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HIGHLY DEGENERATE PARABOLIC BOUNDARY VALUE PROBLEMS*

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Abstract. Of concern are parabolic equations of the form

$$\partial u/\partial t = \phi(x, \nabla u)\Delta u \qquad (x \in \Omega \Subset \mathbb{R}^n,\ t \geq 0)$$

where $\phi(x, \xi) > 0$ on $\Omega \times \mathbb{R}^n$ but $\phi(x, \xi) \to 0$ very rapidly as $x \to \partial\Omega$. By associating the Wentzel boundary condition with this equation, the initial value problem is shown to be well-posed. This is done with the aid of the Crandall-Liggett theorem, applied in the space $C(\overline{\Omega})$.

1. Introduction. In the 1950s, W. Feller [9-11], working from a semigroup point of view, determined all one dimensional Markov processes of diffusion type. If $[a, b]$ denote the underlying spatial interval, Feller classified the boundary points (a and b) as being of regular, exit, entrance, or natural type. (For a nice introduction to these ideas from an analyst's point of view see Yosida's book [27].) Among the attractive interpretations was that a diffusing particle could not reach an entrance boundary in finite time, and consequently no boundary condition need be imposed at such a point. The kind of boundary conditions associated with the (weakly) elliptic generators were sometimes of a nonlocal character, and Feller termed them *lateral conditions* rather than boundary conditions. Soon afterwards, A.D. Wentzel [26] began a program of finding boundary and lateral conditions which characterize multidimensional diffusion. A feature of his work was the use of *second* derivatives in the boundary conditions. Later contributions to this theory were made by Sato and Ueno [21], Taira [22-23], and many others. An earlier contribution from a nonprobabilistic point of view was made by Vishik [24].

A clean semigroup version of these results in one space dimension was obtained recently by Ph. Clément and C.A. Timmermanns [6]. Building upon earlier work of Martini and Boer [17-19], they established the following result.

Let α, β be continuous real functions on the open interval $(0,1)$ with α positive. Define a linear operator A on the real Banach space $X = C[0,1]$ by

$$Au = \alpha u'' + \beta u'$$

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for u in the domain of A, $\mathcal{D}(A)$, which we take to be

$$\mathcal{D}(A) = \{u \in C[0,1] \cap C^2(0,1) : \text{ for } j = 0,1, \ \lim_{x \to j}(\alpha(x)u''(x) + \beta(x)u'(x)) = 0\}.$$

This A is an example of an elliptic operator satisfying Wentzel's boundary condition (which we write for short as $Au\big|_{\partial\Omega} = 0$, regarding Ω as $(0,1)$). The operator A is automatically closed, dissipative, and densely defined on X. It is m-dissipative, i.e., A generates a (C_0) contraction semigroup on X, if and only if the range of $I - A$, $\mathcal{R}(I - A)$, is all of X.

The main result of Clément and Timmermanns is as follows.

$$\mathcal{R}(I - A) = X$$

if and only if, for

$$W(x) = \exp\Big[\int_{1/2}^{x} -\beta(s)\alpha(s)^{-1}\, ds\Big],$$

both (H_0) *and* (H_1) *hold:*

(H_0) $W \in L^1(0, \frac{1}{2})$ *or* $\int_0^{1/2} W(x) \int_0^x \alpha(s)^{-1}W(s)^{-1}\, ds\, dx = \infty$ *or both* ;

(H_1) $W \in L^1(\frac{1}{2}, 1)$ *or* $\int_{1/2}^{1} W(x) \int_x^1 \alpha(s)^{-1}W(s)^{-1}\, ds\, dx = \infty$ *or both* .

In particular, for $\beta \equiv 0$, $\mathcal{R}(I - A) = X$ *holds for every* α.

In earlier papers [14-16] we studied problems of the form

$$\partial u/\partial t = \phi(x, \partial u/\partial x)\partial^2 u/\partial x^2 \tag{1}$$

[14] as well as multidimensional versions of these (cf. [15] and especially [16]), where the diffusion coefficient ϕ degenerates at the boundary.

For simplicity we momentarily consider

$$\partial u/\partial t = \alpha(x)\sigma(x, \partial u/\partial x)\partial^2 u/\partial x^2 \tag{2}$$

where $\alpha \in C[0,1]$, $\sigma \in C^1([0,1] \times \mathbb{R})$, $\alpha > 0$ in $(0,1)$, and $\sigma \geq \epsilon > 0$. By imposing various linear and nonlinear boundary conditions we succeeded in associating with (2) an m-dissipative nonlinear operator A and a strongly continuous contraction (or nonexpansive) semigroup on $\overline{\mathcal{D}(A)} \subset C[0,1]$ provided that $\alpha(x)$ did not approach zero too rapidly as x approached the boundary (i.e., as $x \to 0$ or as $x \to 1$) [14].

