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PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WHICH APPROXIMATE DELAY EQUATIONS

F. KAPPEL†
Institute for Mathematics, University of Graz, A–8010 Graz, Austria

KLAUS SCHMITT‡
Department of Mathematics, University of Utah, Salt Lake City, Utah 84122 USA

1. Introduction. Let $\alpha > 0$ and $f : \mathbf{R} \to \mathbf{R}$ be given. The scalar delay-differential equation

$$\begin{aligned} \dot{x}(t) &= -\alpha f(x(t-1)), \quad && t \geq 0, \\ x(t) &= \phi(t), && -1 \leq t \leq 0, \end{aligned} \tag{1.1}$$

has been investigated thoroughly since the early sixties and many beautiful results have been found concerning existence of nontrivial periodic solution of this equation (see for instance [8,9,12-14] and the references in these papers). During the past decade one can also see an increasing interest in a class of numerical approximation schemes for delay equations based on abstract approximation results for strongly continuous semigroups of transformations (see for instance [2,3,7]). The various schemes developed during the last years have remarkable qualitative properties. For instance, in case of the scheme developed in [2] and in [7], the approximating ordinary differential systems inherit stabilizability, detectability and exponential stability from the linear delay system which is approximated.

In this paper we study a very simple approximation scheme for (1.1) and demonstrate that the approximating ordinary differential systems inherit from (1.1) the occurence of Hopf-bifurcations and the existence of global branches of nontrivial periodic solution.

The approximation scheme used in this paper was developed in [10]. For $N = 1, 2, \ldots$, let $X^N = \{\phi \in C(-1,0;\mathbf{R}) \,|\, \phi$ is a 1st order spline on $[-1,0]$ with knots at the points $-j/N$, $j = 0, \ldots, N\}$. Furthermore, let $\pi^N : C(-1,0;\mathbf{R}) \to X^N$ be defined by interpolation at the meshpoints, i.e.

$$(\pi^N \phi)(-\frac{j}{N}) = \phi(-\frac{j}{N}), \quad j = 0, \ldots, N,$$

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for $\phi \in C(-1,0;\,\mathbf{R})$. Of course, we use the standard norm on $C(-1,0;\,\mathbf{R})$, $\|\phi\| = \sup_{-1\le s\le 0} |\phi(s)|$. For $\phi \in X^N$ we denote by $w^N = \operatorname{col}(w_0^N,\dots,w_N^N) = \operatorname{col}(\phi(0),\phi(-1/N),\dots,\phi(-1))$ the coordinate vector of ϕ with respect to the standard basis e_j^N, $j=0,\dots,N$, of X^N, where the hat functions $e_j^N \in X^N$ are characterized by $e_j^N(-i/N) = \delta_{ij}$ (= Kronecker symbol), $i,j = 0,\dots,N$. If $x(t)$ is a function $[-1,\infty) \to \mathbf{R}$ we use x_t, $t \ge 0$, to denote the function on $[-1,0]$ defined by $x_t(s) = x(t+s)$, $-1 \le s \le 0$.

Let

$$B^N = \begin{bmatrix} 0 & \dots\dots\dots\dots & & & 0 \\ N & -N & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & N & -N \end{bmatrix} \in \mathbf{R}^{(N+1)\times(N+1)}.$$

Then $w^N(t) = w^N(t;\phi) = \operatorname{col}\big(w_0^N(t;\phi),\dots,w_N^N(t;\phi)\big)$ denotes the solution of

$$\begin{aligned} \dot w^N(t) &= B^N w^N(t) + \operatorname{col}\big(-\alpha f(w_N^N(t)),0,\dots,0\big), \quad t \ge 0, \\ w^N(0) &= \operatorname{col}\big(\phi(0),\phi(-\tfrac{1}{N}),\dots,\phi(-1)\big). \end{aligned} \tag{1.2}$$

In case of equation (1.1) Theorem 3.1 of [10] (respectively its corollary) gives the following approximation result:

Theorem 1.1. *Let f be continuous on $\mathbf{R}$ and assume that for any $\phi \in C(-1,0;\mathbf{R})$ there is a unique solution $x(t;\phi)$ of (1.1) with maximal interval of existence $[-1,t_\phi)$, $t_\phi > 0$. Then for any $\bar t \in (0,t_\phi)$*

$$\lim_{N\to\infty} \sum_{j=0}^{N} w_j^N(t;\phi) e_j^N = x_t(\phi) \tag{1.3}$$

uniformly for $t \in [0,\bar t]$.

Note, that $\sum_{j=0}^N w_j^N(t;\phi)e_j^N$ is the spline function with values $w_j^N(t;\phi)$ at the meshpoints $-j/N$. Specifically (1.3) implies $\lim_{N\to\infty} w_0^N(t;\phi) = x(t;\phi)$ and $\lim_{N\to\infty} w_N^N(t;\phi) = x(t-1;\phi)$ uniformly for $t \in [0,\bar t]$. For $w \in X^N$ we choose the norm $|w| = \max_{j=0,\dots,N} |w_j|$, which implies $|w| = \|\sum_{j=0}^N w_j e_j^N\|$.

Our paper is organized as follows. In §2 we investigate the linearization associated with (1.2) and show that nontrivial periodic solutions of (1.2) arise via the Hopf bifurcation theorem. In §3 we then prove existence of nontrivial periodic solutions of (1.2) for any α greater than a critical value using fixed point theorems. Finally, in §4 we show that there exists a subsequence of these nontrivial periodic solutions converging uniformly on compact intervals to a nontrivial periodic solution for the delay equation (1.1).

2. The linearized equation. Linearizing (1.2) around the trivial solution one obtains the linear system

$$\dot w = A^N w, \tag{2.1}$$

where $w = \mathrm{col}(w_0, \dots, w_N)$ and

$$A^N = \begin{bmatrix} 0 & \dots\dots\dots & & 0 & -\alpha \\ N & -N & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & N & -N \end{bmatrix} \in \mathbf{R}^{(N+1)\times(N+1)}.$$

The characteristic equation of (2.1) is given by

$$0 = \det(\lambda I - A^N) = N^N\big[\lambda\big(1+\frac{\lambda}{N}\big)^N + \alpha\big].$$

Thus we wish to discuss the equation

$$f_\alpha(\lambda) = \lambda\big(1+\frac{\lambda}{N}\big)^N + \alpha = 0. \tag{2.2}$$

If $\alpha = 0$ then $\lambda = 0$ is a simple root and $\lambda = -N$ is a root of multiplicity N. In case $\alpha > 0$ the following elementary lemma is concerned with the real roots of (2.2).

Lemma 2.1.

a) *For $\alpha > 0$ all roots of (2.2) are simple with the exception of $\lambda = -N/(N+1)$ which occurs when $\alpha = \big(N/(N+1)\big)^{N+1}$.*

b) *If $0 < \alpha < \big(N/(N+1)\big)^{N+1}$ then equation (2.2) has a simple real root in each of the intervals $\big(-N, -N/(N+1)\big)$ and $\big(-N/(N+1), 0\big)$.*

c) *If N is even, then for any $\alpha > 0$ equation (2.2) has a simple real root in the interval $(-\infty, -N)$.*

d) *There are no other real roots than those appearing in statements a)–c).*

Concerning non-real roots we investigate the possibility for (2.2) to have a root λ with $\mathrm{Re}\lambda = \sigma$ for fixed $\sigma \in \mathbf{R}$. We put $\lambda = \sigma + i\tau$, $\tau \in \mathbf{R}$, and define $r \geq 0$, $\theta \in [-\pi/2, 3\pi/2)$ by

$$re^{i\theta} = 1 + \frac{\sigma + i\tau}{N}. \tag{2.3}$$

Then λ is a root of (2.2) if and only if

$$\mathrm{Re}(\sigma + i\tau)r^N e^{iN\theta} + \alpha = 0 \tag{2.4}$$

and

$$\mathrm{Im}(\sigma + i\tau)r^N e^{iN\theta} = 0. \tag{2.5}$$

We first investigate (2.5) and note that $r = 0$ corresponds to $\tau = 0$ and $\sigma = -N$, which is only possible when $\alpha = 0$. So we may assume that $r > 0$. Then (2.5) yields

$$\sigma \sin N\theta + \tau \cos N\theta = 0. \tag{2.6}$$

We first consider the case $\sigma \neq 0$. Then (2.6) is equivalent to

$$\tau = -\sigma \tan N\theta. \tag{2.7}$$

Note, that $\cos N\theta = 0$ together with (2.5) and $\sigma \neq 0$ would imply $\sin N\theta = 0$ and therefore $\alpha = 0$ by (2.4). Also using (2.3) it is easily seen that $\cos\theta \neq 0$. Then we get from (2.3) that (2.7) is equivalent to

$$(N+\sigma)\tan\theta + N\sigma \tan N\theta = 0. \tag{2.8}$$

If θ_0 is a root of (2.8), then the corresponding value for α is given by

$$\alpha_0 = -\sigma r_0^N \frac{1}{\cos N\theta_0}.$$

Since we are interested in the case $\alpha > 0$ only, we conclude that

$$\sigma \cos N\theta_0 < 0.$$

In case $\sigma = -N$, (2.8) implies $\tan N\theta_0 = 0$, i.e., $\lambda = -N$ by (2.7) and therefore $\alpha = 0$. This implies that for $\alpha > 0$ there is no root of (2.2) on $\mathrm{Re}\lambda = -N$.

From (2.3) we see that

$$r_0 = \frac{1+\sigma/N}{\cos\theta_0},$$

where $\theta_0 \in (-\pi/2\,,\, \pi/2)$ if $\sigma > -N$ and $\theta_0 \in (\pi/2\,,\, 3\pi/2)$ if $\sigma < -N$. It further follows that

$$r_0 = \left|1 + \frac{\sigma}{N}\right| (1+\tan^2\theta_0)^{1/2}. \tag{2.9}$$

These considerations are summarized in the following proposition.

Proposition 2.2.

a) *Let $\sigma \neq 0$. Then $\lambda = N(-1 + r_0 e^{i\theta_0}) = \sigma + i\tau_0$ is a root of (2.2) for $\alpha = \alpha_0 > 0$ if and only if θ_0 and α_0 satisfy*

$$\begin{aligned}
&(1+\frac{\sigma}{N})\tan\theta_0 + \frac{\sigma}{N}\tan N\theta_0 = 0,\\
&\sigma\cos N\theta_0 < 0,\\
&\alpha_0 = -\frac{\sigma}{\cos N\theta_0} r_0^N,
\end{aligned}$$

where $\theta_0 \in (-\pi/2\,,\, \pi/2)$ if $\sigma > -N$ and $\theta_0 \in (\pi/2\,,\, 3\pi/2)$ if $\sigma < -N$. r_0 and τ_0 are given by (2.9) and

$$\tau_0 = -\sigma\tan N\theta_0 = (N+\sigma)\tan\theta_0.$$

b) *If $\alpha > 0$, there is no root of (2.2) with $\mathrm{Re}\lambda = -N$.*

We next consider the case $\sigma = 0$. Equation (2.6) becomes

$$\tau \cos N\theta = 0.$$

Since λ cannot equal zero for $\alpha > 0$, we conclude $\cos N\theta = 0$, i.e., $\theta = \theta_k = (\pi/2N)+(k\pi/N)$, where k is an integer. Observing that $1 + i(\tau/N)$ lies in the right half plane we obtain from (2.3) that $-\pi/2 < \theta_k < \pi/2$. Hence k is restricted to $-(N+1)/2 < k < (N-1)/2$. One easily sees that the corresponding roots λ_k satisfy $\lambda_{-k} = \bar{\lambda}_{k-1}$. So it suffices to consider $0 \leq k < (N-1)/2$. Using (2.3) and (2.4) one may calculate that

$$\alpha_k = (-1)^k \frac{N \tan \theta_k}{(\cos \theta_k)^N}.$$

Since $\alpha_k > 0$, k must be even. These considerations yield the following result.

Proposition 2.3. *Let $N = 4\ell + j + 2$, $0 \leq j \leq 3$, $\ell = 0, 1, 2, \ldots$. Then equation (2.2) has a pair of purely imaginary roots if and only if $\alpha = \alpha_k(N)$, where*

$$\alpha_k(N) = \frac{N \tan \theta_k(N)}{(\cos \theta_k(N))^N},$$
$$\theta_k(N) = \frac{\pi}{2N} + k\frac{\pi}{N}, \quad k = 0, 2, 4, \ldots, 2\ell.$$

These roots are

$$\lambda_k(N) = i\tau_k(N), \qquad \overline{\lambda_k(N)} = -i\tau_k(N),$$

where

$$\tau_k(N) = N \tan \theta_k(N).$$

The following table gives for each N the values of k that need to be considered.

N	1	2	3	4	5	6	7	8	9	10	...
k	-	0	0	0	0	0,2	0,2	0,2	0,2	0,2,4	...

