape of } \Omega. \tag{5.16}$$

Hence, applying Theorem 2.1 to the $\varphi_j = \{v_j\,,\,C_j\}$, with $k=4\,,\ n=2\,,\ m=1$ and $p=2$, we find

$$\int_\Omega (\rho_v+\rho_c)^2\, dx \leq \kappa \sum_{j=1}^N \int_\Omega \Big(|\nabla v^j(x)|^2 + |\nabla C^j(x)|^2\Big)\, dx + \frac{\chi}{\delta(\Omega)^2}\int_\Omega (\rho_v+\rho_c)\, dx\,.$$

Now, thanks to (5.15) and (5.16) this inequality implies

$$\int_\Omega (\rho_v+\rho_c)^2\, dx \leq \kappa_2 \sum_{j=1}^N \int_\Omega \Big(|\nabla v^j(x)|^2 + |\nabla C^j(x)|^2\Big)\, dx \tag{5.17}$$

where $\kappa_2 = \kappa + \chi/\lambda_1\delta(\Omega)^2$ depends only on the shape of Ω. We return to (5.11) and notice that according to Cauchy-Schwarz inequality

$$\left|\int_\Omega \left\{\sum_{j=1}^N \Big((v^j.\nabla)u.v^j - (C^j.\nabla)B.v^j + (v^j.\nabla)B.C^j - (C^j.\nabla)u.C^j\Big)\right\} dx\right|$$

$$\leq \int_\Omega \left\{ |\nabla u| \sum_{j=1}^N \Big(|v^j|^2 + |C^j|^2\Big) + 2|\nabla B| \sum_{j=1}^N |C^j||v^j| \right\} dx$$

$$\leq \int_\Omega (|\nabla u| + |\nabla B|) \sum_{j=1}^N \Big(|v^j|^2 + |C^j|^2\Big)\, dx$$

$$= \int_\Omega \Big(|\nabla u| + |\nabla B|\Big)(\rho_v+\rho_c)\, dx\,.$$

Combining this inequality and (5.17), we deduce from (5.11) that

$$\sigma_N \geq \frac{\nu_\circ}{\kappa_2}\int_\Omega (\rho_v+\rho_c)^2\,dx - \int_\Omega \Big(|\nabla u| + |\nabla B|\Big)(\rho_v+\rho_c)\,dx\,, \tag{5.18}$$

where $\nu_\circ \equiv \min(\nu\,,\,\nu_m)$. Using again Chauchy-Schwarz inequality, it follows from (5.18) that

$$\sigma_N \geq \frac{\nu_\circ}{2\kappa_2}\int_\Omega (\rho_v+\rho_c)^2\,dx - \frac{\kappa_2}{\nu_\circ}\int_\Omega \Big(|\nabla u|^2 + |\nabla B|^2\Big)\,dx\,. \tag{5.19}$$

Since

$$\begin{aligned} N = \int_\Omega (\rho_v+\rho_c)\,dx \leq & \Big(\int_\Omega 1\,dx\Big)^{1/2}\Big(\int_\Omega (\rho_v+\rho_c)^2\,dx\Big)^{1/2} \\ \leq & \delta(\Omega)\Big(\int_\Omega (\rho_v+\rho_c)^2\,dx\Big)^{1/2}, \end{aligned}$$

we deduce from (5.19) that

$$\sigma_N \geq \frac{\nu_\circ N^2}{2\kappa_2\,\delta(\Omega)^2} - \frac{\kappa_2}{\nu_\circ}\int_\Omega \Big(|\nabla u|^2 + |\nabla B|^2\Big)\,dx\,. \tag{5.20}$$

It remains to estimate the last term which only depends on u and B. For that purpose we multiply (5.1) by u, (5.2) by B and integrate the sum of the resulting equalities on Ω. It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|u|^2 + |B|^2\Big)\,dx + \int_\Omega \Big(\nu|\nabla u|^2 + \nu_m|\nabla B|^2\Big)\,dx = \int_\Omega f.u\,dx\,. \tag{5.21}$$

By Cauchy-Schwarz inequality and (5.15),

$$\frac{d}{dt}\int_\Omega \Big(|u|^2 + |B|^2\Big)\,dx + \int_\Omega \Big(\nu|\nabla u|^2 + \nu_m|\nabla B|^2\Big)\,dx \leq \frac{1}{\nu\lambda_1}\int_\Omega |f|^2\,dx\,, \tag{5.22}$$

which shows that

$$\frac{1}{t}\int_0^t\int_\Omega \Big(\nu|\nabla u(x,s)|^2 + \nu_m|\nabla B(x,s)|^2\Big)\,dx\,ds \leq \frac{1}{\nu\lambda_1}\int_\Omega |f|^2\,dx + \frac{1}{t}\int_\Omega \Big(|u_\circ|^2 + |B_\circ|^2\Big)\,dx. \tag{5.23}$$