The result of Clément and Timmermanns suggests that, under a Wentzel type boundary condition, α can be allowed to approach zero *arbitrarily rapidly* . We shall establish this in the context of equation (1) (in its one and several space dimensional versions). The elegant proof of Clément and Timmermanns is decidedly linear and unfortunately does not carry over to our case.

We treat the one dimensional case in §2. Section 3 is devoted to the multidimensional case. Section 4 contains an example from conformal geometry.

2. The one dimensional case. Let $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $\phi(x,\xi) \geq \phi_0(x)$ on $[0,1] \times \mathbb{R}$ where $\phi_0 > 0$ on $(0,1)$ and $\phi_0 \in C[0,1]$. Define an operator A on the real space $X = C[0,1]$ by

$$Au(x) = \phi(x, u'(x))u''(x)$$

for $u \in \mathcal{D}(A) = \{u \in X \cap C^2(0,1) : Au(x) \to 0 \text{ as } x \to j, \ j = 0,1\}$.

Theorem 1. *The operator A defined above is densely defined and m-dissipative on X.*

This means two things. First, the dissipativity inequality

$$\|u - v - \lambda(Au - Av)\| \geq \|u - v\| \tag{3}$$

holds for all u, $v \in \mathcal{D}(A)$ and all $\lambda > 0$. (Here $\|\cdot\|$ is the supremum norm, under which X is a real Banach space.) Next,

$$\mathcal{R}(I - A) \text{ is dense in } X \text{ [resp. } \mathcal{R}(I - A) = X] \tag{4}$$

when A is essentially m-dissipative [resp. when A is m-dissipative].

We identify A with its graph in $X \times X$. When A is essentially m-dissipative, its closure $\overline{A}$ is m-dissipative (cf. e.g. [13]). Thus by the Crandall-Liggett theorem [8],

$$T(t)f = \lim_{m\to\infty} (I - \frac{t}{m}\bar{A})^{-m} f$$

exists for all $f \in \overline{\mathcal{D}(\bar{A})} = \overline{\mathcal{D}(A)}$, and $u(t) = T(t)f$ is the unique mild solution of

$$du(t)dt = Au(t) \quad [t \geq 0], \quad u(0) = f$$

(or of $du/dt \in \bar{A}u$, $u(0) = f$). For details on the theory of nonlinear semigroups and the notion of mild solution see for instance [2], [3], [7], [13].

We now prove Theorem 1. Let u, $v \in \mathcal{D}(A)$, $\lambda > 0$. To avoid trivialities in proving (3), suppose that $u \neq v$. Choose $x_0 \in [0, 1]$ such that

$$\|u - v\| = \pm(u - v)(x_0) \; ; \tag{5}$$

and we may, by interchanging u and v if necessary, suppose that (5) holds with the plus sign.

If $0 < x_0 < 1$, then by the first and second derivative tests of the calculus,

$$(u - v)'(x_0) = 0, \qquad (u - v)''(x_0) \leq 0\,.$$

Whence for all $\lambda > 0$,

$$\begin{aligned}\|u - v\| &= (u - v)(x_0) \leq (u - v)(x_0) - \lambda\phi(x_0, u'(x_0))(u - v)''(x_0) \\ &= [u - v - \lambda(Au - Av)](x_0) \leq \|u - v - \lambda(Au - Av)\|,\end{aligned}$$

which is (3). On the other hand, if $x_0 \in \{0, 1\}$, then $Au(x_0) = Av(x_0) = 0$ and

$$\begin{aligned}\|u - v\| &= (u - v)(x_0) = (u - v)(x_0) - \lambda(Au - Av)(x_0) \\ &\leq \|u - v - \lambda(Au - Av)\|.\end{aligned}$$

It follows that A is dissipative. (Notice how simply Wentzel's boundary condition enters into the calculation.)

If $u \in C^2[0, 1]$ satisfies $u(x) = c_1$ for $0 \leq x \leq \epsilon$ and $u(x) = c_2$ for $1 - \epsilon \leq x \leq 1$ where $\epsilon > 0$, then $u \in \mathcal{D}(A)$; and these u's are dense in X. Thus $\overline{\mathcal{D}(A)} = X$.