Proposition 2.4. *Let $\tau_k(N)$ and $\alpha_k(N)$ be given as in Proposition 2.3. Then for fixed N*

$$0 < \tau_0(N) < \cdots < \tau_{2\ell}(N),$$
$$0 < \alpha_0(N) < \cdots < \alpha_{2\ell}(N),$$

and for fixed $k \in \{0, 2, \ldots, 2\ell\}$ the sequences $\{\tau_k(N)\}$ and $\{\alpha_k(N)\}$ are decreasing with

$$\lim_{N\to\infty} \tau_k(N) = \lim_{N\to\infty} \alpha_k(N) = \frac{\pi}{2} + k\pi.$$

Proof: Since $\theta_0(N) < \theta_2(N) < \ldots$, it follows immediately that $\tau_k(N)$ and $\alpha_k(N)$ are monotonically increasing with respect to k. We write

$$\tau_k(N) = \frac{(2k+1)\pi}{2} \frac{2N}{(2k+1)\pi} \tan \frac{(2k+1)\pi}{2N}$$

and observe that $(1/x)\tan x$ is monotonically increasing for $x \geq 0$ and $\lim_{x\to 0^+}(1/x)\tan x = 1$. This implies that $\tau_k(N)$ is decreasing with respect to N and $\lim_{N\to\infty}\tau_k(N) = (\pi/2)+k\pi$. The result for $\alpha_k(N)$ follows from

$$\alpha_k(N) = \tau_k(N)\Big([\cos\theta_k(N)]^{-1/\theta_k(N)}\Big)^{((2k+1)\pi)/2}.$$

We next present a detailed study for the case $\sigma \neq 0$. Equation (2.8) suggests that we distinguish the cases

$$\sigma < -N, \quad -N < \sigma < 0 \quad \text{and} \quad \sigma > 0.$$

Case 1: $\sigma < -N$

According to Proposition 2.2 we are interested in solutions θ of (2.8) in $(\pi/2\,,\,3\pi/2)$ which also satisfy $\cos N\theta > 0$. Since roots of (2.8) are symmetric with respect to π, we only consider the interval $(\pi/2\,,\,\pi]$.

One easily concludes that we obtain roots θ_j of (2.8) satisfying

$$\theta_j \in \Big(\frac{\pi}{2}+\frac{j\pi}{2N}\,,\,\frac{\pi}{2}+\frac{(j+1)\pi}{2N}\Big),$$

where $j = 0,2,\ldots,N-2$ if N is even and $j = 1,3,\ldots,N-2$ if N is odd. Moreover $\theta = \pi$ is an admissible root if N is even, in which case $\tau = 0$, i.e., $\lambda = \sigma$ and $\alpha = |\sigma|\,|1+\sigma/N|^N$ (compare statement c) of Lemma 2.1).

Since $(N+j)\pi/2 < N\theta_j < (N+j+1)\pi/2$, we see that $\cos N\theta_j > 0$ provided $N+j$ is a multiple of 4. The following table provides admissible values for j for some N :

N	2	3	4	5	6	7	8	9	10	11	12	13	14
j	-	1	0	3	2	1,5	0,4	3,7	2,6	1,5,9	0,4,8	3,7,11	2,6,10

Case 2: $-N < \sigma < 0$.

We are interested in solutions θ of (2.8) satisfying

$$\cos N\theta < 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Again we need only to consider the interval $[0,\pi/2)$.

The solution $\theta_0 = 0$ corresponds to $\tau = 0$ and $\lambda = \sigma$, which is admissible. The corresponding value of α is given by

$$\alpha(\sigma) = -\sigma(1+\frac{\sigma}{N})^N,$$

where $\max_{-N<\sigma<0}\alpha(\sigma) = \big(N/(N+1)\big)^{N+1}$ is attained for $\sigma = -N/(N+1)$ (compare statements a) and b) of Lemma 2.1).

Equation (2.8) has nonzero roots

$$\theta_j \in \Big(\frac{j\pi}{2N}\,,\,\frac{(j+1)\pi}{2N}\Big),$$

where $j = 2, 4, \dots, N-2$ if N is even and $j = 1, 3, \dots, N-3$ if N is odd. Since $j\pi/2 < N\theta_j < (j+1)\pi/2$, we see that θ_j is admissible if j is a multiple of 4. The following table gives a partial list of admissible values for j.

N	2	3	4	5	6	7	8	9	10	...
j	-	-	-	-	4	4	4	4	4,8	...

The intervals $(0, \pi/2N)$ and $(\pi/2 - \pi/2N\,,\, \pi/2)$ demand additional investigation. We find that (2.8) has a root in $(0, \pi/2N)$ if and only if $-N/(N+1) < \sigma < 0$. This root is admissible.

If N is odd there possibly exists a root in $(\pi/2 - \pi/2N\,,\, \pi/2)$. Since

$$\lim_{\theta \uparrow \frac{\pi}{2}} \frac{(N+\sigma)\tan\theta}{-\sigma \tan N\theta} = -\frac{N^2 + N\sigma}{\sigma} \begin{cases} > 1, & \text{if } -\frac{N^2}{N+1} < \sigma < 0, \\ < 1, & \text{if } -N < \sigma < -\frac{N^2}{N+1}, \end{cases}$$

we find there is such a root provided N is odd and $-N < \sigma < -N^2/(N+1)$. This root is admissible if $N = 4\ell + 1$, $\ell = 1, 2, \dots$, because then

$$N\theta \in \Big(\frac{N\pi}{2} - \frac{\pi}{2}, \frac{N\pi}{2}\Big) = (2\ell\pi,\, 2\ell\pi + \frac{\pi}{2}).$$

Moreover $\theta \to \pi^-$ as $\sigma \to -N^2/(N+1)-$. Therefore $\tau \to \infty$ as $\sigma \to -N^2/(N+1)-$.

Case 3: $\sigma > 0$.

We look for solutions θ of (2.8) in $[0, \pi/2)$ satisfying $\cos N\theta < 0$. Thus $\theta = 0$ is never an admissible root. There exist nonzero roots

$$\theta_j \in \Big(\frac{j\pi}{2N}, \frac{(j+1)\pi}{2N}\Big),$$

where $j = 1, 3, \dots, N-1$ if N is even and $j = 1, 3, \dots, N-2$ if N is odd. These roots are only admissible if $j = 4\ell + 1$, $\ell = 0, 1, \dots$.

A complete picture of the behavior of the roots of (2.2) will be obtained if in addition we investigate the dependence of these roots on α. Simple roots depend smoothly on α and

$$\frac{d\lambda}{d\alpha} = \frac{\lambda(1 + \frac{\lambda}{N})}{\alpha(1 + \frac{N+1}{N}\lambda)}.$$

For $\lambda = \sigma + i\tau$ we get

$$\frac{d\sigma}{d\alpha} = \frac{1}{\alpha D(\sigma,\tau)} g(\sigma,\tau), \quad \frac{d\tau}{d\alpha} = \frac{1}{\alpha D(\sigma,\tau)} h(\sigma,\tau),$$

where

$$D(\sigma,\tau) = \big(1 + \frac{N+1}{N}\sigma\big)^2 + \tau^2\big(\frac{N+1}{N}\big)^2,$$

$$g(\sigma,\tau) = \tau^2\big(1 + \frac{N+1}{N^2}\sigma\big) + \sigma + \frac{N+2}{N}\sigma^2 + \frac{N+1}{N^2}\sigma^3,$$

$$h(\sigma,\tau) = \tau\big(1 + \frac{2}{N}\sigma + \frac{N+1}{N^2}\sigma^2 + \frac{N+1}{N^2}\tau^2\big).$$

For $N \geq 2$ the zeros of $g(\sigma, \tau)$ are given by

$$\tau = \pm\big(p(\sigma)\big)^{1/2}, \quad \sigma \in [-N, -\frac{N^2}{N+1}) \cup [-\frac{N}{N+1}, 0],$$

where

$$p(\sigma) = -\sigma \frac{1 + \frac{N+2}{N}\sigma + \frac{N+1}{N^2}\sigma^2}{1 + \frac{N+1}{N^2}\sigma}.$$

Since $1 + (2/N^2)\sigma + \big((N+1)/N^2\big)\sigma^2$ does not have real roots, it follows that $h(\sigma, \tau) > 0$ for $\tau \neq 0$. The sign of g is indicated in Figure 1.

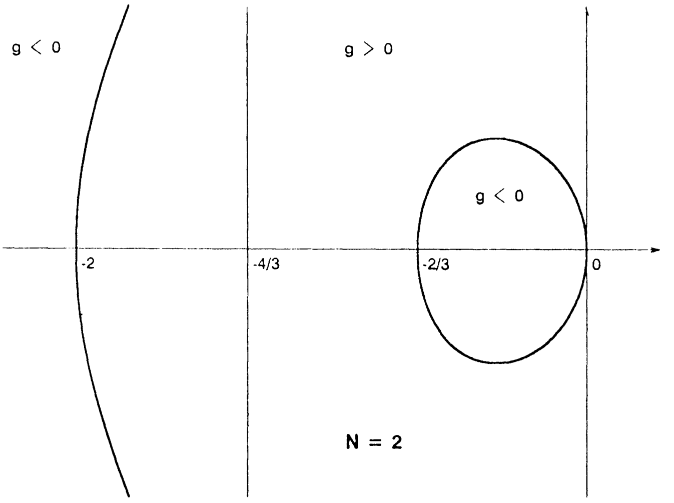

Figure 1.

If $\sigma = 0$ we get

$$\frac{d\sigma}{d\alpha} = \frac{\tau^2}{\alpha\big(1 + (\frac{N+1}{N})^2\tau^2\big)}, \quad \frac{d\tau}{d\alpha} = \frac{\tau + \frac{N+1}{N^2}\tau^3}{\alpha\big(1 + (\frac{N+1}{N})^2\tau^2\big)}.$$

Using these formulas and Proposition 2.3 we immediately get

Proposition 2.5. *For $N = 2, 3, \ldots$, let ℓ be defined as in Proposition 2.3. If α increases through $\alpha_k(N)$, $k = 0, 2, \ldots, 2\ell$, then a pair of conjugate complex roots of (2.2) crosses the imaginary axis from left to right with positive speed,*

$$\frac{d\,Re\,\lambda}{d\alpha}\Big|_{\alpha=\alpha_k(N)} = \frac{N \tan\theta_k(N)}{1 + (N+1)^2 \tan^2\theta_k(N)} [\cos\theta_k(N)]^N > 0,$$

$$\frac{d\,Im\,\lambda}{d\alpha}\Big|_{\alpha=\alpha_k(N)} = \frac{1 + (N+1)\tan^2\theta_k(N)}{1 + (N+1)^2 \tan^2\theta_k(N)} [\cos\theta_k(N)]^N > 0,$$

$k = 0, 2, \dots, 2\ell$. *Moreover,*

$$\lim_{N\to\infty} \frac{d\,Re\,\lambda}{d\alpha}\Big|_{\alpha=\alpha_k(N)} = \left(\frac{\pi}{2} + k\pi\right)\left(1 + \left(\frac{\pi}{2} + k\pi\right)^2\right)^{-1},$$
$$\lim_{N\to\infty} \frac{d\,Im\,\lambda}{d\alpha}\Big|_{\alpha=\alpha_k(N)} = \left(1 + \left(\frac{\pi}{2} + k\pi\right)^2\right)^{-1},$$

$k = 0, 2, \dots, 2\ell$.

Figure 2 depicts the root loci of (2.2) for $N = 9$ and $N = 10$.

