Differential and Integral Equations, Volume 2, Number 2, April 1989, pp. 203-215.

# A SPECTRAL MAPPING THEOREM FOR POLYNOMIAL OPERATOR MATRICES

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(Submitted by : J.A. Goldstein)

**Abstract.** Systems of linear evolution equations can be written as a single equation

$$\dot{u}(t) = \mathcal{A}u(t), \qquad (*)$$

where $u$ is a function with values in a product space $E^n$ and $\mathcal{A} = (A_{ij})_{n\times n}$ is an operator matrix. Often the entries $A_{ij}$ are polynomials $p_{ij}(A)$ with respect to a single (unbounded) operator $A$ on $E$ (see, e.g., [1], [2], [3], [6], [11]). In order to solve (*) one has to determine the properties of the operator matrix $\mathcal{A}$. In particular one has to find an appropriate domain $D(\mathcal{A})$ such that $\mathcal{A}$ is closed. This will be discussed in the first part of this paper. Then it is important to compute the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$. One expects a kind of spectral mapping theorem based on the spectrum $\sigma(A)$ of $A$ and the structure of the matrix $(p_{ij})$. We show in Part 2 in which sense such a spectral mapping theorem holds. An application to stability theory, i.e., the computation of an estimate for the spectral bound $s(\mathcal{A})$ concludes this paper. In a subsequent paper we discuss which operator matrices $(p_{ij}(A))$ generate strongly continuous semigroups on $E^n$ and give applications to systems of differential equations.

**1. How to define an operator matrix?** We study $n \times n$ matrices $\mathcal{A}$ whose entries are polynomials $p_{ij}(A)$ in a fixed – possibly unbounded – operator $A$ on some Banach space $E$.

For bounded $A$ it is obvious that the operator matrix $\mathcal{A}$ defines a bounded operator on the product space $\mathcal{E} := E^n$. The situation for unbounded $A$ is more complicated. In fact, the matrix $\mathcal{A}$ only induces a formal map

$$x = (x_1, \cdots, x_n)^t \mapsto \mathcal{A}x = \left( \sum_{i=1}^{n} p_{1i}(A)x_i, \ldots, \sum_{i=1}^{n} p_{ni}(A)x_i \right)^t,$$

but leaves open a wide choice of possible domains. If one wants "nice" properties of $\mathcal{A}$, such as closedness for example, then a more careful analysis is necessary.

Received July 6, 1988.

†This paper is a result of the author's dissertation, which was supported in part by the Deutsche Forschungsgemeinschaft (DFG).

AMS Subject Classifications: 34G10, 35G10, 47D05, 47A10.

Let us consider the following simple example. Take a closed, unbounded operator $A$ with dense domain $D(A)$ in $E$. If

$$\mathcal{A} = \begin{pmatrix} A & -A \\ 0 & 0 \end{pmatrix},$$

then it might seem reasonable to choose $D(\mathcal{A}) = D(A) \times D(A)$. However, with this domain the operator $(\mathcal{A}, D(\mathcal{A}))$ is not closed. To see this take $x \in E \backslash D(A)$ and let $(x_n) \subset D(A)$ be a sequence converging to $x$. Set $y_n := \begin{pmatrix} x_n \\ x_n \end{pmatrix}$. Then $y_n$ converges to $y = \begin{pmatrix} x \\ x \end{pmatrix}$ and $\mathcal{A}y_n = 0$ for all $n \in \mathbb{N}$. Clearly $y \notin D(\mathcal{A})$ and hence $(\mathcal{A}, D(\mathcal{A}))$ is not closed. As we will show later a more reasonable domain for $\mathcal{A}$ is $D(\mathcal{A}) = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in E^2 : x - y \in D(A) \}$.

From this and other examples we see that in general the appropriate domain $D(\mathcal{A})$ of an operator matrix $\mathcal{A}$ will not be "diagonal", this means, the product of domains in $E$. This causes a number of difficulties and will be carefully studied in the sequel.