To show that $\mathcal{R}(I-A)$ is dense in $X = C[0,1]$, let $h \in C^2[0,1]$ be given and consider the problem

$$\begin{cases} u - \phi(x,u')u'' = h & \text{in } [0,1], \\ u \in \mathcal{D}(A). \end{cases}$$

Let f be the unique linear function on $[0,1]$ satisfying $f'' \equiv 0$, $f(0) = -h(0)$, $f(1) = -h(1)$. Then (6) reduces to the homogeneous Dirichlet problem

$$v - \phi(x,\ v' - f')v'' = h + f,$$
$$v(0) = 0, \quad v(1) = 0$$

for $v = u + f$. Let now

$$\psi(x,\xi) = \phi(x,\ \xi - f'(x))$$

for $(x,\xi) \in [0,1] \times \mathbb{R}$ and let $g = h + f$. We then must solve

$$\begin{cases} v - \psi(x,v')v'' = g & \text{in } [0,1], \\ v(0) = 0, \quad v(1) = 0. \end{cases} \tag{7}$$

Here $g \in C^2[0,1]$ vanishes at both end points and ψ satisfies the same assumptions as does ϕ. A solution to (7) gives a solution to (6) for $u = v - f$. Thus it suffices to solve (7).

We regularize by replacing ψ by ψ_m, where for a given positive integer m,

$$\psi_m(x,\xi) = \begin{cases} \psi(x,\xi) & \text{if } \psi(x,\xi) \geq 1/m, \\ 1/m & \text{if } \psi(x,\xi) < 1/m. \end{cases}$$

According to [5] or [14],

$$v_m - \psi_m(x,v_m')v_m'' = g\,,$$
$$v_m(0) = 0, \quad v_m(1) = 0$$

has a unique solution v_m. By the maximum principle,

$$\|v_m\|_{L^\infty[0,1]} \leq \|g\|_{L^\infty[0,1]} \tag{8}$$

holds for $m = 1, 2, \ldots$. Let $0 < \epsilon < 1/2$ and let $J = [\epsilon, 1-\epsilon]$. Since $\psi(x,\xi) \geq \phi_0(x) > 0$ in $(0,1) \times \mathbb{R}$ where $\phi_0 \in C[0,1]$, there is an M_ϵ such that $\psi_m(x,\xi) = \psi(x,\xi) > 1/m$ on $J \times \mathbb{R}$ for all $m \geq M_\epsilon$. For such m, $v_m'' = (v_m - g)\psi(x,v_m')^{-1}$ in J implies

$$\|v_m''\|_{L^\infty(J)} \leq 2\|g\|_{L^\infty[0,1]}k^{-1}\,, \tag{9}$$

where

$$k = \min\{\phi_0(x) : x \in J\} > 0\,.$$

From

$$v_m(x) - v_m(\epsilon) = \int_\epsilon^x v_m'(s)\,ds \quad v_m'(x) - v_m'(\epsilon) = \int_\epsilon^x v_m''(s)\,ds$$

in J, together with (8) and (9), it follows that

$$\|v_m''\|_{W^{2,\infty}(J)} \leq \text{ constant}$$

for all $m \geq M_\epsilon$. Using the compactness arguments of our earlier paper [14], $\{v_m\}$ has a subsequence which converges in $C^1_{\text{loc}}(0,1)$ to a function $v \in C^2(0,1)$ which satisfies

$$v - \psi(x, v')v'' = g \quad \text{in } (0,1).$$

It only remains to show that $v \in C[0,1]$ and $v(0) = v(1) = 0$.

To that end, we choose a function w in $C^2[0,1]$ satisfying

$$\begin{aligned} &w \geq 0 \quad \text{in } [0,1]\,, \\ &w(0) = w(1) = 0\,, \\ &w''(x) \leq 0 \quad \text{for all } x \text{ in } [0,1], \\ &w(x) \geq |g(x)| \quad \text{for all } x \text{ in } [0,1]. \end{aligned}$$

Such a w can be chosen since $g \in C^2[0,1]$. (We may, in fact, take $w(x) = kx(1-x)$ for k a sufficiently large constant.) Then

$$|v_m'' - \phi_m(x, v_m')^{-1}v_m| = \phi_m(x, v_m')^{-1}|g| \leq -w'' + \phi_m(x, v_m')^{-1}w,$$

which implies

$$(w \pm v_m)'' - \phi_m(x, v_m')^{-1}(w \pm v_m) \leq 0 \quad \text{in } (0,1)\,.$$

But also

$$(w \pm v_m)(0) = (w \pm v_m)(1) = 0\,,$$

consequently the maximum principle [20] yields

$$w \pm v_m \geq 0 \quad \text{on } [0,1]\,,$$

that is,

$$|v_m(x)| \leq w(x) \quad \text{for all } x \text{ in } [0,1]\,.$$

A passage to the limit ($m \to \infty$) gives $v \in C[0,1]$, $v(0) = v(1) = 0$.