Using (5.11), (5.12), (5.20) and (5.23) we find

$$q_N \geq \frac{\nu_\circ N^2}{2\kappa_2\,\delta(\Omega)^2} - \frac{\kappa_2}{\nu_\circ^2\,\nu\lambda_1}\int_\Omega |f(x)|^2\,dx\,,\ \ \forall\ N \geq 1\,. \tag{5.24}$$

Hence for sufficiently large N, $q_N > 0$. We introduce the nondimensional Grashoff number (recall that $\nu_\circ = \min(\nu\,,\,\nu_m)$)

$$G_m = \frac{1}{\nu^{1/2}\,\nu_\circ^{3/2}\,\lambda_1}\Big(\int_\Omega |f(x)|^2\,dx\Big)^{1/2} \tag{5.25}$$

so that (5.24) reads

$$q_N \geq \frac{\nu_\circ}{2\kappa_2\,\delta(\Omega)^2}\Big(N^2 - \kappa_3\,G_m^2\Big) \tag{5.26}$$

where $\kappa_3 = 2\lambda_1\,\delta(\Omega)^2\,\kappa_2^2$ depends only on the geometry of Ω according to (5.16). Taking $N_\circ =$ the integer part of $\sqrt{2\kappa_3}\,G_m$, we have according to (5.14) and (5.26)

Theorem 5.1. *The fractal and Hausdorff dimensions of the universal attractor describing the long time behavior of the two-dimensional magnetohydrodynamic equations are less than or equal to*

$$c_\circ\,(1+G_m)$$

where $c_\circ$ is a constant which only depends on the shape of Ω.

Remark 5.2. A similar analysis can be carried out for Navier-Stokes equations with boundary conditions:

$$u.n=0\,,\quad \sigma(u,p)\times n=0 \;\; on \;\; \Gamma\,, \tag{5.27}$$

where the stress tensor σ reads

$$\sigma_{i\,j}=\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-p\delta_{i\,j}\,.$$

Here again the estimate on the dimension is linear with respect to the Grashoff number.

5.2. Navier-Stokes equations on a two-dimensional manifold. In order to estimate the dimension of the corresponding attractor, we first show how Theorem 2.1 can be generalized to functions defined on a Riemannian manifold.

5.2.1. A functional inequality in a Riemannian manifold. We consider an n-dimensional compact and oriented Riemannian manifold (M,g) and an open regular subset W of M. In order to formulate the problem, we introduce an atlas $\{(U_i,f_i)\}_{i=1}^r$ on M, compatible with the metrics g and a partition of unity $\{\theta_i\}_{i=1}^r$ associated with this atlas. We recall that the Sobolev spaces of functions on W (and M), $H^m(W)$, $L^2(W)$ are defined by transport into $\mathbf{R}^n$ on each chart, using the canonical volume element $dM=\sqrt{G}\,dx_1\dots dx_n$ with $G=\det(g)$. The Sobolev spaces of tangent vector fields $H^m(TW)$, $L^2(TW)$ are defined in the same manner (see T. Aubin [2]). For example,

$$\int_W|\varphi|^2\,dM=\sum_{i=1}^r\int_{\Omega_i}\left(\theta_i|\varphi|^2\sqrt{G}\right)\circ f_i^{-1}\,dx\,,$$

where $\Omega_i=f_i(W\cap U_i)$ and, in (5.30) below, $\int_W|\nabla^{(m)}\varphi|^2\,dM$ is the usual Sobolev semi norm on $H^m(W)$ ([2]).

We assume that W is m-regular in the sense that for every $i=1,\dots,r$

$$\Omega_i=f_i(W\cap U_i)\;\; \text{satisfies } (2.1)_m\,. \tag{5.28$_m$}$$

We can then state

Proposition 5.3. *Let W be an open subset of an n-dimensional compact and oriented Riemannian manifold (M,g) satisfying $(5.28)_m$. For every p with*

$$\max(1,\frac{n}{2m})<p\le 1+\frac{n}{2m}\,, \tag{5.29}$$

there exist two positive constants $\kappa(W)$ and $\chi(W)$ such that for every finite family in $H^m(W)$ which is suborthonormal in $L^2(W)$, we have

$$\left(\int_W \rho^{p/(p-1)}\, dM\right)^{2m(p-1)/n} \leq \kappa(W) \sum_{j=1}^{N} \int_W |\nabla^{(m)} \varphi_j|^2\, dM + \chi(W) \int_W \rho\, dM\,. \tag{5.30}$$

The constants $\kappa(W)$ and $\chi(W)$ depend on m, n, p, W and on (M, g).

Remark 5.4. We have stated the result for scalar functions on W. It is of course valid for tensors of arbitrary order,

$$\rho(x) = \sum_{j=1}^{N} |\varphi_j(x)|^2$$

denoting then the sum of the norms of these tensors. Actually, in the next Section, we will use (5.30) for vector fields.