Throughout this paper $A$ denotes a linear, densely defined operator with domain $D(A)$ in some (complex) Banach space $E$. We also assume that the resolvent set $\rho(A)$ of $A$ is not empty. Before defining operator matrices and their domains we need a functional calculus for certain functions on the spectrum $\sigma(A)$ of $A$.

Let $\mathbb{C}[x]$ be the ring of polynomials in $x$ over $\mathbb{C}$, $\mathbb{C}(x)$ its quotient field and $\mathbb{C}_0[x] := \{p \in \mathbb{C}[x] : p(0) = 0\}$. For an element $f = \frac{p}{q} \in \mathbb{C}(x)$ we define $[f] := \deg(p) - \deg(q)$ where $\deg(p)$ is the degree of the polynomial $p$. For each $\lambda \in \mathbb{C}$ we define the linear polynomial $\xi_\lambda$ by $\xi_\lambda(x) := x - \lambda$. The polynomial $\xi_0$ is denoted simply by $\xi$.

We now consider a class of functions for which a functional calculus for unbounded operators exists.

**Definition 1.1.** *For a given linear operator $A$ we consider the following classes of functions:*

$$\mathcal{G} = \mathcal{G}(A) := \{f \in \mathbb{C}(x) : f \textit{ holomorphic in a neighborhood of } \sigma(A)\},$$
$$\mathcal{N} = \mathcal{N}(A) := \{f \in \mathcal{G} : [f] \leq 0\}.$$

Then $\mathcal{G}$ is a commutative algebra and $\mathcal{N}$ a subalgebra of $\mathcal{G}$. Since $\mathcal{G} = \mathbb{C}_0[x] \oplus \mathcal{N}$ it is possible to decompose every $f \in \mathcal{G}$ into a unique sum of an "unbounded" part in $\mathbb{C}_0[x]$ and a "bounded" part in $\mathcal{N}$. As we have shown in Theorem A.2 it is possible to define $f(A)$ with domain $D(f(A)) := D(A^m)$ for $m = \max\{0, [f]\}$ for all $f \in \mathcal{G}$. Then $f(A)$ is a closed, densely defined operator on $E$. Moreover the following holds (for the details see the Appendix).

> *The functional calculus $\Phi : f \mapsto \Phi(f) := f(A)$ is an "algebra homomorphism" from $\mathcal{G}$ into the set of all linear, densely defined, closed operators on $E$. Moreover, for unbounded $A$, the operator $f(A)$ is bounded if and only if $f \in \mathcal{N}$.*

Concerning matrices we need some further definitions. The ring of all $n \times n$ matrices with entries from a set $L$ is denoted by $M_n(L)$. Thus $M_n(\mathbb{C}[x])$, $M_n(\mathbb{C}(x))$, $M_n(\mathcal{G})$, $M_n(\mathcal{N})$ and so on are all well defined. Clearly $M_n(\mathcal{G})$ and $M_n(\mathcal{N})$ are subrings of $M_n(\mathbb{C}(x))$ and we have $M_n(\mathcal{G}) = M_n(\mathbb{C}_0[x]) \oplus M_n(\mathcal{N})$. Hence, every matrix in $M_n(\mathcal{G})$ can be written as a unique sum of an "unbounded" part in $M_n(\mathbb{C}_0[x])$ and a "bounded" part in $M_n(\mathcal{N})$. For an element $M = \sum_{k=0}^{m} \xi^k \cdot M_k$ of $M_n(\mathbb{C}[x])$ we call $M_m \in M_n(\mathbb{C})$ the *principal coefficient matrix* of $M$.