It only remains to show that A is closed, which implies that $\mathcal{R}(I - A)$ (which is dense) is closed. Let $(u, v) \in \bar{A}$, thus there exists $u_m \in \mathcal{D}(A)$ such that, as $m \to \infty$, $u_m \to u$ and $Au_m \to v$ in X. Since $\phi \geq \delta > 0$ on $[\epsilon, 1-\epsilon] \times \mathbb{R}$ for arbitrary $\epsilon > 0$ and suitable $\delta > 0$, it follows that $\{u_m\}$ is bounded in $C^2[\epsilon, 1-\epsilon]$ and relatively compact in $C^1[\epsilon, 1-\epsilon]$, i.e., in $C^1_{\text{loc}}(0,1)$. Thus, on $(0,1)$ it follows from $\phi(x, u_m')u_m'' \to v$ that $u \in C^2$ and $v(x) = \phi(x, u'(x))u''(x)$. From $Au_m \to v$ uniformly on $[0,1]$ and $Au_m(0) = Au_m(1) = 0$ it follows that $v(0) = v(1) = 0$. Thus $u \in \mathcal{D}(A)$ and $Au = v$.

The proof is complete. ∎

Corollary 2. *Let $\phi \in C([0,1] \times \mathbb{R})$ be positive on $(0,1) \times \mathbb{R}$, and satisfy $\phi(x, \xi) \geq \phi_0(x) > 0$ on $(0,1) \times \mathbb{R}$ where $\phi_0 \in C[0,1]$. Define A on*

$$Y = C_0(0,1) = \{u \in C[0,1] : u(0) = u(1) = 0\}$$

by

$$\begin{aligned} Au(x) &= \phi(x, u'(x))u''(x) \qquad (x \in [0,1]), \\ \mathcal{D}(A) &= \{u \in Y \cap C^2(0,1) : Au \in Y\}. \end{aligned}$$

Then A is densely defined and m-dissipative on Y.

Thus for the *homogeneous Dirichlet problem* as well as for the Wentzel problem, ϕ can appraoch zero *arbitrarily rapidly* as the spatial variable x approaches the boundary.

That A is dissipative follows from the dissipativity proof of Theorem 1. $\mathcal{D}(A)$ contains the C^2 functions with compact support in $(0,1)$, and so $\overline{\mathcal{D}(A)} = Y$. The range condition (or m-part) also follows from the proof of Theorem 1. The point of the introduction of f into the previous proof was to reduce to the homogeneous boundary condition. In other words, for $\Omega = (0,1)$, the Wentzel condition $Au|_{\partial\Omega} = 0$ for $u - \lambda Au = h$ is equivalent to $u|_{\partial\Omega} = h|_{\partial\Omega}$. When h vanishes on $\partial\Omega$ this reduces to $u|_{\partial\Omega} = 0$. Thus Corollary 2 follows from the proof of Theorem 1.

3. Multidimensional problems.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be positive and Hölder continuous with Hölder exponent $\beta \in (0,1)$ in the sense that for any $M > 0$ there is a constant $K(M)$ such that

$$|\phi(x,\xi) - \phi(y,\eta)| \leq K(M)\{|x-y|^\beta + |\xi - \eta|^\beta\}$$

for all $x, y \in \Omega$ and all $\xi, \eta \in \mathbb{R}$ with $|\xi| \leq M$, $|\eta| \leq M$. Thus ϕ is locally Hölder continuous on $\overline{\Omega} \times \mathbb{R}^n$ but possibly $\phi(x,\xi) \to 0$ as $x \to x_0 \in \partial\Omega$. Define

$$Au(x) = \phi(x, \nabla u(x))\Delta u(x)$$

with

$$\mathcal{D}(A) = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : Au \in C(\overline{\Omega}),\ Au = 0 \text{ on } \partial\Omega\}.$$

We take X to be the real Banach space $C(\overline{\Omega})$ with the supremum norm $\|\cdot\|$, and we assume that $\phi(x,\xi) \geq \phi_0(x) > 0$ on $\Omega \times \mathbb{R}^n$, where $\phi_0 \in C(\overline{\Omega})$.

Theorem 3. *The operator A defined above is densely defined and essentially m-dissipative on X.*

Thus the semigroup

$$T(t)f = \lim_{m\to\infty} (I - \frac{t}{m}\bar{A})^{-m} f$$

gives the unique mild solution of the well-posed mixed problem

$$\begin{aligned} \partial u/\partial t &= \phi(x,\nabla u)\Delta u \quad && (x \in \Omega,\ t \geq 0), \\ \phi(x,\nabla u)\Delta u &= 0 && (x \in \partial\Omega,\ t \geq 0), \\ u(x,0) &= f(x) && (x \in \Omega) \end{aligned}$$

for any $f \in X$.