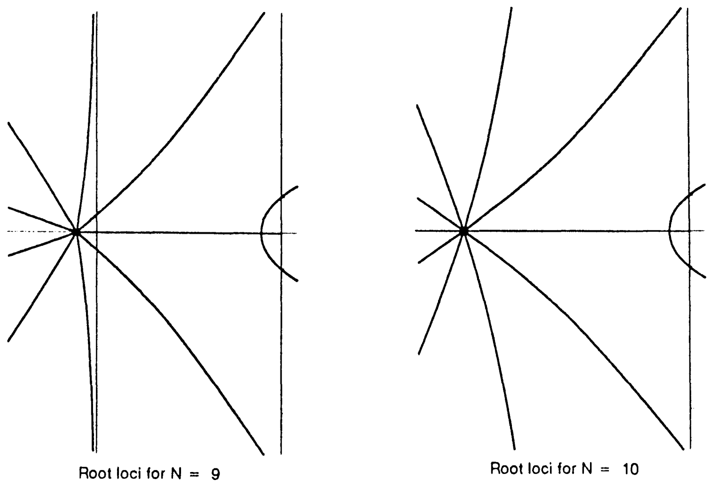

Figure 2

Proposition 2.5 shows that under appropriate smoothness assumptions on f Hopf bifurcations occur for system (1.2), $N = 2, 3, \dots$, at $\alpha = \alpha_k(N)$, $k = 0, 2, \dots, 2\ell$. An application of the Hopf bifurcation theorem as e.g. stated in [4], p. 99, gives the following result:

Theorem 2.6. *Let f be a C^1-function $\mathbf{R} \to \mathbf{R}$ with $f(0) = 0$, $f'(0) = 1$. Then for $N = 2, 3, \dots$ and $k = 0, 2, \dots, 2\ell$ ($N = 4\ell + j + 2$, $0 \le j \le 3$) there exist positive constants $s_{k,N}$, $\epsilon_{k,N}$, $\delta_{k,N}$, $\gamma_{k,N}$, real valued C^1-functions $\alpha_{k,N}(s)$, $\omega_{k,N}(s)$, $0 \le s \le s_{k,N}$, with $\alpha_{k,N}(0) = \alpha_k(N)$, $\omega_{k,N}(0) = 2\pi/\tau_k(N)$ ($\alpha_k(N)$ and $\tau_k(N)$ given in Proposition 2.3) and $\omega_{k,N}(s)$-periodic C^1-functions $w^{k,N}(t; s) = col(w_0^{k,N}(t; s), \dots, w_N^{k,N}(t; s))$ with $|w^{k,N}(t; s)| < \gamma_{k,N}$ for $t \in \mathbf{R}$, $0 \le s < s_{k,N}$, solving (1.2) when $\alpha = \alpha_{k,N}(s)$. Furthermore, as $s \to 0$,*

$$w^{k,N}(t; s) = s\ col(\cos \frac{2\pi}{\omega_{k,N}(s)}t,\ \cos^2\theta_k(N) \cos \frac{2\pi}{\omega_{k,N}(s)}t +$$
$$+ \sin\theta_k(N) \cos\theta_k(N) \sin \frac{2\pi}{\omega_{k,N}(s)}t,\ 0, \dots, 0) + o(s)$$

and for $|\alpha - \alpha_k(N)| < \epsilon_{k,N}$, $|\omega - \frac{2\pi}{\tau_k(N)}| < \delta_{k,N}$ every ω-periodic solution $w(t)$ of (1.2) with $\sup_{\mathbf{R}} |w(t)| < \gamma_{k,N}$ except for translation in phase is given by one of the solutions $w^{k,N}(t; s)$, $0 \le s < s_{k,N}$.

Because of the relatively simple structure of system (1.2) it is possible to compute the coefficients $\alpha_2^{k,N}$, $\omega_2^{k,N}$ and $\beta_2^{k,N}$ in the expansions

$$\begin{aligned}\alpha_{k,N}(s) &= \alpha_k(N) + \alpha_2^{k,N}s^2 + \mathrm{O}(s^3),\\ \omega_{k,N}(s) &= \frac{2\pi}{\tau_k(N)}\big(1 + \omega_2^{k,N}s^2 + \mathrm{O}(s^3)\big),\\ \beta_{k,N}(s) &= \beta_2^{k,N}s^2 + \mathrm{O}(s^3)\end{aligned}$$

for $0 \le s < s_{k,N}$, which are valid if $f \in C^4$ for instance. Here $\beta_{k,N}(s)$ is the nonzero characteristic exponent corresponding to the variational equation for $w^{k,N}(t;s)$ (after reduction to a center manifold), which determines the orbital stability behavior of this periodic solution in the center manifold. The bifurcation formulas given in [6] were used to get the results stated below. We do not reproduce any of the elementary but rather lengthy and tedious computations [11]. Let

$$\frac{1}{1 + 2i(\cos\theta_k(N) + 2i\sin\theta_k(N))^N} = H_1 + iH_2.$$

Then

$$\begin{aligned}\beta_2^{k,N} &= \frac{N^2\sin^2\theta_k(N)[\cos\theta_k(N)]^N}{1 + N(N+2)\sin^2\theta_k(N)} \times \Big[f'''(0)+\\ &\quad + f''(0)^2\Big(\frac{1 + N(N+2)\sin^2\theta_k(N)}{N\sin\theta_k(N)\cos\theta_k(N)}H_2 - 2 - H_1\Big)\Big],\\ \alpha_2^{k,N} &= -\frac{1}{2}N\sin\theta_k(N)[\cos\theta_k(N)]^{N-1} \times \Big[f'''(0)+\\ &\quad + f''(0)^2\Big(\frac{1 + N(N+2)\sin^2\theta_k(N)}{N\sin\theta_k(N)\cos\theta_k(N)}H_2 - 2 - H_1\Big)\Big],\\ \omega_2^{k,N} &= \frac{1}{2}\frac{[\cos\theta_k(N)]^N}{N\sin\theta_k(N)\cos\theta_k(N)}H_2 f''(0)^2.\end{aligned}$$

It is interesting to consider the limits of these expressions as $N \to \infty$ for fixed $k = 0, 2, \cdots$:

$$\beta_2^k = \lim_{N\to\infty}\beta_2^{k,N} = \frac{\theta_k^2}{1+\theta_k^2}f'''(0) - \frac{(11\theta_k - 2)\theta_k}{5(1+\theta_k^2)}f''(0)^2, \tag{2.10}$$

$$\alpha_2^k = \lim_{N\to\infty}\alpha_2^{k,N} = -\frac{1}{2}\theta_k f'''(0) + \frac{11\theta_k - 2}{10}f''(0)^2, \tag{2.11}$$

$$\omega_2^k = \lim_{N\to\infty}\omega_2^{k,N} = \frac{1}{5\theta_k}f''(0)^2, \tag{2.12}$$

where $\theta_k = \pi/2 + k\pi$. Note, that in case $f''(0) = 0$ and $f'''(0) > 0$ the bifurcations are in the direction of decreasing α and the bifurcating periodic solutions of (1.2) are all orbitally unstable, $N = 2, 3, \dots$, $k = 0, 2, \dots, 2\ell$.

It is well known that Wright's equation,

$$\dot{x}(t) = -\alpha x(t-1)(1 + x(t)), \quad t \ge 0,$$

can be transformed to an equation of type (1.1). In fact the transformation $x(t) = e^{y(t)} - 1$, gives

$$\dot{y}(t) = -\alpha\big(e^{y(t-1)} - 1\big), \quad t \geq 0.$$

Since in this case $f^{(j)}(0) = 1$, $j = 1, 2, \ldots$, we obtain from (2.10)–(2.12) in case $k = 0$ the values

$$\beta_2^0 = -\frac{\pi}{1 + \pi^2/4} \cdot \frac{3\pi - 2}{10}, \quad \alpha_2^0 = \frac{3\pi - 2}{10}, \quad \omega_2^0 = \frac{2}{5\pi},$$

which, coincide with those given in [6]. Using $\alpha - \pi/2$ as parameter, which is possible because of $\alpha_2^0 > 0$, we get for instance the well known (see [6]) expansion

$$4\big[1 + \frac{4}{\pi(3\pi - 2)}(\alpha - \frac{\pi}{2}) + \mathrm{O}((\alpha - \frac{\pi}{2})^2)\big], \quad \alpha > \frac{\pi}{2},$$

for the periods of the bifurcating solutions in case $k = 0$, which are orbitally asymptotically stable.

It is interesting to note, that the ordinary differential equation given in [6], pp. 221-223, can be obtained from (1.2) by a change of variables. Indeed, taking

$$\begin{aligned} y_0 &= w_0, \\ y_1 &= w_N, \\ y_j &= N^{j-1} \sum_{k=N-j+1}^{N} (-1)^{k+j-N-1} \binom{j-1}{k-1} w_k, \quad j = 2, \ldots, N, \end{aligned}$$

we obtain

$$\begin{aligned} \dot{y}_0 &= -\alpha f(y_1), \\ \dot{y}_j &= y_{j+1}, \quad j = 1, \ldots, N-1, \\ \dot{y}_N &= N^N y_0 - \sum_{j=1}^{N} N^{N-j+1} \binom{N}{N-j+1} y_j. \end{aligned} \tag{2.13}$$

In [6] there is only a heuristic indication that solutions of (2.13) approximate solutions of (1.1).

3. Global branches of periodic solutions. In this section we consider equation (1.2) under the following assumption on f:

$$\begin{aligned} &f : \mathbf{R} \to \mathbf{R} \ \text{ is a } C^1\text{-function satisfying} \\ &f(0) = 0, \ f'(0) = 1 \ \text{ and} \\ &uf(u) > 0 \ \text{ for } \ u \neq 0. \end{aligned}$$

As before we denote by $w(t) = \mathrm{col}(w_0(t), \ldots, w_N(t))$ the unique solution of (1.2) with initial value $w(0) = \mathrm{col}(w_0(0), \ldots, w_N(0))$. We note that the coordinate $w_j(t)$ of $w(t)$ is $(j+2)$-times continuously differentiable.

Let $\mathcal{K}^N = \{w \in \mathbf{R}^{N+1} \,|\, w_0 > \cdots > w_N = 0\}$. We shall show that for $\alpha > 0$, sufficiently large, solutions of (1.2) define a Poincaré map $\mathcal{K}^N \to \mathcal{K}^N$. In order to accomplish this we need to establish a sequence of lemmas. Because $uf(u) > 0$ for $u \neq 0$, the dual versions of the following lemmas are also valid, where the duality is understood by replacing the ordering of $\mathcal{K}^N$ by that of $-\mathcal{K}^N$.

Lemma 3.1. *Let $j \in \{1, \dots, N\}$ and assume that, for some $T > 0$, $w_{j-1}(t) > w_j(t)$ on $[0, T)$ and $w_j(0) > w_{j+1}(0)$. Then also $w_j(t) > w_{j+1}(t)$ on $[0, T)$.*

Proof: Assume there exists a $t_0 \in (0, T)$ such that $w_j(t_0) = w_{j+1}(t_0)$ and $w_j(t) > w_{j+1}(t)$ on $[0, t_0)$. Using (1.2) we see that $\dot{w}_{j+1}(t_0) = 0$. By assumption $\dot{w}_j(t_0) > 0$. Hence there exists a $\delta \in (0, t_0)$ such that

$$w_j(t_0 - \delta) - w_{j+1}(t_0 - \delta) < 0.$$

However, since $w_j(0) - w_{j+1}(0) > 0$, there exists a $t_1 \in (0,\, t_0 - \delta)$ such that $w_j(t_1) = w_{j+1}(t_1)$ contradicting the choice of t_0. ∎

Lemma 3.2. *If $w_0(0) > \cdots > w_{N-1}(0) > w_N(0) \geq 0$, then there exists a $t_0 > 0$ such that $w_0(t_0) = w_1(t_0)$.*

Proof: Assume the result is false. Then by Lemma 3.1 we conclude that

$$w_0(t) > \cdots > w_N(t) \geq 0, \quad t \geq 0.$$

Hence $w_0(t)$ is decreasing, while $w_j(t)$, $1 \leq j \leq N$, are increasing for $t \geq 0$. Fix $\tau > 0$. Then $0 < w_N(\tau) \leq w_N(t) < w_0(0)$, $t \geq \tau$, and

$$\dot{w}_0(t) = -\alpha f(w_N(t)) \leq -\alpha \min\left\{ f(u) \,|\, w_N(\tau) \leq u \leq w_0(0) \right\} < 0.$$

Therefore $w_0(t)$ and $w_1(t)$ must intersect. ∎

Lemma 3.3. *The following situation cannot occur: There exists a $t_0 > 0$ and a $j \in \{1, \dots, N-1\}$ with*

$$\begin{aligned} &w_{j-1}(t) > w_j(t) > w_{j+1}(t) \quad \text{on} \quad [0, t_0), \\ &w_{j-1}(t_0) = w_j(t_0) = w_{j+1}(t_0), \\ &\dot{w}_{j-1}(t_0) < 0. \end{aligned}$$

Proof: Again we argue indirectly. Using (1.2) we get

$$\dot{w}_j(t_0) = \dot{w}_{j+1}(t_0) = 0$$

and

$$\begin{aligned} \ddot{w}_j(t_0) &= N(\dot{w}_{j-1}(t_0) - \dot{w}_j(t_0)) = \beta < 0, \\ \ddot{w}_{j+1}(t_0) &= N(\dot{w}_j(t_0) - \dot{w}_{j+1}(t_0)) = 0. \end{aligned}$$

By continuity of $\ddot{w}_k(t)$ for $k = 1, 2, \dots$ we may choose $\delta > 0$ such that

$$\ddot{w}_j(t) < \frac{\beta}{2} \quad \text{and} \quad \ddot{w}_{j+1}(t) > \frac{\beta}{2} \quad \text{on} \quad [t_0 - \delta, t_0].$$

By Taylor's formula we get

$$\begin{aligned} w_j(t_0 - \delta) &= w_j(t_0) + \frac{\delta^2}{2} \ddot{w}_j(t_1), \\ w_{j+1}(t_0 - \delta) &= w_{j+1}(t_0) + \frac{\delta^2}{2} \ddot{w}_{j+1}(t_2), \end{aligned}$$

where t_1, $t_2 \in (t_0 - \delta,\, t_0)$. Hence we get

$$\begin{aligned} w_j(t_0 - \delta) - w_{j+1}(t_0 - \delta) &= \frac{\delta^2}{2} (\ddot{w}_j(t_1) - \ddot{w}_{j+1}(t_2)) \\ &< \frac{\delta^2}{2} (\frac{\beta}{2} - \frac{\beta}{2}) = 0, \end{aligned}$$

a contradiction to our assumption. ∎

Lemma 3.4. *Assume that, for some* $j \in \{1, \ldots, N\}$, $w_{j-1}(t)$ *is monotone on* $[T, \infty)$ *and that* $\lim_{t\to\infty} w_j(t) = c$. *Then also* $\lim_{t\to\infty} w_{j-1}(t) = c$.