It is our aim to define a closed operator $M(A)$ for matrices $M = (m_{ik}) \in M_n(\mathcal{G})$. First assume $M \in M_n(\mathcal{N})$. Then by Theorem A.2 the operator matrix $M(A)$ defined by $M(A) :=$

$(m_{ik}(A))$ is bounded on $\mathcal{E}$.
Now take $M \in M_n(\mathcal{G})$ and $m := \max_{ik}[m_{ik}] > 0$. Let $\lambda \in \rho(A)$ and $U_\lambda := \mathrm{diag}(\xi_\lambda^m, \dots, \xi_\lambda^m)$. Then $U_\lambda$ has an inverse in $M_n(\mathcal{N})$ and $M = U_\lambda(MU_\lambda^{-1}) =: U_\lambda B_\lambda$ where $B_\lambda \in M_n(\mathcal{N})$. Hence, $B_\lambda(A)$ is bounded on $\mathcal{E}$. On the other hand $U_\lambda(A)$ with domain $D(U_\lambda(A)) = D(A^m)^n$ is invertible and therefore closed. In the following definition we use this decomposition in order to define $\mathcal{A} := M(A)$ as a closed operator on $\mathcal{E}$.

**Definition 1.2.** *Let $A$ be a densely defined operator with non-empty resolvent set $\rho(A)$. For a matrix $M = (m_{ik}) \in M_n(\mathcal{G})$ decomposed as $M = U_\lambda \cdot B_\lambda$ for some $\lambda \in \rho(A)$ (see above) the operator matrix $\mathcal{A}_\lambda$ is defined by*

$$\mathcal{A}_\lambda := U_\lambda(A) \cdot B_\lambda(A)$$

*with domain*

$$D(\mathcal{A}_\lambda) := \{x \in \mathcal{E} : B_\lambda(A)x \in D(U_\lambda(A))\}.$$

*Here $D(U_\lambda(A)) = D(A^m)^n$ where $m := \max\{0, [m_{ik}]; i, k = 1, \dots, n\}$.*

We show that the above definition is independent of $\lambda \in \rho(A)$ and yields a closed operator.

**Proposition 1.3.** *Let $\lambda \in \rho(A)$ and $\mathcal{A}_\lambda$ be defined as in Definition 1.2. Then*

(i) *$(\mathcal{A}_\lambda, D(\mathcal{A}_\lambda))$ is a closed, densely defined operator and*
(ii) *$(\mathcal{A}_\lambda, D(\mathcal{A}_\lambda)) = (\mathcal{A}_\mu, D(\mathcal{A}_\mu))$ for all $\mu \in \rho(A)$.*

**Proof:** (i) The operator $U_\lambda(A)$ is invertible in $\mathcal{L}(\mathcal{E})$ and therefore closed. Thus the product of the operator $U_\lambda(A)$ with the bounded operator $B_\lambda(A)$ is closed (see Chapter III.5, [7]). Since $D(A^l)^n \subset D(\mathcal{A})$ for all $l \in \mathbb{N}$, $l > \max_{i,k}[m_{ik}]$ we have shown (i).
(ii) Take $\mu \in \rho(A)$ and define $C_{\lambda,\mu} := \mathrm{diag}\big((\frac{\xi_\lambda}{\xi_\mu})^m, \cdots, (\frac{\xi_\lambda}{\xi_\mu})^m\big)$. Apparently $C_{\lambda,\mu}$ and its inverse $C_{\mu,\lambda}$ are elements of $M_n(\mathcal{N})$. From Theorem A.2 it follows that $U_\lambda(A)C_{\mu,\lambda}(A) = U_\mu(A)$ and $C_{\lambda,\mu}(A)B_\lambda(A) = B_\mu(A)$. Hence,

$$\begin{aligned}\mathcal{A}_\lambda &= U_\lambda(A)B_\lambda(A) = (U_\lambda(A)C_{\mu,\lambda}(A)) \cdot (C_{\lambda,\mu}(A)B_\lambda(A)) \\ &= U_\mu(A)B_\mu(A) = \mathcal{A}_\mu.\end{aligned}$$

∎

Thus, for all $M \in M_n(\mathcal{G})$ we can define the operator matrix $\mathcal{A} = M(A)$ by $M(A) := U_\lambda(A)B_\lambda(A)$ for some $\lambda \in \rho(A)$. By $\Phi_n$ we denote the *functional calculus*

$$\Phi_n : M \mapsto M(A), \quad M \in M_n(\mathcal{G}).$$

We remark that $\Phi_n$ assumes its values in the set of all linear, densely defined, closed operators on $\mathcal{E}$.