The conclusion of Theorem 3 is weaker than that of Theorem 1 since the word "essentially" appears. The operator A of Theorem 3 is not closed.

First we establish the dissipativity of A. Let $u, v \in \mathcal{D}(A)$ and let $w = u - v$. Choose $x_0 \in \overline{\Omega}$ such that $w(x_0) = \pm\|w\|$; assume the plus sign holds (otherwise interchange u and v). If $x_0 \in \Omega$, then $\nabla w(x_0) = 0$, $\Delta w(x_0) \leq 0$. Thus if $u \neq v$ and $\lambda > 0$,

$$\begin{aligned} \|u - v\| &\leq w(x_0) - \lambda\phi(x_0, \nabla u(x_0))\Delta w(x_0 \\ &= [u - v - \lambda(Au - Av)](x_0) \\ &\leq \|u - v - \lambda(Au - Av)\|. \end{aligned} \tag{10}$$

If $x_0 \in \partial\Omega$, then the first inequality in (10) holds as an equality by the Wentzel boundary condition. Thus A is dissipative.

Let g be a C^2 function in $\mathbb{R}^n$ which is harmonic in a neighborhood N of $\partial\Omega$. Let $f \in C^2(\overline{\Omega})$ be such that $f \equiv g$ in $N \cap \Omega$. Then $f \in \mathcal{D}(A)$ since $\phi(x, \nabla f)\Delta f = 0$ on $\partial\Omega$. Since $f|_{\partial\Omega} = g|_{\partial\Omega}$, the resulting set of functions $f|_{\partial\Omega}$ (as g varies) is dense in $C(\partial\Omega)$ by linear elliptic theory [12]. It follows that the set $\{u + f\}$, where f is as above and u is in $C^2(\Omega)$ and has compact support, is contained in $\mathcal{D}(A)$ and is dense in X. Thus $\overline{\mathcal{D}(A)} = X$.

To show that the range condition (4) holds, it suffices to solve

$$u - \phi(x, \nabla u)\Delta u = h, \quad u \in \mathcal{D}(A) \tag{11}$$

for h an arbitrary given function in $C^{2+\beta}(\overline{\Omega})$ where β is as in the Hölder condition hypothesis. By linear elliptic theory [12], there exists a unique $f \in C^{2+\beta}(\overline{\Omega})$ such that

$$\Delta f = 0 \quad \text{in } \Omega, \quad f|_{\partial\Omega} = -h|_{\partial\Omega}.$$

Let

$$v = u + f.$$

Then solving (11) reduces to solving the homogeneous Dirichlet problem

$$\begin{cases} v - \psi(x, \nabla v)\Delta v' = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $g = h + f$ and

$$\psi(x, \xi) = \phi(x, \xi - \nabla f(x)) \quad \text{for } (x, \xi) \in \Omega \times \mathbb{R}^n.$$

As in the proof of Theorem 1, the boundary condition $\phi(x, \nabla u) = 0$ on $\partial\Omega$ means $u = h$ on $\partial\Omega$, which in turn is equivalent to $v = 0$ on $\partial\Omega$.

Suppose first that $\phi(x, \xi) \geq \delta > 0$ holds for some $\delta > 0$ and all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$. Then $\psi(x, \xi) \geq \delta$ for all (x, ξ). The existence of the solution v of (12) follows from techniques developed by Lin [16]. We sketch the proof now.

Define $S : C^2(\overline{\Omega}) \to C^\beta(\overline{\Omega})$ by

$$Sv = (v - g)/\psi(x, \nabla v).$$

Letting Δ_D denote the Laplacian on Ω equipped with homogeneous Dirichlet boundary conditions, we see that (12) is equivalent to

$$v = \Delta_D^{-1} S v. \tag{13}$$

We solve (13) with the aid of the Schauder fixed point theorem. For each $m = 1, 2, \ldots$ let $S_m v$ be Sv or $S(mv/\|v\|_{C^2})$, according as $\|v\|_{C^2} \leq m$ on $\|v\|_{C^2} > m$. Then $\Delta_D^{-1} S_m : C^\beta(\overline{\Omega})$ is compact, continuous, and uniformly bounded. Thus by the Schauder fixed point theorem, for each m there is a v_m such that $\Delta_D^{-1} S_m v_m = v_m$. If we can show that $\|v_\nu\|_{C^2} \leq \nu$ holds for some ν, then $v = v_\nu$ is the desired solution of (13), i.e., of (12).