The inequality (5.30) is proved by applying Theorem 2.1 on each chart. Indeed, we first note that, for every $i = 1, \ldots, r$, $\{\theta_i^{1/2} \varphi_j\}_{j=1}^N$, is suborthonormal in $L^2(W)$, or equivalently, the families $\{\psi_j^i\}_{j=1}^N$, $i = 1, \ldots, r$, where $\psi_j^i = (\theta_i^{1/2} G^{1/4} \varphi_j) \circ f_i^{-1}$, are suborthonormal in $L^2(\Omega_i)$. For $\xi \in \mathbf{R}^N$, we have, using the suborthonormality of the φ_j's in $L^2(W)$,

$$\sum_{\ell=1}^{N} \xi_\ell^2 \geq \sum_{j,\ell=1}^{N} \xi_\ell\, \xi_j \int_W \varphi_j \varphi_\ell\, dM\,,$$

which can be expressed as

$$\begin{aligned} \sum_{\ell=1}^{N} \xi_\ell^2 &\geq \sum_{i=1}^{r} \sum_{j,\ell=1}^{N} \xi_\ell \xi_j \int_{\Omega_i} \psi_j^i\, \psi_\ell^i\, dx \geq \sum_{i=1}^{r} \int_{\Omega_i} \left(\sum_{\ell=1}^{N} \xi_\ell\, \psi_\ell^i\right)^2 dx \\ &\geq \int_{\Omega_i} \left(\sum_{\ell=1}^{N} \xi_\ell\, \psi_\ell^i\right)^2 dx\,, \quad \text{for } i \in \{1, \ldots, r\}\,. \end{aligned}$$

Therefore, Theorem 2.1 applies to the family $\{\psi_j^i\}_{j=1}^N$ on Ω_i, for $i = 1, \ldots, r$. Adding the corresponding inequalities and using Young inequality, one obtains easily (5.30).

Remark 5.5. In (5.30), we have not mentioned how $\kappa(W)$ and $\chi(W)$ depend on W. It is easy to check (either directly or in the course of the proof) that, when $W = M$, $\kappa(M)$ and $\mathrm{Vol}(M)^{2m/n} \chi(M)$ depend only on the shape of M.

5.2.2. Application to Navier-Stokes equations. We consider Navier-Stokes equations on a two-dimensional manifold M. This problem arises when considering the flow of air around the earth (geophysical flows) and has of course some interest in meteorology. The equations read (see D. Ebin and J. Marsden [7] and also A. Avez and Y. Bamberger [3] for $M = S^2$)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + D_u u + \nabla p = f\,, \\ \operatorname{div} u = 0\,, \end{cases} \tag{5.31}$$

where f is a given tangent vector field on M, the Riemannian manifold on which the flow occurs. The velocity u and the pressure p are respectively a tangent vector field and a scalar field on M.

The existence of a finite dimensional attractor describing the long time behavior of (5.31) in the case of a two dimensional manifold was proved in [13], under the hypothesis that Poincaré inequality

$$\operatorname{Inf} \frac{\int_M |\nabla u|^2 \, dM}{\int_M |u|^2 \, dM} > 0 \tag{5.32}$$

holds for tangent vector fields on M. This property depends on M; it is satisfied when $M = S^2$ with the usual metrics and in fact (see [13]), (5.32) holds as soon as the Euler-Poincaré characteristic of M is not zero. Or equivalently when

$$\text{Every continuous vector field on } M \text{ vanishes at one point at least.} \tag{5.33}$$

We notice that (5.33) is a topological condition on M which is independent of the metrics and persists, when e.g., M is a slight perturbation of S^2.

We deduce from (5.30) with $W = M$, $n = 2$, $m = 1$, $p = 2$ and (5.32), that there exists a constant $\kappa(M)$ such that for every finite family $\{v_j\}_{j=1}^N$ in $H^1(TM)$ which is suborthonormal in $L^2(TM)$ we have

$$\int_M \left(\sum_{j=1}^N |v_j|^2 \right)^2 dM \leq \kappa(M) \sum_{j=1}^N \int_M |\nabla v_j|^2 \, dM \,, \tag{5.34}$$

where $|\cdot|$ denotes the norm induced by g on the tangent spaces. The constant $\kappa(M)$ depends only on the shape of M (and on g). Introducing the Grashoff number

$$G_M = \frac{\operatorname{Vol}(M)}{\nu^2} \left(\int_M |f|^2 \, dM \right)^{1/2}$$

and using the techniques of Section 5.1, one derives the following estimate of the dimension of the attractor.

Theorem 5.6. *The fractal and Hausdorff dimensions of the universal attractor describing the long time behavior of the equations (5.31) are less than or equal to*

$$c_\circ(1 + G_M) \,,$$

where $c_\circ$ is a constant which only depends on the shape of M (and on g).

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