Proof: Since $\lim_{t\to\infty} w_j(t) = c$ there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ and $\lim_{k\to\infty} \dot{w}_j(t_k) = 0$. Then it follows from (1.2) that

$$\lim_{k\to\infty} w_{j-1}(t_k) = \lim_{k\to\infty} w_j(t_k) + \frac{1}{N} \lim_{k\to\infty} \dot{w}_j(t_k) = c.$$

The result follows by monotonicity of $w_{j-1}(t)$. ∎

Lemma 3.5. *Assume that* $w_0(t_0) < w_1(t_0)$ *and* $w_N(t) > 0$ *on* $[t_0, T)$. *Then* $w_0(t) < w_1(t)$ *on* $[t_0, T)$.

Proof: Suppose there exists a $t_1 \in (t_0, T)$ such that $w_0(t_1) = w_1(t_1)$ and $w_0(t) < w_1(t)$ on $[t_0, t_1)$. Then $\dot{w}_1(t_1) = 0$ and $\dot{w}_0(t_1) = -\alpha f(w_N(t_1)) < 0$. This implies $\dot{w}_0(t_1) - \dot{w}_1(t_1) < 0$, i.e., there exists a $\delta \in (0,\, t_1 - t_0)$ such that $w_0(t_1 - \delta) > w_1(t_1 - \delta)$ which contradicts the choice of t_1. ∎

Lemma 3.6. *If* $w_N(0) > 0$ *and* $w_0(0) < \cdots < w_N(0)$, *then either*

(i) $0 < w_0(t) < \cdots < w_N(t)$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} w_j(t) = 0$, $j = 0, 1, \ldots, N$, *or*

(ii) *There exists a* $t^* > 0$ *such that* $w_0(t) < \cdots < w_N(t)$ *on* $[0, T^*]$ *and* $w_N(t^*) = 0$.

Proof: Assume that $w_0(t) > 0$ for all $t \geq 0$, then Lemma 3.5 implies $w_0(t) < w_1(t)$ as long as $w_N(t) > 0$. But then by the dual version of Lemma 3.1 also $w_0(t) < w_1(t) < \cdots < w_N(t)$ as long as $w_N(t) > 0$. Therefore $w_N(t)$ never can get zero by assumption on $w_0(t)$. This proves the first part of (i). Since all the components $w_j(t)$ are monotone (see system (1.2)) it follows that

$$c_j = \lim_{t\to\infty} w_j(t) \geq 0, \quad j = 0, \ldots, N.$$

By Lemma 3.4 $c_0 = c_1 = \cdots = c_N =: c$. Hence $f(c) = 0$ which implies $c = 0$.

Assume next that there exists a $t_0 > 0$ such that $w_0(t_0) = 0$ and $w_0(t) > 0$ on $[0, t_0)$. Then $w_N(t) > \cdots > w_0(t)$ on $[0, t_0)$ by the dual version of Lemma 3.1. $w_N(t_0) = 0$ would imply $w_0(t_0) = \cdots = w_N(t_0) = 0$ and by uniqueness of solutions $w(t) \equiv 0$. Therefore $w_N(t_0) > 0$ and $\dot{w}_0(t_0) = -\alpha f(w_N(t_0)) < 0$. Hence by shifting the time scale we may, without loss of generality, assume that $w_N(0) > 0 > w_0(0)$. Suppose then that $w_N(t) > 0$ for all $t \geq 0$. Then $w_N(t) > \cdots > w_0(t)$ (cf. Lemmas 3.5 and 3.1) which implies that the components of $w(t)$ are decreasing. Therefore $\lim_{t\to\infty} w_N(t) = c \geq 0$. Applying Lemma 3.4 repeatedly we see that $\lim_{t\to\infty} w_0(t) = c$, contradicting $w_0(t) \leq w_0(0) < 0$ for $t \geq 0$. Hence there exists a $t^* > t_0$ such that $w_N(t^*) = 0$ and $w_N(t) > 0$ on $[0, t^*)$. It follows from Lemmas 3.5 and 3.1 that

$$w_N(t) > \cdots > w_0(t) \quad \text{on} \quad [0, t^*). \tag{3.1}$$

Suppose that $w_N(t^*) = w_{N-1}(t^*) = 0$. Because of $w_0(t^*) < 0$ (this follows from $t^* > t_0$) there exists a $j \in \{2, \ldots, N-1\}$, such that

$$0 = w_N(t^*) = \cdots = w_{N-j}(t^*) > w_{N-j-1}(t^*).$$

Therefore $\dot{w}_{N-j-1}(t^*) = 0$ and $\dot{w}_{N-j}(t^*) < 0$ and there exists a $\delta > 0$ such that $w_{N-j}(t^* - \delta) - w_{N-j+1}(t^* - \delta) > 0$, a contradiction to (3.1). This implies $w_{N-1}(t^*) < 0$ and $\dot{w}_N(t^*) < 0$. Suppose now that $w_0(t^*) = w_1(t^*)$. Then

$$\ddot{w}_0(t^*) = -\alpha f'(0)\dot{w}_N(t^*) > 0.$$

On the other hand $\dot{w}_0(t^*) = -\alpha f(0) = 0$, $\dot{w}_1(t^*) = N(w_0(t^*) - w_1(t^*)) = 0$ and $\ddot{w}_1(t^*) = N(\dot{w}_0(t^*) - \dot{w}_1(t^*)) = 0$. Hence there exists a $\delta > 0$ such that

$$w_1(t^* - \delta) - w_0(t^* - \delta) < 0$$

in contradiction to (3.1). Therefore $w_1(t^*) > w_0(t^*)$ and by repeated application of Lemma 3.1 also

$$0 = w_N(t^*) > \cdots > w_0(t^*),$$

i.e., (ii) holds. ∎

Lemma 3.7. *Assume that* $\alpha > 1$. *If* $w_0(0) < \cdots < w_N(0)$ *and* $w_N(0) > 0$, *then there exists a* $t^* > 0$ *such that*

$$w_0(t^*) < \cdots < w_N(t^*) = 0.$$

Proof: Using Lemma 3.6 we need only to show that there exists a $\tilde{t} > 0$ such that $w_0(\tilde{t}) = 0$ and $w_0(t) > 0$ on $[0, \tilde{t})$. Assuming that $w_0(t) > 0$ on $[0, \infty)$ we get

$$\alpha \int_0^t f(w_N(s))\, ds < w_0(0), \quad t \geq 0. \tag{3.2}$$

For $\epsilon \in (0, 1)$ there exists a $\delta > 0$ such that

$$(1 - \epsilon)u \leq f(u) \leq (1 + \epsilon)u, \quad u \in [0, \delta].$$

By Lemma 3.6, (i), there exists a $t_0 > 0$ such that $0 < w_0(t) < \cdots < w_N(t) \leq \delta$ for $t \geq t_0$. Again by shifting the time axis we may assume the latter inequality for all $t \geq 0$. Hence we get

$$\alpha \int_0^t f(w_N(s))\, ds \geq \alpha(1 - \epsilon) \int_0^t w_N(s)\, ds, \quad t \geq 0.$$

Using (1.2) we obtain

$$\begin{aligned} w_N(t) &= e^{-Nt}\Big(w_N(0) + Ntw_{N-1}(0) + \cdots + \frac{(Nt)^{N-1}}{(N-1)!} w_1(0)\Big) \\ &\quad + N \int_0^t \frac{[N(t-s)]^{N-1}}{(N-1)!} e^{-N(t-s)} w_0(s)\, ds \\ &> e^{-Nt}\Big(1 + Nt + \cdots + \frac{(Nt)^{N-1}}{(N-1)!}\Big) w_0(0), \ \ t \geq 0, \end{aligned}$$

since the $w_j(0)$ are decreasing, $j = 1, \ldots, N$. Therefore

$$\int_0^t w_N(s)\, ds > 1 - e^{-Nt} \sum_{j=0}^{N-1} (1 - \frac{j}{N}) \frac{(Nt)^j}{j!} =: g_N(t).$$

Since $\dot{g}_N(t) = e^{-Nt}\sum_{j=0}^{N-1}\frac{(Nt)^j}{j!}$, we see that $g_N(t)$ is increasing and $\lim_{t\to\infty} g_N(t) = 1$. Thus there exists a unique $\tilde{t} > 0$ such that $g_N(\tilde{t}) = 1 - \epsilon$. For $t = \tilde{t}$ we get

$$\alpha \int_0^{\tilde{t}} f(w_N(s))\, ds > \alpha(1-\epsilon)^2 w_0(0).$$

Now choose ϵ such that $\alpha(1-\epsilon)^2 > 1$. Then

$$\alpha \int_0^{\tilde{t}} f(w_N(s))\, ds > w_0(0),$$

which contradicts (3.2). ∎

Using the above lemmas and their dual versions we obtain

Theorem 3.8. *Let $w(0) \in \mathcal{K}^N$. Then either*

(i) *there exists a $t_0 > 0$ such that either $w_N(t) > \cdots > w_0(t) > 0$ for all $t \geq t_0$ or $w_N(t) < \cdots < w_0(t) < 0$ for all $t \geq t_0$ and $\lim_{t\to\infty} w_j(t) = 0$, $j = 0, \ldots, N$, or*

(ii) *the set $P(w(0)) = \{t > 0 \mid w(t) \in \mathcal{K}^N\}$ is nonempty and discrete.*

If $\alpha > 1$, then only the alternative (ii) is possible and, moreover, $P(w(0))$ is unbounded.

Proof: We first show that there exist numbers $0 < t_1 < \cdots < t_N$ such that

$$w_j(t_j) = w_{j-1}(t_j) \quad \text{and} \quad w_{j-1}(t) > w_j(t) \ \text{ on } \ [0, t_j), \ j = 1, \ldots, N, \tag{3.3}$$

$$w_j(t) > w_{j+1}(t) > \cdots > w_N(t) > 0 \quad \text{on } (0, t_j], \ j = 1, \ldots, N, \text{ and} \tag{3.4}$$

$$w_{j-1}(t) < w_j(t) \ \text{ on } \ (t_j, t_N], \quad j = 1, \ldots, N-1. \tag{3.5}$$

Furthermore, for any sufficiently small $\delta > 0$

$$w_0(t_N + \delta) < \cdots < w_N(t_N + \delta) \ \text{ and } \ w_N(t_N + \delta) > 0. \tag{3.6}$$

By Lemma 3.2 there exists a $t_1 > 0$ such that $w_0(t_1) = w_1(t_1)$ and $w_0(t) > w_1(t)$ on $[0, t_1)$, which is (3.3) for $j = 1$. From Lemma 3.1 (successively applied for $j = 1, \ldots, N-1$) we get

$$w_1(t) > w_2(t) > \cdots > w_N(t) \quad \text{on} \quad [0, t_1).$$

Since $\dot{w}_N(t) > 0$ on $[0, t_1)$, we see that $w_N(t_1) > 0$ and hence $\dot{w}_0(t_1) < 0$. Thus Lemma 3.3 shows that $w_1(t_1) > w_2(t_1)$ and therefore by Lemma 3.1 $w_1(t) > \cdots > w_N(t) > 0$, $t \in (0, t_1]$, which is (3.4) for $j = 1$. Since $\dot{w}_0(t_1) - \dot{w}_1(t_1) < 0$, we see that $w_0(t) < w_1(t)$ for $t \in (t_1, t_1 + \delta)$, $\delta > 0$ sufficiently small. Then Lemma 3.5 implies that $w_0(t) < w_1(t)$ as long as $w_N(t) > 0$.