Thus assume $\|v_m\|_{C^2} > m$ holds for all m; we seek a contradiction. We obtain this by establishing a $C^{2+\beta}(\overline{\Omega})$ bound for v_m which is independent of m. To that end, note that $\Delta_D^{-1} S_m v_m = v_m$ becomes

$$v_m = \psi(x, \nabla w_m)^{-1}(w_m - g) \tag{14}$$

where $w_m = mv_m/\|v_m\|_{C^2}$. Choose $x_0 \in \overline{\Omega}$ such that $\|v_m\|_\infty = |v_m(x_0)|$. Then $x_0 \in \Omega$ since $v_m = 0$ on $\partial\Omega$. By the second derivative test, $v_m(x_0)\Delta v_m(x_0) \le 0$ and $\nabla v_m(x_0) = 0$. Multiplying (14) by v_m and evaluating at x_0, we obtain $\|w_m\|_\infty \le \|g\|_\infty$, which is independent of m.

We next bound $\|\nabla v_m\|_\infty$. The solution of (14) is given by

$$v_m(x) = \int_\Omega G(x,y)\psi(y, \nabla w_m(y))^{-1}(w_m(y) - g(y))\, dy$$

where G is the Green's function for Δ_D on Ω, cf. [12]. Consequently

$$\nabla v_m(x) = \int_\Omega \nabla_x G(x,y)[\psi(y, \nabla w_m)^{-1}(w_m - g)](y)\, dy,$$

and so $\|\nabla v_m\|_\infty$ is uniformly bounded, independently of m (since $\psi \ge \delta > 0$). Finally, the uniform $C^{2+\beta}(\overline{\Omega})$ bound of v_m independently of m) follows from the global Schauder estimate [12]. We thus have obtained contradiction to $\|v_\nu\|_{C^2} > \nu$ for all ν, and so (12) is solved (provided $\psi \ge \delta > 0$).

We now remove the uniform ellipticity restriction that $\psi \ge \delta > 0$. For each $m = 1, 2, \ldots$ let

$$\psi_m(x,\xi) = \begin{cases} \psi(x,\xi) & \text{if } \psi(x,\xi) \ge 1/m, \\ 1/m & \text{if } \psi(x,\xi) < 1/m. \end{cases}$$

Then $\psi_m \ge 1/m$ on $\overline{\Omega} \times \mathbb{R}^n$, whence

$$v - \psi_m(x, \nabla v)\Delta v = g \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \tag{15}$$

has a unique solution v_m by the above proof.

Now let $\{\Omega_k\}$ be a sequence of smooth domains in Ω such that $\Omega_k \subset \Omega_{k+1}$, $\bigcup_{k=1}^\infty \Omega_k = \Omega$, and $\psi(x,\xi) \ge 1/k$ for all $(x,\xi) \in \overline{\Omega}_k \times \mathbb{R}^n$. Then v_m satisfies

$$v_m - \psi(x, \nabla v_m)\Delta v_m = g \quad \text{in } \Omega_k$$

for each $m \ge k$. The maximum principle applied to (15) gives $\|v_m\|_\infty \le \|g\|_\infty$. Compactness arguments (see the remarks surrounding Corollary 6.3 in [12]) imply the existence of a subsequence of $\{v_m\}$ converging in $C^2_{\text{loc}}(\Omega)$ to a $v \in C^2(\Omega)$ satisfying

$$v - \psi(x, \nabla v)\Delta v = g \quad \text{in } \Omega.$$

It remains to show that $v \in C(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$.

For this purpose we choose $w \in C^2(\overline{\Omega})$ to satisfy

$$\begin{aligned} &w \ge 0 && \text{in } \overline{\Omega}, \\ &w = 0 && \text{on } \partial\Omega, \\ &\Delta w \le 0 && \text{in } \Omega, \\ &w(x) \ge |g(x)| && \text{for all } x \in \overline{\Omega}. \end{aligned}$$

This reduces to the construction of Theorem 1 when $n = 1$ and Ω is an interval (since $g = 0$ on $\partial\Omega$). It suffices to let w be a large multiple of the *ground state* of the Dirichlet

Laplacian, i.e., the positive eigenfunction corresponding to the bottom of the spectrum of $-\Delta_D$ on $L^2(\Omega)$. We have

$$\begin{aligned}|\Delta v_m - \psi_m(x,\nabla v_m)^{-1} v_m| = \psi_m(x,\nabla v_m)^{-1}|g(x)| &\le \psi_m(x,\nabla v_m)^{-1} w(x)\\ &\le -\Delta w + \psi_m(x,\nabla v_m)^{-1} w\,,\end{aligned}$$

and consequently

$$\Delta(w \pm v_m) - \psi_m(x,\nabla v_m)(w \pm v_m) \le 0 \quad \text{in } \Omega\,.$$

But also

$$w \pm v_m = 0 \quad \text{on } \partial\Omega\,,$$

whence by the maximum principle [20],

$$w \pm v_m \ge 0 \quad \text{on } \overline{\Omega}\,.$$

In other words, $|v_m(x)| \le w(x)$ for all $x \in \overline{\Omega}$. Therefore $v \in C(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$; the proof is complete. ∎

Analogous to Corollary 2 we readily obtain the following result.