**Remark.** Using the same construction we can also define non-quadratic operator matrices. For example let $R = (r_1, \dots, r_n) \in (\mathbb{C}[x])^n$. If $\lambda \in \rho(A)$, then

$$R = \underbrace{\xi_\lambda^m}_{\in \mathbb{C}[x]} \cdot \underbrace{\left(\frac{r_1}{\xi_\lambda^m}, \dots, \frac{r_n}{\xi_\lambda^m}\right)}_{\in \mathcal{N}^n}$$

for $m := \max\{[r_1], \dots, [r_n]\}$. Since $\xi_\lambda^m(A)$ is closed and $\big(\frac{r_1}{\xi_\lambda^m}(A), \dots, \frac{r_n}{\xi_\lambda^m}(A)\big)$ is bounded the operator

$$R(A) := \xi_\lambda^m(A)\Big(\frac{r_1}{\xi_\lambda^m}(A), \dots, \frac{r_n}{\xi_\lambda^m}(A)\Big)$$

with the domain

$$D(R(A)) := \left\{ x \in E^n : \left( \frac{r_1}{\xi_\lambda^m}(A), \dots, \frac{r_n}{\xi_\lambda^m}(A) \right) x \in D(A^m) \right\}$$

is closed.

The next result is central for the following. It shows that in some sense the functional calculus $\Phi_n$ is an "algebra" homomorphism.

**Proposition 1.4.** *Let $S$, $T$, $L$, $M \in M_n(\mathcal{G})$ and set $\mathcal{S} := S(A)$, $\mathcal{T} := T(A)$, $\mathcal{L} := L(A)$ and $\mathcal{A} := M(A)$. Then*

(i) $(L+M)(A) \supseteq \mathcal{L} + \mathcal{A}$,

(ii) $(L+M)(A) = \mathcal{L} + \mathcal{A}$ *if* $L \in M_n(\mathcal{N})$,

(iii) $(S \cdot M \cdot T)(A) \supseteq \mathcal{S} \cdot \mathcal{A} \cdot \mathcal{T}$,

(iv) $(S \cdot M \cdot T)(A) = \mathcal{S} \cdot \mathcal{A} \cdot \mathcal{T}$ *if* $S$, $S^{-1}$, $T \in M_n(\mathcal{N})$.

**Proof:** (i) and (iii) follow immediately from Theorem A.2. Since $\mathcal{L}$ is bounded for $L \in M_n(\mathcal{N})$, (ii) is an easy consequence of (i).
It remains to prove (iv). It suffices to show that

$$(SM)(A) = \mathcal{S}\mathcal{A} \text{ if } S, S^{-1} \in M_n(\mathcal{N}) \tag{1.1}$$

and $(MT)(A) = \mathcal{A}\mathcal{T}$ if $T \in M_n(\mathcal{N})$. The second assertion is clear by (iii) and Definition 1.2, thus consider (1.1). Let $\mathcal{A} = \mathcal{U} \cdot \mathcal{B}$, where $\mathcal{U} = \mathcal{U}_\lambda = \operatorname{diag}\big((A-\lambda)^m, \dots, (A-\lambda)^m\big)$ and $\mathcal{B} = \mathcal{B}_\lambda \in \mathcal{L}(\mathcal{E})$ as in Definition 1.2. We only have to show that $\mathcal{S} \cdot \mathcal{U} = \mathcal{U} \cdot \mathcal{S}$, i.e., $D(\mathcal{U}) = D(\mathcal{U}\mathcal{S})$. Since $S^{-1} \in M_n(\mathcal{N})$ the operator $\mathcal{S}$ is invertible in $\mathcal{L}(\mathcal{E})$ and we obtain $\mathcal{S}D(\mathcal{U}) \subset D(\mathcal{U})$ and $\mathcal{S}^{-1}D(\mathcal{U}) \subset D(\mathcal{U})$. Hence, $D(\mathcal{U}) = \mathcal{S}^{-1}D(\mathcal{U}) = D(\mathcal{U}\mathcal{S})$. ∎