Suppose now that we have proved the existence of numbers $0 < t_1 < \cdots < t_k$, $k < N$, with (3.3) and (3.4) holding for $j = 1, \ldots, k$ and

$$w_{j-1}(t) < w_j(t) \ \text{ for } \ t > t_j \ \text{ as long as } \ w_N(t) > 0, \ j = 1, \ldots, k. \tag{3.7}$$

We note that (3.4) for $j = k$ implies $w_N(t) > 0$ on $(0, t_k]$. By assumption we have $w_{k-1}(t) > w_k(t) > w_{k+1}(t)$ for $t \in [0, t_k)$, $w_k(t_k) = w_{k-1}(t_k)$ and $\dot{w}_{k-1}(t_k) = N(w_{k-2}(t_k) - w_{k-1}(t_k)) < 0$. Then Lemma 3.3 shows $w_k(t_k) > w_{k+1}(t_k)$. If now $w_k(t) > w_{k+1}(t)$ for all $t \geq 0$, then by Lemma 3.1 it is true that $w_k(t) > \cdots > w_k(t) > 0$ for all $t > 0$. Using (1.2) we see that all functions $w_{k+1}(t), \ldots, w_N(t)$ are strictly increasing for $t \geq 0$. Using (3.7) for $t = t_k + \delta$, $\delta > 0$ sufficiently small, we get by Lemma 3.5 and the dual version of Lemma 3.1 that

$$w_0(t) < \cdots < w_k(t), \quad t > t_k.$$

This shows that the function $w_1(t), \ldots, w_k(t)$ are strictly decreasing for $t > t_k$. The same is true for $w_0(t)$, because $\dot{w}_0(t) = -\alpha f(w_N(t)) < 0$ on $t > 0$. Using Lemma 3.4 and the monotonicity of the components $w_j(t)$ we infer that

$$\lim_{t \to \infty} w_0(t) = \cdots = \lim_{t \to \infty} w_N(t) = c > 0.$$

Since $w_N(t)$ is strictly increasing for $t \geq 0$ we find that there exists a t^* such that

$$c \geq w_N(t) \geq \frac{c}{2}, \quad t \geq t^*,$$

and therefore

$$\dot{w}_0(t) \leq -\alpha \min_{c/2 \leq u \leq c} f(u), \quad t \geq 0,$$

i.e., $\lim_{t \to \infty} w_0(t) = -\infty$, a contradiction. Hence there exists a $t_{k+1} > t_k$ such that $w_{k+1}(t_{k+1}) = w_k(t_{k+1})$ and $w_{k+1}(t) < w_k(t)$ on $[0, t_{k+1})$. Thus (3.3) holds for $j = k+1$. Now using Lemma 3.1 we get that $w_{k+1}(t) > \cdots > w_N(t) > 0$ on $[0, t_{k+1})$ and by Lemma 3.3 $w_{k+1}(t_{k+1}) > w_{k+2}(t_{k+1})$ which by Lemma 3.1 implies $w_{k+1}(t) > \cdots > w_N(t)$ on $[0, t_{k+1}]$. This establishes (3.4) for $j = k+1$. If $k+1 < N$, we must show that $w_k(t) < w_{k+1}(t)$ for $t > t_{k+1}$ as long as $w_N(t) > 0$.

Inequality (3.7) for $j = k$ implies that

$$\dot{w}_k(t_{k+1}) - \dot{w}_{k+1}(t_{k+1}) = \dot{w}_k(t_{k+1}) < 0,$$

i.e., $w_k(t) < w_{k+1}(t)$ for $t \in (t_{k+1}, t_{k+1} + \epsilon)$, ϵ some positive constant. From (3.7) for $j = 1$ we get $w_0(t_{k+1}) < w_1(t_{k+1})$. Then by Lemmas 3.5 and 3.1 we obtain

$$w_k(t) < w_{k+1}(t) \quad \text{for} \quad t > t_{k+1} \quad \text{as long as} \quad w_N(t) > 0,$$

which is (3.7) for $j = k+1$. If $k+1 = N$, the same arguments prove (3.6). Using (3.6) and Lemma 3.6 we get either the first part of the theorem or the existence of a $t^* > t_N$ such that

$$w_0(t^*) < \cdots < w_N(t^*) = 0.$$

But then applying the dual arguments which led to (3.6) we get a $t^{**} > t^*$ such that

$$w_0(t^{**}) > \cdots > w_N(t^{**}) \quad \text{and} \quad w_N(t^{**}) < 0.$$

Using the dual version of Lemma 3.6 we now get either (i) of the theorem or a $\tilde{t} > t^{**}$ such that

$$w_0(\tilde{t}) > \cdots > w_N(\tilde{t}) = 0,$$

which shows that $P(w(0))$ is not empty. Since $\dot{w}_N(t) > 0$ for any $t \in P(w(0))$ this set is discrete.

If $\alpha > 1$, then Lemma 3.7 shows that conclusion (i) of Lemma 3.6 cannot hold, hence only part (ii) of the theorem is possible. For $\alpha > 1$ the set $P(w(0))$ is clearly infinite. Assume that $P(w(0))$ is bounded. Then there exists a $\tilde{t} > 0$ and a sequence $t_n \in P(w(0))$ such that $t_n \to \tilde{t}$. Obviously $\lim_{n\to\infty} \dot{w}_N(t_n) = 0$ and $w_N(t_n) = 0$ for all n. Using (1.2) we see that also $\lim_{n\to\infty} w_{N-1}(t_n) = 0$. But then $\lim_{n\to\infty} \dot{w}_{n-1}(t_n) = 0$. In a finite number of steps we arrive at $\lim_{n\to\infty} w_j(t_n) = 0$, $j = 0, \dots, N$, i.e., $w(t) \equiv 0$, which is a contradiction. This proves that $P(w(0))$ is unbounded. ∎

In case $\alpha > 1$ we have established the existence of a mapping $\mathcal{T}^N : \mathcal{K}^N \to \mathcal{K}^N$ defined by

$$\mathcal{T}^N w(0) = w(t^*), \quad w(0) \in \mathcal{K}^N, \tag{3.8}$$

where $t^* = \min P(w(0))$. Since solutions depend continuously on initial data, it follows that $\mathcal{T}^N$ is continuous on $\mathcal{K}^N$. We next wish to extend $\mathcal{T}^N$ to the closure $\bar{\mathcal{K}}^N$ of $\mathcal{K}^N$. To accomplish this we shall use the following lemma.

Lemma 3.9. *Let $w(0) \in \bar{\mathcal{K}}^N \backslash \{0\}$. Then there exists an index $j_0 \in \{0, \dots, N-1\}$ such that*

$$0 < w_0(t) < \cdots < w_{j_0}(t) \tag{3.9}$$

and

$$w_{j_0}(t) > \cdots > w_N(t) > 0 \tag{3.10}$$

for $t > 0$ sufficiently small.

Proof: If $w(0) \in \mathcal{K}^N$, then $j_0 = 0$ and the result is obvious. Let $w(0) \in \bar{\mathcal{K}}^N \backslash \mathcal{K}^N$ and $w(0) \neq 0$. Then there exist two indices $j_1 \in \{1, \dots, N\}$ and $j_0 \in \{0, \dots, N-1\}$ with $j_0 < j_1$ such that

$$w_0(0) = \cdots = w_{j_0}(0) > w_{j_0+1}(0) \geq \cdots \geq w_{j_1-1}(0) > w_{j_1}(0) = \cdots = w_N(0) = 0.$$

From (1.2) we see that

$$w_j(t) = e^{-Nt} \sum_{k=1}^{j} \frac{(Nt)^{j-k}}{(j-k)!} w_k(0) + N \int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!} e^{-N(t-s)} w_0(s)\, ds, \quad t \geq 0, \tag{3.11}$$

for $j = 1, \dots, N$ and therefore

$$\begin{aligned} w_j(t) - w_{j+1}(t) = {} & e^{-Nt} \sum_{k=1}^{j} \frac{(Nt)^{j-k}}{(j-k)!} \big(w_k(0) - w_{k+1}(0)\big) - e^{-Nt} \frac{(Nt)^j}{j!} w_1(0) \\ & + N \int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!} \Big(1 - \frac{N(t-s)}{j}\Big) e^{-N(t-s)} w_0(s)\, ds, \quad t \geq 0, \end{aligned} \tag{3.12}$$

for $j = 1, \dots, N-1$. We choose $\delta > 0$ such that $w_0(s) \geq (1/2) w_0(0)$ for $s \in [0, \delta]$. Then

$$N \int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!} \Big(1 - \frac{N(t-s)}{j}\Big) e^{-N(t-s)} w_0(s)\, ds \geq \frac{1}{2} w_0(0) \frac{(Nt)^j}{j!} e^{-Nt}$$

and therefore

$$w_j(t) - w_{j+1}(t) \geq e^{-Nt} \sum_{k=1}^{j} \frac{(Nt)^{j-k}}{(j-k)!}\big(w_k(0) - w_{k+1}(0)\big) + \big(\frac{1}{2}w_0(0) - w_1(0)\big)\frac{(Nt)^j}{j!}e^{-Nt}, \quad t \in [0,\delta],$$

$j = 1, \ldots, N-1$. For $j \geq \max(1, j_0)$ this implies

$$w_j(t) > w_{j+1}(t)$$

for $t > 0$ sufficiently small. From (3.11) for $j = N$ we see that $w_N(t) > 0$ for $t > 0$ sufficiently small. If $j_0 = 0$, i.e., $w_0(0) > w_1(0)$, then obviously $w_0(t) > w_1(t)$ also for $t > 0$ sufficiently small. Thus (3.10) is established. In case $j_0 \geq 1$ we have to prove (3.9). From

$$\begin{aligned} w_j(t) &= w_0(0)e^{-Nt}\sum_{k=0}^{j-1}\frac{(Nt)^k}{k!} + N\int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!}e^{-N(t-s)}w_0(s)\,ds \\ &= w_0(0) + N\int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!}e^{-N(t-s)}(w_0(s) - w_0(0))\,ds \end{aligned} \tag{3.13}$$

for $t \geq 0$ and $j = 1, \ldots, j_0$ we obtain

$$w_j(t) - w_{j+1}(t) = N\int_0^t \frac{(N(t-s))^{j-1}}{(j-1)!}\big(1 - \frac{N(t-s)}{j}\big)e^{-N(t-s)}(w_0(s) - w_0(0))\,ds$$

for $t \geq 0$ and $j = 1, \ldots, j_0 - 1$. From (1.2) and $w^N(t) > 0$ for $t > 0$ sufficiently small we see that $w_0(s)$ is strictly decreasing for small t. Therefore $w_j(t) < w_{j+1}(t)$ for $j = 1, \ldots, j_0 - 1$ and $t > 0$ sufficiently small. From (3.13) for $j = 1$ we get using the mean value theorem for integrals

$$w_1(t) - w_0(0) = N\int_0^t e^{-N(t-s)}(w_0(s) - w_0(0))\,ds = (w_0(\theta t) - w_0(0))(1 - e^{-Nt})$$

for $t > 0$ sufficiently small, $0 < \theta < 1$. By monotonicity of $w_0(t)$ this implies

$$w_1(t) = w_0(\theta t) + (w_0(0) - w_0(\theta t))e^{-Nt} > w_0(t)$$

for $t > 0$ sufficiently small. Hence (3.9) is established. ∎

On the basis of Lemma 3.9 we see that Theorem 3.8 is also valid for $w(0) \in \bar{\mathcal{K}}^N \backslash \{0\}$. The proof is quite analogous to that part of the proof of Theorem 3.8 which assumes existence of numbers $0 < t_1 < \cdots < t_k$, $k < N$, such that (3.3), (3.4) and (3.7) are satisfied. Because of Lemma 3.9 we can take $k = j_0$.

Hence the definition of $\mathcal{T}^N$ given in (3.8) is also valid for $w(0) \in \bar{\mathcal{K}}^N \backslash \{0\}$. Of course, we put $\mathcal{T}^N 0 = 0$. Thus $\mathcal{T}^N$ is a mapping $\bar{\mathcal{K}}^N \to \bar{\mathcal{K}}^N$ which obviously is continuous on $\bar{\mathcal{K}}^N \backslash \{0\}$ with $\mathcal{T}\big(\bar{\mathcal{K}}^N \backslash \{0\}\big) \subset \mathcal{K}^N$.

The following estimates will be fundamental for our further investigations.