Corollary 4. *Let $\phi \in C(\overline{\Omega} \times \mathbb{R}^n)$ be positive on $\Omega \times \mathbb{R}^n$ and locally Hölder continuous with Hölder exponent $\beta \in (0,1)$. Let $\phi(x,\xi) \ge \phi_0(x) > 0$ on $\Omega \times \mathbb{R}^n$, where $\phi_0 \in C(\overline{\Omega})$. Define A on*

$$Y = C_0(\Omega) = \{u \in C(\overline{\Omega}) : u = 0 \;\; on \;\; \partial\Omega\}$$

by

$$\begin{aligned}Au(x) &= \phi(x,\nabla u(x))\Delta u(x) \qquad (x \in \overline{\Omega}),\\ \mathcal{D}(A) &= \{u \in Y \cap C^2(\Omega) : Au \in Y\}\,.\end{aligned}$$

Then A is densely defined and essentially m-dissipative on Y.

Again, the point is that the Wentzel condition for $(I - A)u = h$, viz. $Au = 0$ on $\partial\Omega$, reduces to $u = h$ on $\partial\Omega$, which becomes $u = 0$ on $\partial\Omega$ when $h \in Y$.

4. An example. We consider a problem coming from Riemannian geometry where operators of the form

$$Au = \alpha(x)\Delta u + H(x,u)$$

arise naturally. We indicate how the Wentzel boundary condition and Theorem 3 are tied in with this problem.

As a special case of the operator treated in Theorem 3, consider

$$Au = \alpha(x)\Delta u + F(x)h(u) + G(x)$$

where x is in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, α, F, and G are in $C(\overline{\Omega})$, α, h, and $-F$ are nonnegative, h is a monotone nondecreasing continuous function on $\mathbb{R}$, and α is positive on Ω. Of especial concern is when Ω is the unit ball $\{x : |x| < 1\}$ in $\mathbb{R}^2$, $\alpha(x) = (1 - |x|^2)^2$,

$G(x) \equiv 1$, and $h(u) \equiv e^{2u}$. If $F(x)$ is the Gauss curvature of a metric g conformally equivalent to the hyperbolic metric

$$\tilde{h} = \alpha^{-1} dx = (1 - |x|^2)^{-2} dx_1 dx_2 ,$$

then g, is of the form

$$g = e^{-2u} \alpha^{-1} \, dx$$

where u satisfies

$$Au = 0 .$$

For $n \geq 3$, the n-dimensional problem involves the critical Sobolev exponent ($h(u) = u_+^{(n+2)/(n-2)}$) rather than the exponential function.

Bland and Kalka [4] and Aviles and McOwen [1] approached this problem by assuming that F satisfies certain conditions and then solving for g, which is equivalent to solving for u.

Under the hypotheses stated above, A, equipped with the Wentzel boundary condition (i.e., $Au = 0$ on $\partial\Omega$ for $u \in \mathcal{D}(A)$), is essentially m-dissipative on $C(\overline{\Omega})$. This follows from a standard perturbation theorem in semigroup theory, namely, $A_0 u = \alpha \Delta u$ (with Wentzel boundary condition) is linear and essentially m-dissipative on $C(\overline{\Omega})$ (by Theorem 3) and $Bu = Fh(u) + G$ is everywhere defined, continuous and dissipative on $C(\overline{\Omega})$. Thus $A_0 + B = A$, equipped with Wentzel boundary condition, is densely defined and essentially m-dissipative on $C(\overline{\Omega})$ (by Webb [25]). Thus for each $\epsilon > 0$, the range of $\epsilon I - \bar{A}$ is all of $C(\overline{\Omega})$. So for each $\epsilon > 0$, we can uniquely solve

$$\alpha(x) \Delta u_\epsilon + F(x) h(u_\epsilon) + G(x) - \epsilon u_\epsilon = 0 .$$

A natural approach to finding a solution of $Au = 0$ is to obtain first some compactness results and then let u be the limit of u_ϵ as a tends to zero through a suitable sequence.