**Lemma 1.5.** *Let $M \in M_n(\mathcal{G})$. Then there exist invertible matrices $S$, $T \in M_n(\mathcal{N})$ such that $SMT = D + N$. Here $D \in M_n(\mathbb{C}_0[x])$ is diagonal and $N$ belongs to $M_n(\mathcal{N})$.*

**Proof:** After multiplication of $M$ with suitable permutation matrices we can suppose $[m_{11}] = \max_{i,k}[m_{ik}]$. By Proposition A.3 there are elements $\alpha_2, \dots, \alpha_n$, $\beta_2, \dots, \beta_n \in \mathcal{N}$ such that $\alpha_k m_{11} + m_{k1} \in \mathcal{N}$ and $\beta_k m_{11} + m_{1k} \in \mathcal{N}$ for $k = 2, \dots n$. Set

$$S_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ \alpha_2 & \ddots & & \vdots \\ \vdots & 0 & \ddots & 0 \\ \alpha_n & & & 1 \end{pmatrix}, \quad T_1 := \begin{pmatrix} 1 & \beta_2 & \dots & \beta_n \\ 0 & \ddots & 0 & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{pmatrix} \in M_n(\mathcal{N})$$

and $M_1 := S_1 M T_1 = (m_{ij}^{(1)})$. Then $m_{1j}^{(1)}$, $m_{j1}^{(1)} \in \mathcal{N}$ for $j \neq 1$.

Now we apply the same procedure to the truncated matrix $(m_{ij}^{(1)})_{i,j=2}^n$ and obtain matrices $\tilde{S}_2$ and $\tilde{T}_2 \in M_{n-1}(\mathcal{N})$. Next define

$$S_2 := \begin{pmatrix} 1 & 0 \\ 0 & \tilde{S}_2 \end{pmatrix} \in M_n(\mathcal{N}) \quad \text{and} \quad T_2 := \begin{pmatrix} 1 & 0 \\ 0 & \tilde{T}_2 \end{pmatrix} \in M_n(\mathcal{N}).$$

Iterating this $(n-1)$-times yields matrices

$$S := S_{n-1} \cdot \ldots \cdot S_1 \in M_n(\mathcal{N})$$

and

$$T := T_1 \cdot \ldots \cdot T_{n-1} \in M_n(\mathcal{N}).$$

Let $L := (l_{ij}) = SMT$; then by construction $l_{ij} \in \mathcal{N}$ for $i \neq j$. Therefore we can decompose $L \in M_n(\mathcal{G}) = M_n(\mathbb{C}_0[x]) \oplus M_n(\mathcal{N})$ into a sum $L = D + N$ where $D \in M_n(\mathbb{C}_0[x])$ is diagonal and $N$ belongs to $M_n(\mathcal{N})$. Since $\det S = \det T = \pm 1$ both $S$ and $T$ are invertible in $M_n(\mathcal{N})$ which completes the proof. ∎

**Corollary 1.6.** *Let $M = (m_{ik}) \in M_n(\mathcal{G})$ and define $\mathcal{A} := M(A)$. If $l \geq \max\{0, [m_{ik}] : 1 \leq i, k \leq n\}$, then $D(A^l)^n \subset D(\mathcal{A})$ is a core of $\mathcal{A}$. In particular in the situation of Proposition 1.4 we have*

(i′) $(L + M)(A) = \overline{\mathcal{L} + \mathcal{A}}$,

(iii′) $(SMT)(A) = \overline{\mathcal{SAT}}$,

(iv′) $(SMT)(A) = \mathcal{SAT}$ *if* $S^{-1}, T \in M_n(\mathcal{N})$.

