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ON A SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. In this note we consider a nonlinear differential equation of second order which has been proposed as a model for the periodic motion of a satellite in its elliptical orbit. We use some elementary methods to demonstrate the existence of an odd periodic solution for a considerably larger parameter set than considered earlier.

Consider the following periodic boundary value problem

$$(1 + a\cos t)x''(t) - 2a\sin t x' + \alpha \sin x = 4a \sin t \tag{1}$$

$$x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi). \tag{2}$$

For the parameter set $\{(a, \alpha) : 0 < a < 1,\ |\alpha| \le 3\}$, this problem has been proposed in [1] as a mathematical model for the periodic motions of a satellite in the plane of its elliptical orbit. Recently, Petryshin and Yu [4] have studied this problem using the generalized topological degree for A-proper mappings, and have shown the existence of a solution of (1)–(2) for the parameter range

$$0 \le a < \frac{2}{\pi}|\alpha|, \qquad (8\sqrt{2} + 3)a + 2|\alpha| < 1.$$

In a previous paper [2] using variational methods, we showed that in fact, problem (1)–(2) has a solution for $|a| < 1$ and arbitrary α. The aim of this paper is to show that problem (1)–(2) has an odd solution in case $|a| < 1$ and α arbitrary. Since, as will be shown, an odd solution may be obtained by a monotone iteration scheme, our method of proof is in fact constructive. Since we wish to establish the existence of an odd solution, we shall consider equation (1) subject to the boundary condition

$$x(0) = x(\pi) = 0. \tag{3}$$

Introducing the new variable

$$y = (1 + a\cos t)x,$$

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(1) and (3) become

$$y'' + g(t,y) = 0, \qquad y(0) = y(\pi) = 0\,, \tag{4}$$

where

$$g(t,y) = \frac{a\cos t}{1+a\cos t}y + \alpha\sin\frac{y}{1+a\cos t} - 4a\sin t\,.$$

We note that g satisfies the Lipschitz condition

$$|g(t,y_1) - g(t,y_2)| \le M^2|y_1 - y_2|\,, \tag{5}$$

where

$$M^2 = \frac{|a| + |\alpha|}{1 - |a|}\,.$$

Thus, let us state and prove our main result.

Theorem. *Let $|a| < 1$ and α be any real number. Then the problem (4) has a solution y which may be obtained by a monotone iteration scheme. Letting*

$$x(t) = \begin{cases} \dfrac{y(t)}{1+a\cos t}\,, & t \in [0,\pi] \\ -\dfrac{y(2\pi - t)}{1+a\cos t}\,, & t \in [\pi, 2\pi]\,, \end{cases} \tag{6}$$

then x is a solution of (1), (2).

Proof: For each $\eta \in C[0,\pi]$ the linear problem

$$y'' + g(t,\eta) - M^2(y - \eta) = 0, \quad y(0) = y(\pi)\,, \tag{7}$$

has a unique solution

$$y = A\eta\,,$$

which is given by

$$y(t) = C[e^{Mt} - e^{-Mt}] - \frac{e^{Mt}}{2M}\int_0^t \sigma(s)e^{-Ms}\,ds + \frac{e^{-Mt}}{2M}\int_0^t \sigma(s)e^{Ms}\,ds\,,$$

where

$$\sigma(t) = g(t,\eta) + M^2\eta$$

and C is so chosen that $y(\pi) = 0$.

We now consider the iteration scheme

$$y_0 = 0, \quad y_n = Ay_{n-1}, \quad n \ge 1\,. \tag{8}$$

The cases a nonpositive and a negative must be treated separately. In the former the iteration scheme will generate a decreasing sequence and in the latter, an increasing one. We shall only consider the case that $a \ge 0$, the other case is proved in much the same way. Thus, since $a \ge 0$, it follows that $y_0 = 0$ is an upper solution (supersolution) of problem

(4) and hence the iteration scheme (8) will generate a decreasing sequence (see [3], [5]). It follows from the Lipschitz condition (5) that each y_n is an upper solution of (4), i.e.,

$$y_n'' + g(t, y_n) \le 0, \qquad y_n(0) = y_n(\pi) = 0. \tag{9}$$

Multiplying (9) by y_n and integrating over $[0, \pi]$, we obtain

$$\int_0^\pi y_n'(t)^2\,dt - \int_0^\pi \frac{a\cos t}{1 + a\cos t} y_n(t)^2\,dt \le (|\alpha| + 4a)\int_0^\pi |y_n(t)|\,dt. \tag{10}$$

Since

$$\int_0^\pi y_n(t)^2\,dt \le \int_0^\pi y_n'(t)^2\,dt$$

(Poincaré's inequality), then

$$\int_0^\pi |y_n(t)|\,dt \le \sqrt{\pi}\Big(\int_0^\pi y_n(t)^2\,dt\Big)^{1/2},$$

and since

$$\frac{a\cos t}{1 + a\cos t} \le \frac{a}{1+a},$$

it follows from (10) that

$$\frac{1}{1+a}\int_0^\pi y_n'(t)^2\,dt \le \sqrt{\pi}(|\alpha| + 4a)\Big(\int_0^\pi y_n'(t)^2\,dt\Big)^{1/2}. \tag{11}$$

From (11) and the Sobolev imbedding theorem, we get

$$\max_{[0,\pi]} |y_n(t)| \le K, \tag{12}$$

where K is a constant that only depends upon α and a. Since

$$y_n'' + g(t, y_{n-1}) - M^2(y_n - y_{n-1}) = 0, \quad y_n(0) = y_n(\pi) = 0,$$

it follows that $\{y_n\}$ is bounded in $C^2[0, \pi]$. Hence using the Ascoli-Arzelà theorem, we obtain that the sequence $\{y_n\}$ converges to a solution of (4). It is easily seen that x defined by (6) is a solution of (1)–(2), completing the proof.

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