Lemma 3.10.

a) *For all* $N = 1, 2, \ldots$ *and all* $p = 2, 3, \ldots$

$$e^{-Np} \sum_{k=0}^{N-1} \frac{(Np)^k}{k!} \leq \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{(p-1)\sqrt{N}} \prod_{k=2}^{p-1} (\frac{k}{p})^N,$$

where as usual $\prod_{k=2}^{1} = 1$.

b) *For all* $\rho \geq 0$ *and all* $N = 1, 2, \ldots$

$$\rho e^{-Nt} \sum_{k=0}^{N-1} \frac{(Nt)^k}{k!} \leq \frac{1}{2\sqrt{\pi N}} \qquad \text{for } t \geq 2 + \rho.$$

Proof: a) According to Sterling's formula (see for instance [1], p. 257)

$$e^{-n} \leq \frac{1}{\sqrt{2\pi n}} \cdot \frac{n!}{n^n}, \qquad n = 1, 2, \ldots.$$

Therefore

$$e^{-Np} \sum_{k=0}^{N-1} \frac{(Np)^k}{k!} \leq \frac{1}{2\sqrt{\pi N}} \sum_{k=0}^{N-1} \frac{(k+1)(k+2)\ldots Np}{(Np)^{Np-k}}$$

for $N = 1, 2, \ldots$ and $p = 2, 3, \ldots$. If we observe that

$$\frac{(k+1)(k+2)\ldots Np}{(Np)^{Np-k}} \leq (\frac{1}{p})^{N-k} (\frac{2}{p})^N \ldots (\frac{p-1}{p})^N,$$

then the result follows immediately.

b) Let p be an integer such that $p \leq 2 + \rho < p + 1$. Since $p \geq 2$ we get from a) that

$$e^{-Np} \sum_{k=0}^{N-1} \frac{(Np)^k}{k!} \leq \frac{1}{2\sqrt{\pi N}(p-1)}.$$

Observing $\rho < p - 1$ and that the function $e^{-Nt} \sum_{k=0}^{N-1} \left((Nt)^k\right)/(k!)$ is decreasing on $t \geq 0$ we get

$$\rho e^{-Nt} \sum_{k=0}^{N-1} \frac{(Nt)^k}{k!} \leq \frac{1}{2\sqrt{\pi N}}. \qquad \text{for all } t \geq 2 + \rho. \quad \blacksquare$$

In order to state the next lemma we introduce some notation. Assume that $\alpha > 1$, which makes $\mathcal{T}^N$ well-defined for all N. For $w(0) \in \bar{\mathcal{K}}^N \backslash \{0\}$ and $w(0) \in -\bar{\mathcal{K}}^N \backslash \{0\}$ we put $t^* = t^*(w(0)) = \min\{t > 0 \,|\, w(t) \in -\mathcal{K}^N\}$ and $t^{**} = t^{**}(w(0)) = \min\{t > 0 \,|\, w(t) \in \mathcal{K}^N\}$, respectively. Furthermore, for any $\sigma > 0$ we define

$$\beta^+(\sigma) = \max_{0 \leq u \leq \sigma} f(u), \quad \beta^-(\sigma) = -\min_{-\sigma \leq u \leq 0} f(u),$$

and

$$\mathcal{K}_\sigma^N = \{w \in \mathcal{K}^N \mid |w| < \sigma\}, \quad \bar{\mathcal{K}}_\sigma^N = \overline{\mathcal{K}_\sigma^N}.$$

Lemma 3.11. *Let $\alpha > 1$. Then for all $N = 1, 2, \ldots$*

$$|w(t^*)| = -w_0(t^*) < 3\alpha\beta^+(\sigma)$$

provided $w(0) \in \bar{\mathcal{K}}_\sigma^N \backslash \{0\}$ and

$$|w(t^{**})| = w_0(t^{**}) < 3\alpha\beta^-(\sigma)$$

provided $w(0) \in -\bar{\mathcal{K}}_\sigma^N \backslash \{0\}$. Furthermore,

$$|w(t)| \leq 3\alpha\beta^-(3\alpha\beta^+(\sigma)) \quad \text{for} \quad 0 \leq t \leq t^*(w(0)) + t^{**}(w(t^*)) \tag{3.14}$$

and

$$|\mathcal{T}^N w(0)| \leq 3\alpha\beta^-\big(3\alpha\beta^+(|w(0)|)\big) \tag{3.15}$$

provided $w(0) \in \bar{\mathcal{K}}^N \backslash \{0\}$.

Proof: By duality it is sufficient to prove the result for $w(0) \in \bar{\mathcal{K}}_\sigma^N \backslash \{0\}$. The following characterization of t^* follows from the results leading to the definition of $\mathcal{T}^N$:

$$w_N(t^*) = 0 \quad \text{and} \quad 0 < w_N(t) < \sigma \quad \text{for} \quad t \in (0, t^*). \tag{3.16}$$

Therefore

$$\dot{w}(t) \geq -\alpha\beta^+(\sigma) \quad \text{for} \quad t \in (0, t^*). \tag{3.17}$$

Assume that

$$w_0(t^*) \leq -3\alpha\beta^+(\sigma).$$

Since $w_0(t)$ is strictly decreasing on $(0, t^*)$, there exist unique $t_1 \in (0, t^*)$ and $t_2 \in (t_1, t^*)$ such that $w_0(t_1) = 0$ and

$$w_0(t) \leq -\alpha\beta^+(\sigma) \quad \text{for} \quad t \in [t_2, t^*]. \tag{3.18}$$

From (3.17) we get

$$t_1 > \frac{w_0(0)}{\alpha\beta^+(\sigma)}, \quad t^* - t_1 \geq 3 \quad \text{and} \quad t^* - t_2 \geq 2. \tag{3.19}$$

Monotonicity of $w_0(t)$ and (3.17) imply

$$w_0(t) \leq w_0(0) \quad \text{for} \quad t \in [0,\, t_1 - \frac{w_0(0)}{\alpha\beta^+(\sigma)}] \tag{3.20}$$

and

$$w_0(t) \leq -\alpha\beta^+(\sigma)(t - t_1) \quad \text{for} \quad t \in [t_1 - \frac{w_0(0)}{\alpha\beta^+(\sigma)}\,, t_1]. \tag{3.21}$$

Let $\bar{t} = t_1 - (w_0(0))/(\alpha\beta^+(\sigma))$. Using (3.11) for $j = N$ and (3.18), (3.20), (3.21) we see that

$$\begin{aligned}
w_N(t^*) &= e^{-Nt}\sum_{k=1}^{N}\frac{(Nt^*)^{N-k}}{(N-k)!}w_k(0) + N\int_0^{t^*}\frac{(N(t^*-s))^{N-1}}{(N-1)!}e^{-N(t^*-s)}w_0(s)\,ds \\
&< w_0(0)e^{-Nt^*}\sum_{k=0}^{N-1}\frac{(Nt^*)^k}{k!} + w_0(0)N\int_0^{\bar{t}}\frac{(N(t^*-s))^{N-1}}{(N-1)!}e^{-N(t^*-s)}\,ds \\
&\quad - \alpha\beta^+(\sigma)N\int_{\bar{t}}^{t_1}\frac{(N(t^*-s))^{N-1}}{(N-1)!}e^{-N(t^*-s)}(s-t_1)\,ds \\
&\quad - \alpha\beta^+(\sigma)N\int_{t_2}^{t^*}\frac{(N(t^*-s))^{N-1}}{(N-1)!}e^{-N(t^*-s)}\,ds \\
&=: a_1 + \cdots + a_4.
\end{aligned}$$

Repeated integration by parts gives

$$\begin{aligned}
a_2 &= w_0(0)e^{-N(t^*-\bar{t})}\sum_{k=0}^{N-1}\frac{(N(t^*-\bar{t}))^k}{k!} - w_0(0)e^{-Nt^*}\sum_{k=0}^{N-1}\frac{(Nt^*)^k}{k!}, \\
a_3 &= -w_0(0)e^{-N(t^*-\bar{t})}\sum_{k=0}^{N-1}\frac{(N(t^*-\bar{t}))^k}{k!} + \frac{1}{N}\alpha\beta^+(\sigma)e^{-N(t^*-t_1)}\sum_{j=0}^{N-1}\sum_{k=0}^{j}\frac{(N(t^*-t_1))^k}{k!} \\
&\quad - \frac{1}{N}\alpha\beta^+(\sigma)e^{-N(t^*-\bar{t})}\sum_{j=0}^{N-1}\sum_{k=0}^{j}\frac{(N(t^*-\bar{t}))^k}{k!}, \\
a_4 &= -\alpha\beta^+(\sigma)\Big(1 - e^{-N(t^*-t_2)}\sum_{k=0}^{N-1}\frac{(N(t^*-t_2))^k}{k!}\Big).
\end{aligned}$$

Thus we get

$$w_N(t^*) < -\alpha\beta^+(\sigma) + 2\alpha\beta^+(\sigma)e^{-N(t^*-t_2)}\sum_{k=0}^{N-1}\frac{(N(t^*-t_2))^k}{k!} + w_0(0)e^{-Nt^*}\sum_{k=0}^{N-1}\frac{(Nt^*)^k}{k!}.$$

Observing (3.19) we obtain from Lemma 3.10 that

$$w_N(t^*) < \alpha\beta^+(\sigma)(-1 + \frac{3}{2\sqrt{\pi N}}) < 0,$$

for all N, a contradiction to (3.16). The estimates (3.14) and (3.15) follow immediately using the monotonicity properties of $w_0^N(t)$ and $\mathcal{T}^N w(0) = w(t^{**}(w(t^*)))$. ∎

The following two corollaries are immediate consequences of Lemma 3.11. Note that $\lim_{\sigma\downarrow 0}\beta^+(\sigma) = \lim_{\sigma\downarrow 0}\beta^-(\sigma) = 0$.

Corollary 3.12. *Assume $\alpha > 1$. Then for any $N = 1, 2, \dots$ the mapping $\mathcal{T}^N$ is also continuous at $w(0) = 0$, i.e., $\mathcal{T}^N$ is continuous $\bar{\mathcal{K}}^N \to \bar{\mathcal{K}}^N$.*

Corollary 3.13. *Let $\alpha > 1$.*

a) *Assume that there exists a constant $\beta_0 > 0$ such that*

$$f(u) \geq -\beta_0 \quad \text{for} \quad u \in \mathbf{R}. \tag{3.22}$$

Then for all $N = 1, 2, \dots$ and all $w(0) \in \bar{\mathcal{K}}^N$

$$|\mathcal{T}^N w(0)| < 3\alpha\beta_0.$$

b) *Assume that for a constant $\beta_1 > 0$*

$$f(u) \leq \beta_1 \quad \text{for all} \quad u \in \mathbf{R}. \tag{3.23}$$

Then for all $N = 1, 2, \dots$ and all $w(0) \in \bar{\mathcal{K}}^N$

$$|\mathcal{T}^N w(0)| \leq 3\alpha\beta^-(3\alpha\beta_1).$$

We want to show that for any $N = 2, 3, \dots$ the operator $\mathcal{T}^N$ has a fixed point in $\bar{\mathcal{K}}^N \setminus \{0\}$, i.e., system (1.2) has a nontrivial periodic solution. In order to avoid the fixed point 0 the concept of ejectivity will be important. In the terminology of [13] (see also [5]) $w = 0$ is called an ejective fixed point of $\mathcal{T}^N$ if there exists an $r > 0$ such that for any $w(0) \in \bar{\mathcal{K}}_r^N \setminus \{0\}$ there exists a positive integer $m = m(w(0))$ with $|(\mathcal{T}^N)^m w(0)| > r$.

Lemma 3.14. *For any $N = 2, 3, \dots$ and any $\alpha > \alpha_0(N) = \big(N \tan(\pi/2N)\big)/[\cos(\pi/2N)]^N$, $w = 0$ is an ejective fixed point of $\mathcal{T}^N$.*

Proof: Suppose 0 is not ejective. Then there exist a sequence (r_n), $r_n > 0$, and a sequence $(w^n(0)) \subset \bar{\mathcal{K}}^N \setminus \{0\}$ such that $r_n \to 0$, $|w^n(0)| \leq r_n$ and $|(\mathcal{T}^N)^k w^n(0)| \leq r_n$ for all $k = 1, 2, \dots$. Let $w^n(t)$ denote the solution of (1.2) with initial value $w^n(0)$. Then Lemma 3.11 implies that $|w^n(t)| \leq \max(3\alpha\beta^+(r_n), r_n)$ for all $t \geq 0$, i.e., $\lim_{n\to\infty} w^n(t) = 0$ uniformly for $t \geq 0$.

Let $\eta > 0$ and choose $\delta > 0$ such that $|f(u)| \leq (1+\eta)|u|$ for $|u| \leq \delta$. Without restriction of generality we can assume that

$$\gamma_n = \sup_{t \geq 0} |w^n(t)| \leq \delta \quad \text{for all} \quad n.$$

The local maxima of $|w^n(t)|$ occur at the points where $w^n(t) \in \mathcal{K}^N$ or $w^n(t) \in -\mathcal{K}^N$. Therefore there exists a point t_1 with $w^n(t_1) \in -\mathcal{K}^N$ and $|w^n(t_1)| > (\gamma_n)/2$ or a point t_2 with $w^n(t_2) \in -\mathcal{K}^N$ and $|w^n(t_2)| > (\gamma_n)/2$. In the second case let $t_n^* = \max\{t < t_2 \mid w^n(t) \in \mathcal{K}^N\}$. Then according to Lemma 3.11, $|w^n(t_n^*)| > 1/\big(3\alpha(1+\eta)\big)(\gamma_n)/2$. Thus in both cases we find a $t_n^* \geq 0$ such that $|w^n(t_n^*)| > \gamma_n/\big(6\alpha(1+\eta)\big)$. Shifting the time axis we can assume that $|w^n(0)| > \gamma_n/\big(6\alpha(1+\eta)\big)$ for all n. We put

$$u^n(t) = \frac{1}{\gamma_n} w^n(t), \quad t \geq 0.$$

Then $(u^n(t))$ is a sequence of continuous functions, which is uniformly bounded on $[0,\infty)$. From $\dot{u}^n(t) = \gamma_n^{-1}[B^N w^n(t) + \mathrm{col}(f(w_N^n(t)), 0, \dots, 0)]$ we see that also $(\dot{u}^n(t))$ is uniformly bounded on $[0,\infty)$. By Arzela-Ascoli's theorem there exists a subsequence, which we again denote by $(u^n(t))$, and a continuous function $u(t)$ on $[0,\infty)$ such that

$$u(t) = \lim_{n\to\infty} u^n(t)$$

uniformly on compact subintervals of $[0,\infty)$. Because of $\sup_{t\geq 0}|u^n(t)| = 1$ for all n we also have $\sup_{t\geq 0}|u(t)| = 1$. Moreover, $|u(0)| \geq [6\alpha(1+\eta)]^{-1}$. From

$$w^n(t) = w^n(0) + \int_0^t \left[B^N w^n(\tau) + \mathrm{col}(f(w_N^n(\tau)), 0, \dots, 0)\right] d\tau, \quad t \geq 0,$$

and differentiability of f at zero we see that $u(t)$ is a nontrivial solution of the linearized system (2.1) with $w(0) \in \bar{K}^N\backslash\{0\}$. The rest of the proof will show that for $\alpha > \alpha_0(N)$, $N = 2, 3, \dots$, any solution of (2.1) with initial value in $\bar{K}^N\backslash\{0\}$ is unbounded on $[0,\infty)$, thus leading to a contradiction with $\sup_{t\geq 0}|u(t)| = 1$.