**Proof:** First we show that $G := D(A^l)^n$ is a core of $\mathcal{A}$. Let $SMT = D + N$ as in the previous lemma. Define $\mathcal{S} := S(A)$, $\mathcal{T} := T(A)$ and $\mathcal{D} := \operatorname{diag}(d_1(A), \ldots, d_n(A))$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then by Proposition 1.4 we have $\mathcal{SAT} = \mathcal{D} + N(A)$. Since $\mathcal{T}$ is bounded, invertible and commutes with $\operatorname{diag}(R(\lambda, A)^l, \ldots, R(\lambda, A)^l)$ for $\lambda \in \rho(A)$ we obtain that $G = \mathcal{T}^{-1}G$ is contained in the domain $D(\mathcal{D})$ of $\mathcal{D}$. Using the fact that $\mathcal{D}$ is diagonal and $N(A)$ is bounded it is clear that $G$ is a core of $\mathcal{D} + N(A) = \mathcal{SAT}$. This implies that $G$ is a core of $\mathcal{AT}$. Next we show that $\mathcal{T}G$ is a core of $\mathcal{A}$: Take $x \in D(\mathcal{A})$, then there exists a sequence $(y_n) \subset G$ such that $y_n \to \mathcal{T}^{-1}x$ and $(\mathcal{AT})y_n \to (\mathcal{AT})\mathcal{T}^{-1}x = \mathcal{A}x$ as $n \to \infty$. Now define $x_n := \mathcal{T}y_n$, then $(x_n) \subset \mathcal{T}G$, $x_n \to x$ and $\mathcal{A}x_n = \mathcal{AT}y_n \to \mathcal{A}x$. Therefore $\mathcal{T}G = G$ is a core of $\mathcal{A}$.
Next consider (i′), (iii′) and (iv′). Since $D(A^l)^n \subset D(\mathcal{L} + \mathcal{A})$ and $D(A^l)^n \subset D(\mathcal{SAT})$ for $l \in \mathbb{N}$ big enough we obtain from Proposition 1.4 the assertions (i′) and (iii′). To prove (iv′) note that $\mathcal{SA}$ is closed whenever $\mathcal{S}$ is invertible with bounded inverse. (see [7], Chapter III.2). ∎

In the next step we want to obtain more precise information on the size of the domain $D(\mathcal{A})$. It is clear that $D(\mathcal{A})$ depends only on the unbounded part of $\mathcal{A} = M(A)$. So let us assume $M \in M_n(\mathbb{C}_0[x])$. Then it is possible to write $M = \sum_{k=1}^m \xi^k \cdot M_k$ where $M_k \in M_n(\mathbb{C})$ and $m = \max_{i,k}[m_{ik}]$. After the canonical identification of $E^n$ with $E \otimes \mathbb{C}^n$ we obtain $\mathcal{A} = \sum_{k=1}^m A^k \otimes M_k$. Now the following holds.

**Proposition 1.7.** *Let $M = \sum_{k=1}^m \xi^k \cdot M^k \in M_n(\mathbb{C}[x])$ and $\mathcal{A} := M(A)$. Then*

$$D := \bigcap_{k=1}^{m} D(A^k \otimes M_k) \subset D(\mathcal{A}) \subset D(A \otimes M_m).$$

*If the principal coefficient matrix $M_m$ of $M$ is invertible, then $D(\mathcal{A}) = D(A^m)^n$.*

**Proof:** Assuming $0 \in \rho(A)$ we decompose

$$\mathcal{A} = (A^m \otimes Id) \cdot \Big( \sum_{k=1}^{m} A^{k-m} \otimes M_k \Big) = \mathcal{U} \cdot \mathcal{B}$$

as in Definition 1.2 and obtain $D(\mathcal{A}) = \{x \in \mathcal{E} : \mathcal{B}x \in D(A^m)^n\}$. First we show that $\mathcal{B}D \subset D(A^m)^n$. For $x \in D$ it follows that $(Id \otimes M_k)x \in D(A^k)^n$ for all $k = 1, \ldots, m$. Therefore $(A^{k-m} \otimes M_k)x \in D(A^m)^n$ and $\mathcal{B}x \in D(A^m)^n$. To show the second inclusion take $x \in D(\mathcal{A})$. Then we have

$$\underbrace{\sum_{k=1}^{m-1} (A^{k-m} \otimes M_k)x}_{\in D(A)^n} + (Id \otimes M_m)x \quad \in D(A^m)^n.$$

Therefore $(Id \otimes M_m)x \in D(A)^n$, i.e., $x \in D(A \otimes M_m)$.