In general, u_ϵ will not converge to a limit without some assumption on F, G, h, and α. We proceed very informally, keeping in mind the case of $G(x) \equiv 1$ and $F(x) \leq -\epsilon_0 < 0$ on $\overline{\Omega}$. Since $\alpha(x)\Delta u_\epsilon(x) \to 0$ as $x \to \partial\Omega$, on $\partial\Omega$ we have $Fh(u_\epsilon) + G - \epsilon u_\epsilon = 0$ on $\partial\Omega$. If we assume that $F\big|_{\partial\Omega} < 0$ and $sh(s) \to \infty$ as $s \to \infty$, then $u_\epsilon\big|_{\partial\Omega}$ is necessarily uniformly bounded above (as $\epsilon \to 0^+$). If $G\big|_{\partial\Omega} > 0$, then $u_\epsilon\big|_{\partial\Omega}$ is uniformly bounded below provided that $h(s) \to 0$ as $s \to -\infty$. Let $x_\epsilon \in \overline{\Omega}$ be such that $\pm u_\epsilon(x_\epsilon) = \|u_\epsilon\|_\infty$. The case of $x_\epsilon \in \partial\Omega$ is covered by the above discussion, so assume $x_\epsilon \in \Omega$. The second derivative test applies since u_ϵ, $\Delta u_\epsilon \in C(\Omega)$. Thus $\pm\Delta u_\epsilon(x_\epsilon) \leq 0$, whence from

$$\alpha(x_\epsilon)\Delta u_\epsilon(x_\epsilon) + F(x_\epsilon) h(u_\epsilon(x_\epsilon)) + G(x_\epsilon) - \epsilon u_\epsilon(x_\epsilon) = 0 , \tag{16}$$

we deduce

$$h(u_\epsilon(x_\epsilon)) \leq [G/(-F)](x_\epsilon)$$

in case $u_\epsilon(x_\epsilon) > 0$. If $h(s) \to \infty$ as $s \to \infty$ and $G/(-F) \in L^\infty(\Omega)$, then u_ϵ is uniformly bounded above. On the other hand, if $x_\epsilon \in \Omega$ and $u_\epsilon(x_\epsilon) < 0$, then from (16) we deduce that u_ϵ is uniformly bounded below, provided that $G \geq \epsilon_1 > 0$ in Ω and $h(s) \to 0$ as $s \to -\infty$.

Once we have a uniform bound on $\{u_\epsilon\}$ on Ω [or on compact subsets of Ω], from $\alpha\Delta u_\epsilon + Fh(u_\epsilon) + G - \epsilon u_\epsilon + 0$, we deduce that

$$\Delta u_\epsilon = \alpha^{-1}\{-Fh(u_\epsilon) - G + \epsilon u_\epsilon\}$$

is uniformly bounded (in ϵ) on compact subsets of Ω. From this follows local bounds on $\{\nabla u_\epsilon\}$, and then we can pass to the limit to find $u \in C(\Omega)$ satisfying

$$\alpha(x)\Delta u + F(x)h(u) + G(x) = 0$$

in Ω. The boundary regularity of u is a matter that requires further analysis.

But rather than pursue this informal discussion we want to make some remarks of a general nature.

The Bland-Kalka approach was to solve for $u = v_\epsilon$ in the ball, $\{x \in \mathbb{R}^2 : |x| < 1 - \epsilon\} = \Omega_\epsilon$ and to impose an inhomogeneous Dirichlet condition ($v_\epsilon = h_\epsilon$) on $\partial\Omega_\epsilon$. One point worth emphasizing here is that an inhomogeneous Dirichlet condition often reduces to a homogeneous Wentzel condition.

Another point is that Theorem 3 and not Corollary 4 is an appropriate tool for this problem. The reason is that the operator

$$u \mapsto F(x)h(u) + G(x)$$

(under our hypotheses) will always map $C(\overline{\Omega})$ into $C(\overline{\Omega})$ but will rarely map $C_0(\Omega)$ into $C_0(\Omega)$. When $h(u) = e^{2u}$ and $G \equiv 1$, mapping $C_0(\Omega)$ into $C_0(\Omega)$ implies that $F(x) = -1$ for all $x \in \partial\Omega$, but this condition excludes many examples of geometric interest. With the Wentzel boundary condition F can restrict to *any* given negative continuous function on $\partial\Omega$. This illustrates the difference between the Wentzel and the homogeneous Dirichlet boundary conditions. Another related important difference, hinted at above, is that the Wentzel boundary condition defines a semigroup generator (in the Crandall-Liggett sense) that is densely defined in $C(\overline{\Omega})$ rather than in the smaller space $C_0(\overline{\Omega})$.

The question of regularity at the boundary is also nontrivial. A technique for boundary regularity was developed by H. Brezis and P.-L. Lions [28].

Our point here is neither to rederive nor improve the results of [1] and [4]. Rather, we wish to emphasize that the context of this paper fits these geometric problems and ties them in with semigroup theory. Moreover, at the geometric level, it is not obvious which boundary conditions are appropriate for the underlying degenerate elliptic equation. The Wentzel condition seems to fit naturally into this framework.

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