For $\alpha > \alpha_0(N)$ there is a pair of conjugate complex eigenvalues λ_0, $\bar{\lambda}_0$ of A^N in the right half plane (see Proposition 2.3 and the discussion of the case $\sigma = \mathrm{Re}\,\lambda > 0$ in §2). λ_0 is given by

$$\lambda_0 = N(-1 + r_0 e^{i\theta_0}), \quad \theta_0 \in \left(\frac{\pi}{2N}, \frac{\pi}{N}\right).$$

Let $x \in \bar{K}^N\backslash\{0\}$. Without restriction of generality we assume that $x = \mathrm{col}(1, x_1, \dots, x_N)$ with $1 \geq x_1 \geq \dots \geq x_N = 0$. It is sufficient to show that x has a nonzero component in the eigenspace of A^N corresponding to λ_0. Note that for $\alpha > \alpha_0(N)$ all eigenvalues of A^N are simple. Let c be an eigenvector of $(A^N)^* = (\overline{A^N})^T = (A^N)^T$ corresponding to the eigenvalue $\bar{\lambda}_0$ and b be an eigenvector of A^N corresponding to λ_0. Then $x = \gamma b + \sum_{j=1}^N \gamma_j b_j$, where the b_j's are the eigenvectors of A^N corresponding to eigenvalues different from λ_0. Taking inner products and observing $c^*b \neq 0$, $c^*b_j = 0$, $j = 1, \dots, N$, we have

$$\gamma = \frac{c^*x}{c^*b}.$$

An easy computation shows that an eigenvector of $(A^N)^T$ corresponding to $\bar{\lambda}_0$ is given by

$$c = \mathrm{col}\Big(1, \frac{\bar{\lambda}_0}{N}, \frac{\bar{\lambda}_0}{N}(1 + \frac{\bar{\lambda}_0}{N}), \dots, \frac{\bar{\lambda}_0}{N}(1 + \frac{\bar{\lambda}_0}{N})^{N-1}\Big).$$

Then (observe $\lambda_0(1+\lambda_0/N)^N + \alpha = 0$)

$$\begin{aligned} c^*x &= 1 + \frac{\lambda_0}{N}\sum_{k=1}^{N-1}(1 + \frac{\lambda_0}{N})^{k-1}x_k = 1 - \frac{\alpha}{N}\sum_{k=1}^{N-1}(1 + \frac{\lambda_0}{N})^{k-1-N}x_k \\ &= 1 - \frac{\alpha}{N}\sum_{k=1}^{N-1} r_0^{k-1-N} e^{i(k-1-N)\theta_0} x_k. \end{aligned}$$

From Proposition 2.2 we get, $\sigma_0 = \operatorname{Re}\lambda_0 > 0$,

$$\frac{\alpha}{r_0^N} = -\frac{\sigma_0}{\cos N\theta_0},$$

which implies

$$c^*x = 1 + \frac{\sigma_0}{\cos N\theta} \sum_{k=1}^{N-1} r_0^{k-1} e^{i(k-1-N)\theta_0} x_k.$$

If $x_1 = \cdots = x_N = 0$, Then $c^*x = 1$. Otherwise there exists a k_0 such that $0 = x_N = \cdots = x_{k_0+1} < x_{k_0} \leq \cdots \leq x_1$ and

$$\operatorname{Im} c^*x = -\frac{\sigma_0}{\cos N\theta_0} \sum_{k=1}^{k_0} x_k r_0^{k-1} \sin(N-k+1)\theta_0 > 0.$$

Note that $\cos N\theta_0 > 0$ and $(N-k+1)\theta_0 \in (0,\pi)$ for $k = 1,\ldots,N$. Thus we have shown that $\gamma \neq 0$ which implies that the solution $u(t)$ of (2.1) with $u(0) = x$ has the unbounded component $2\operatorname{Re}(\gamma b e^{\lambda_0 t})$, $t \geq 0$. ∎

Clearly, 0 is an extreme point of $\bar{\mathcal{K}}_R^N$, $R > 0$ (i.e., $0 = \delta a_1 + (1-\delta)a_2$ with $a_1, a_2 \in \bar{\mathcal{K}}_R^N$ and $0 < \delta < 1$ implies $a_1 = a_2 = 0$). This together with Corollaries 3.12, 3.13 and Lemma 3.14 shows that all assumptions of Theorem 2.1 in [5], §11, can be satisfied for $\bar{\mathcal{K}}_R^N$. If we in addition observe that $\mathcal{T}^N(\bar{\mathcal{K}}^N \backslash \{0\}) \subset \mathcal{K}^N$, we get

Theorem 3.15. *Let f satisfy the assumptions stated at the beginning of this section and assume that (3.22) or (3.23) hold. Then for any $N = 2,3,\ldots$ and $\alpha > \alpha_0(N)$ there exists a fixed point of $\mathcal{T}^N$ in $\mathcal{K}_R^N$, where $R = 2\alpha\beta_0$ if (3.22) is satisfied and $R = 3\alpha\beta^-(3\alpha\beta_1)$ if (3.23) is satisfied. Consequently, system (1.2) has a nontrivial periodic solution with initial value in $\mathcal{K}_R^N$.*

4. Approximation of nontrivial periodic solutions of the delay equation. It is an interesting and important question if the periodic solutions of the approximating systems (1.2), which exist according to Theorem 3.15, converge to nontrivial periodic solutions of the delay equation (1.1). In this section we give a positive answer to this question. In principle this gives an existence proof for nontrivial periodic solutions of equation(1.1). But one should have in mind that it is very improbable that one would get new results concerning periodic solutions of (1.1) using this approach, because the methods developed directly for (1.1) are already very powerful. The convergence result proved in this section is interesting for a different reason. It shows that the approximating systems inherit the existence of nontrivial periodic solutions from the delay equation.

We first establish some technical results. For $N = 1,2,\ldots$, $j \in \{1,\ldots,N\}$ and $t \geq 0$ we define the function

$$f_{N,j}(t,s) = N\frac{(N(t-s))^{j-1}}{(j-1)!}e^{-N(t-s)}, \quad 0 \leq s \leq t.$$

Lemma 4.1.

a) $\lim_{N\to\infty} \int_0^t f_{N,j_N}(t,s)\,ds = 1$ *uniformly for* $t \geq 2$ *and sequences* (j_N) *with* $1 \leq j_N \leq N$.

b) *Let* $t \geq 2$ *and* $\epsilon \in (0,1)$. *Then for all* $N = 1, 2, \ldots$ *and* $j = 1, \ldots, N$

$$f_{N,j}(t,\, t - \frac{j}{N} - \epsilon) \leq \frac{1}{\epsilon\sqrt{2\pi}} \sqrt{N} e^{-N(\epsilon - \ln(1+\epsilon))}.$$

If in addition $j \geq 1 + N\epsilon$ *(i.e.,* $t - j/N + 1/N + \epsilon \leq t$*), then*

$$f_{N,j}(t,\, t - \frac{j-1}{N} + \epsilon) \leq \frac{1}{\sqrt{2\pi\epsilon}} \sqrt{N} e^{N(\epsilon + \ln(1-\epsilon))}.$$

Proof: The result a) follows immediately from

$$1 > \int_0^t f_{N,j}(t,s)\,ds = 1 - e^{-Nt} \sum_{k=0}^{j-1} \frac{(Nt)^k}{k!} > 1 - e^{-Nt} \sum_{k=0}^{N-1} \frac{(Nt)^k}{k!} \quad \text{for} \quad j = 1, \ldots, N$$

and Lemma 3.10.

In order to prove b) we use Sterling's formula and get

$$\begin{aligned} f_{N,j}(t,\, t - \frac{j}{N} - \epsilon) &= N \frac{(j + \epsilon N)^{j-1}}{(j-1)!} e^{-(j+\epsilon N)} \\ &\leq \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{N}}{\sqrt{j/N}(1 + \epsilon N/j)} \exp\big[-\epsilon N \big(1 - \frac{j}{\epsilon N} \ln(1 + \frac{\epsilon N}{j})\big)\big] \\ &\leq \frac{1}{\epsilon\sqrt{2\pi}} \sqrt{N} \exp\big[-\epsilon N \big(1 - \frac{j}{\epsilon N} \ln(1 + \frac{\epsilon N}{j})\big)\big]. \end{aligned}$$

It is not difficult to see that the function $x\ln(1 + 1/x)$ is monotonically increasing from 0 at $x = 0$ to 1 as $x \to \infty$. Since $j/(\epsilon N) \leq 1/\epsilon$ the result follows immediately. The estimate for $f_{N,j}(t,\, t - (j-1)/N + \epsilon)$ follows analogously. One has to use that $x\ln(1 - 1/x)$ is monotonically increasing on $(1, \infty)$ and bounded above by -1. ∎

Theorem 4.2. *Let* f *be as in Theorem 3.15. For* $\alpha > \pi/2$ *choose* N_0 *such that* $\alpha_0(N_0) \in (\pi/2\,, \alpha)$ *(which is possible in view of Proposition 2.4). For any* $N \geq N_0$ *let* $w^N(t)$ *be a nontrivial periodic solution of (1.2) with initial value in* $\mathcal{K}_R^N$ *(existing according to Theorem 3.15). Then there exists a subsequence* $(w^{N_k}(t))$ *such that*

$$\lim_{k\to\infty} w_0^{N_k}(t) = y(t), \quad t \in \mathbf{R}$$

uniformly on compact intervals, where $y(t)$ *is a nontrivial periodic solution of (1.1). Moreover,*

$$\lim_{k\to 0} \Big\| \sum_{j=0}^{N_k} w_j^{N_k}(t) e_j^{N_k} - y_t \Big\| = 0, \quad t \in \mathbf{R},$$

uniformly on compact t-intervals.

Proof: For $N \geq N_0$ let $w^N(t) = \text{col}(w_0^N(t), \dots, w_N^N(t))$ a nontrivial solution of (1.2) with initial value $w^N(0) \in \mathcal{K}_R^N$, where R is given in Theorem 3.15. The monotonicity behavior of the coordinates $w_j^N(t)$ and Lemma 3.11 imply

$$\sup_{t\in\mathbf{R}} |w^N(t)| = \sup_{t\in\mathbf{R}} w_0^N(t)| \leq R \tag{4.1}$$

for all $N \geq N_0$. Using this and $\dot{w}_0^N(t) = -\alpha f(w_N^N(t))$ we see that also $(\dot{w}_0^N(t))_{N\geq N_0}$ is uniformly bounded on $\mathbf{R}$. Then by Arzela-Ascoli's theorem there exists a subsequence, which we again write as $(w^N(t))_{N\geq N_0}$, such that

$$\lim_{N\to\infty} w_0^N(t) = y(t), \quad t \in \mathbf{R}, \tag{4.2}$$

uniformly on compact intervals. It is clear that $y(t)$ satisfies a Lipschitz-condition on $\mathbf{R}$ with Lipschitz-constant $L = \sup_{\substack{N\geq N_0\\ t\in\mathbf{R}}} |\dot{w}_0^N(t)|$.

For $\tau \in [0,1]$ let ($[\sigma]$ is the integer part of σ)

$$j_N = \max(1,\, [N\tau]), \quad N = 1, 2, \dots.$$

Then $\lim_{N\to\infty} j_N/N = \tau$. We shall show that $\lim_{N\to\infty} w_{j_N}^N(t) = y(t-\tau)$ uniformly for $\tau \in [0,1]$ and t in compact intervals. Since we can shift the time axis it is enough to consider compact t-intervals in $[2,\infty)$. From (3.11) we get

$$w_{j_N}^N(t) = e^{-Nt} \sum_{k=1}^{j_N} \frac{(Nt)^{j_N-k}}{(j_N-k)!} w_k^N(0) + N \int_0^t \frac{(N(t-s))^{j_N-1}}{(j_N-1)!} e^{-N(t-s)} w_0^N(s)\, ds, \quad t \geq 0.$$

Lemma 3.10 implies

$$\left| e^{-Nt} \sum_{k=1}^{j_N} \frac{(Nt)^{j_N-k}}{(j_N-k)!} w_k^N(0) \right| \leq \frac{R}{2\sqrt{\pi}} N^{-1/2}$$

for $t \geq 2$ and all $\tau \in [0,1]$. We next estimate

$$\begin{aligned} &\left| \int_0^t f_{N,j_N}(t,s) w_0^N(s)\, ds - y(t-\tau) \right| \\ &\leq \left| \int_0^t f_{N,j_N}(t,s)\big(w_0^N(s) - y(s)\big)\, ds \right| + \left| \int_0^t f_{N,j_N}(t,s) y(s)\, ds - y(t-\tau) \right| \\ &=: a_1^N + a_2^N. \end{aligned}$$

Lemma 4.1, a) and (4.2) imply

$$\lim_{N\to\infty} a_1^N = 0$$

uniformly for t in compact subintervals of $[2,\infty)$ and $\tau \in [0,1]$.

In order to estimate a_2^N we choose $\epsilon \in (0, 1/2)$, $T > 2$, $\tau \in [0,1]$ and $N \geq \max(2/\epsilon,\, N_0)$. Then $t - (j_N/N) - \epsilon > t - \tau - (1/N) - \epsilon > t - \tau - 2\epsilon \geq 0$ and $t - (j_N - 1)/N + \epsilon <$

$t-\tau+(2/N)+\epsilon<t-\tau+2\epsilon$. We put $t_\epsilon^- = t-\tau-2\epsilon$ and $t_\epsilon^+ = \min(t,\, t-\tau+2\epsilon)$. Since $f_{N,j_N}(t,s)$ has a unique maximum at $s_N = t-(j_N-1)/N$, we obtain from Lemma 4.1, b)

$$\Big|\int_0^{t_\epsilon^-} f_{N,j_N}(t,s)y(s)\,ds\Big| \le \frac{RT}{\epsilon\sqrt{2\pi}}\sqrt{N}e^{-N(\epsilon-\ln(1+\epsilon))} \tag{4.4}$$

and

$$\Big|\int_{t_\epsilon^+}^{t} f_{N,j_N}(t,s)y(s)\,ds\Big| \le \frac{RT}{\sqrt{2\pi\epsilon}}\sqrt{N}e^{N(\epsilon+\ln(1-\epsilon))} \tag{4.5}$$

for all $t\in[2,T]$, $\tau\in[0,1]$ and $N\ge\max(N_0, 2/\epsilon)$.

The estimates (4.4) and (4.5) together with Lemma 4.1, a) show that

$$\lim_{N\to\infty}\int_{t_\epsilon^-}^{t_\epsilon^+} f_{N,j_N}(t,s)\,ds = 1 \tag{4.6}$$

uniformly for $t\in[2,T]$, $\tau\in[0,1]$. Using the mean value theorem for integrals and the Lipschitz-condition for $y(t)$ we obtain

$$\begin{aligned} a_2^N \le{}& \Big|\int_0^{t_\epsilon^-} f_{N,j_N}(t,s)y(s)\,ds\Big| + \Big|\int_{t_\epsilon^+}^{t} f_{N,j_N}(t,s)y(s)\,ds\Big| \\ &+ 2\epsilon L\int_{t_\epsilon^-}^{t_\epsilon^+} f_{N,j_N}(t,s)\,ds + R\Big(1-\int_{t_\epsilon^-}^{t_\epsilon^+} f_{N,j_N}(t,s)\,ds\Big). \end{aligned}$$

The estimates (4.4)–(4.6) prove that we can find an $N_1 = N_1(\epsilon)$ such that

$$a_2^N \le 2\epsilon L + R\epsilon + 2\epsilon$$

for all $N\ge N_1$, $t\in[2,T]$ and $\tau\in[0,1]$, i.e.,

$$\lim_{N\to\infty} a_2^N = 0$$

uniformly for $t\in[2,T]$ and $\tau\in[0,1]$.

Thus we have established that

$$\lim_{N\to\infty} w_{j_N}^N(t) = y(t-\tau) \tag{4.7}$$

uniformly for t in compact intervals and $\tau\in[0,1]$. It is not difficult to prove that (4.7) implies

$$\lim_{N\to\infty}\Big\|\sum_{j=0}^{N} w_j^N(t)e_j^N - y_t\Big\| = 0 \tag{4.8}$$

uniformly for t in compact intervals. One only has to observe that, for fixed N, $j_N = j$ for all $\tau\in[j/N,\,(j+1)/N)$ if $j=2,\dots,N-1$, $j_N = N$ for $\tau=1$ and $j_N=1$ for $\tau\in[0,2/N)$.

Since (4.7) specifically implies $\lim_{N\to\infty} w_N^N(t) = y(t-1)$ uniformly on compact intervals, we immediately get from $w_0^N(t) = w_0^N(0) - \alpha\int_0^t f(w_N^N(\tau))\,d\tau$ that $y(t) = y(0) - \alpha\int_0^t f(y(\tau-1))\,d\tau$, $t \in \mathbf{R}$, i.e., $y(t)$ is a solution of (1.1) on $\mathbf{R}$ with $y(0) \in \bar{\mathcal{K}}_R = \{\phi \in \bar{\mathcal{K}} \mid \|\phi\| \le R\}$, $\mathcal{K} = \{\phi \in C(0,1;\mathbf{R}) \mid \phi(-1) = 0,\ \phi \text{ increasing}\}$.

We next prove that $y(t) \not\equiv 0$. Assume that this is not true, i.e., $\lim_{N\to\infty} w_0^N(t) = 0$ uniformly on compact intervals. But then $\lim_{N\to\infty} w^N(0) = 0$ because $|w^N(0)| = w_0^N(0)$. Lemma 3.11 implies $|w^N(t)| \le 3\alpha\beta^-(3\alpha\beta^+(|w^N(0)|))$ on $[0,\ t^{**}(w^N(t^*))]$, which by periodicity of $w^N(t)$ implies

$$|w^N(t)| \le 3\alpha\beta^-\big(3\alpha\beta^+(|w^N(0)|)\big), \quad t \in \mathbf{R}, \tag{4.9}$$

i.e., $\lim_{N\to\infty} w^N(t) = 0$ uniformly on $\mathbf{R}$.

For $\gamma \in (0,1)$ we choose $\delta > 0$ such that $|f(u)| \le (1+\gamma)|u|$ for $|u| \le \delta$. This is possible because $f'(0) = 1$. Without restriction of generality we can assume that $|w^N(0)| \le \delta$ for all N. Then (4.9) implies

$$|w^N(t)| \le 9\alpha^2(1+\gamma)^2|w^N(0)|, \quad t \in \mathbf{R}. \tag{4.10}$$

For any N we define

$$u^N(t) = \big(9\alpha^2(1+\gamma)^2|w^N(0)|\big)^{-1} w^N(t), \quad t \in \mathbf{R}.$$

Then (4.10) implies that $|u^N(t)| \le 1$ on $\mathbf{R}$ and

$$|u^N(0)| = \frac{1}{9\alpha^2(1+\gamma)^2} \quad \text{for all} \quad N. \tag{4.11}$$

From

$$\begin{aligned} |\dot{u}_0^N(t)| &\le \alpha\big(9\alpha^2(1+\gamma)^2|w^N(0)|\big)^{-1}(1+\gamma)|w_N^N(t)| \\ &= \alpha(1+\gamma)|u_N^N(t)|, \quad t \in \mathbf{R}, \end{aligned}$$

we see that $(\dot{u}_0^N(t))$ is uniformly bounded on $\mathbf{R}$. By Arzela-Ascoli's theorem there exists a subsequence (denoted by $(u^N(t))$ again) such that

$$\lim_{N\to\infty} u_0^N(t) = u(t)$$

uniformly on compact intervals. Because of (4.11) $|u(0)| \ne 0$, i.e., $u(t) \not\equiv 0$. Moreover, $u(t)$ is bounded on $\mathbf{R}$. Since (4.3) implies

$$u_N^N(t) = e^{-Nt}\sum_{k=1}^{N}\frac{(Nt)^{N-k}}{(N-k)!}u_k^N(0) + N\int_0^t \frac{(N(t-s))^{N-1}}{(N-1)!}e^{-N(t-s)}u_0^N(s)\,ds,$$

the same proof which led to (4.7) also shows

$$\lim_{N\to\infty} u_N^N(t) = u(t-1)$$

uniformly for t in compact intervals. Thus using $w_0^N(t) = w_0^N(0) - \alpha \int_0^t f(w_N^N(\tau))\, d\tau$, $t \in \mathbf{R}$, and differentiability of f at zero we obtain that $u(t)$ satisfies $u(t) = u(0) - \alpha \int_0^t f'(0)u(\tau)\, d\tau$, $\tau \in \mathbf{R}$, i.e., $u(t)$ is a bounded nontrivial solution of the linear delay equation

$$\dot{x}(t) = -\alpha x(t-1) \tag{4.12}$$

with initial function $\phi(t) = u(t)$, $-1 \le t \le 0$. Note that

$$\phi = \lim_{N\to\infty} \sum_{j=0}^{N} u_j^N(0) e_j^N \in \bar{\mathcal{K}}\backslash\{0\}.$$

Let λ_0 be the eigenvalue of (4.12) satisfying $\operatorname{Re}\lambda_0 > 0$ and $0 < \operatorname{Im}\lambda_0 < \pi$. This eigenvalue exists for $\alpha > \pi/2$ and is given by $\lambda_0 = \sigma_0 + i\tau_0$, where τ_0 is the solution of $\tau = -\sigma \tan\tau$ in the interval $(0, \pi)$.

The projection of a function $\phi \in C(-1, 0; \mathbf{R})$ into eigenspace of (4.12) corresponding to λ_0 is given by (see [5])

$$(\pi_{\lambda_0}\phi)(s) = \operatorname*{Re}_{\lambda=\lambda_0} s e^{\lambda s} \Delta^{-1}(\lambda) p(\lambda;\phi), \quad -1 \le s \le 0,$$

where $\Delta(\lambda) = \lambda + \alpha e^{-\lambda}$ and $p(\lambda;\phi) = \phi(0) - \alpha \int_{-1}^{0} e^{-\lambda(\theta+1)}\phi(\theta)\, d\theta$. A short computation shows

$$(\pi_{\lambda_0}\phi)(s) = (1 - \alpha e^{-\lambda_0})^{-1}\Big(\phi(0) - \alpha \int_{-1}^{0} e^{-\lambda_0(\theta+1)}\phi(\theta)\, d\theta\Big) e^{\lambda_0 s}.$$

Obviously $1 - \alpha e^{-\lambda_0} \ne 0$. For $\phi \in \bar{\mathcal{K}}$ we get

$$\operatorname{Im}\Big(\phi(0) - \alpha \int_{-1}^{0} e^{-\lambda_0(\theta+1)}\phi(\theta)\, d\theta\Big) = \alpha \int_{-1}^{0} e^{-\sigma_0(\theta+1)} \sin\tau_0(\theta+1)\phi(\theta)\, d\theta > 0,$$

because $\tau_0(\theta+1) \in [0, \pi)$ for $\theta \in [-1, 0]$ and $\phi(0) > 0$. Therefore ϕ has a nonzero component in the eigenspace corresponding to λ_0, which implies that $u(t)$ is unbounded on $t \ge 0$. This contradiction proves that $y(t)$ obtained in (4.2) is a nontrivial solution of (1.1) which is bounded on $\mathbf{R}$. Moreover, $y(0) \in \bar{\mathcal{K}}\backslash\{0\}$.

It remains to prove that $y(t)$ is a periodic solution of (1.1). The first observation is that in case $\alpha > 1$ for any $\phi \in \bar{\mathcal{K}}\backslash\{0\}$ there exists $t^* = \min\{t > 0 \mid x_t(\phi) \in \mathcal{K}\}$ where $x(t;\phi)$ denotes the solution of (1.1). Moreover, $t^* > 2$ and $\dot{x}(t^* - 1;\phi) > 0$. The proof for these facts is much easier than in case of the approximating systems (see for instance [13], [5]).

Let now $t^* = \min\{t > 0 \mid y_t \in \mathcal{K}\}$ and choose $\epsilon > 0$. From $\lim_{N\to\infty} w_N^N(t) = y(t-1)$ uniformly for t in compact intervals, $\dot{y}(t^* - 1) > 0$ we see that there exists an N_0 such that $|t_N^* - t^*| < \epsilon/L$ for $N \ge N_0$, where L is the Lipschitz-constant for $y(t)$ and t_N^* is the second positive zero of $w_N^N(t)$. This means that $w^N(0) = \mathcal{T} w^N(0) = w^N(t_N^*)$. Using (4.8) we can choose N_0 such that in addition $\| \sum_{j=0}^N w_j^N(t) e_j^N - y_t \| < \epsilon$ for t in compact intervals. Then the estimate

$$\|y_{t^*} - \phi\| \le \|y_t - y_{t_N^*}\| + \|y_{t_N^*} - \sum_{j=0}^{N} w_j^N(t_N^*) e_j^N\| + \|\sum_{j=0}^{N} w_j^N(0) e_j^N - \phi\| < 3\epsilon$$

proves $y_{t^*} = \phi$. ∎

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