Assume now $M_m$ to be invertible and $x \in D(\mathcal{A})$. By the same arguments as above we obtain $(Id \otimes M_m)x \in D(A)^n$ and conclude, after left multiplication by $(Id \otimes M_m^{-1})$, that $x \in D(A)^n$. Let $x \in D(A^l)^n$ for some $l = 1, \dots, m-1$. Then from

$$(A^l \otimes Id)\mathcal{B}x = \underbrace{\sum_{k=1}^{m-1}(A^{k+l-m} \otimes M_k)x}_{\in D(A)^n} + (A^l \otimes M_m)x \quad \in D(A^{m-l})^n$$

it follows that $(A^l \otimes M_m)x \in D(A)^n$. As above we deduce $x \in D(A^{l+1})^n$ and by induction $D(\mathcal{A}) = D(A^m)^n$. ∎

**Example.** Take

$$M = \begin{pmatrix} \xi^2 & \xi \\ 0 & 0 \end{pmatrix} = \xi^2 \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=:M_2} + \xi \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=:M_1}$$

and

$$\mathcal{A} = \begin{pmatrix} A^2 & A \\ 0 & 0 \end{pmatrix}.$$

An easy computation shows

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in E^2 : x \in D(A),\ Ax + y \in D(A) \right\}.$$

Since $D(A \otimes M_2) = D(A) \times E$ and $D(A^2 \otimes M_2) \cap D(A \otimes M_1) = D(A^2) \times D(A)$, we have

$$D = \Big(D(A^2 \otimes M_2) \cap D(A \otimes M_1)\Big) \neq D(\mathcal{A}) \neq D(A \otimes M_2).$$

**2. The spectral mapping theorem for operator matrices.** In this part we will calculate the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ with domain $D(\mathcal{A})$ as defined in Part 1. Moreover we give a formula for the resolvent $R(\lambda, \mathcal{A})$ if $\lambda \in \rho(\mathcal{A})$ and derive an estimate for the spectral bound $\mathrm{s}(\mathcal{A})$ of $\mathcal{A}$.

The main tool in the sequel is the operator-valued determinant $\Delta(\mathcal{A})$ which we define first. Recall that for the fixed operator $A$ on $E$ we have

$$\mathcal{G} = \{f \in \mathbb{C}(x) : f \text{ holomorphic in a neighborhood of } \sigma(A)\} \text{ and } \mathcal{N} = \{f \in \mathcal{G} : [f] \leq 0\}.$$

**Definition 2.1.** *Consider $M \in M_n(\mathcal{G})$ and its determinant $detM$ which is an element of $\mathcal{G}$. If $\mathcal{A} = M(A)$, then we call the operator*

$$\Delta(\mathcal{A}) := (detM)(A)$$

*on $E$ the determinant of $\mathcal{A}$.*

Next we will show that for bounded $A$ the operator matrix $\mathcal{A}$ is invertible in $\mathcal{L}(\mathcal{E})$ if and only if $\Delta(\mathcal{A})$ is invertible in $\mathcal{L}(E)$. More generally we consider bounded operator matrices with commuting entries.

**Lemma 2.2.** *Let $\mathcal{B} = (B_{ik}) \in \mathcal{L}(\mathcal{E})$ be an operator matrix with commuting entries. Then $\mathcal{B}$ is invertible in $\mathcal{L}(\mathcal{E})$ if and only if $\det\mathcal{B}$ is invertible in $\mathcal{L}(E)$.*

**Proof:** Let $\det\mathcal{B}$ be invertible in $\mathcal{L}(E)$. Then $\mathcal{B}^{-1} = (\det\mathcal{B})^{-1}\,\mathrm{Adj}(\mathcal{B})$ by [8], Proposition 8, p.334, where $\mathrm{Adj}(\mathcal{B})$ denotes the co-factor matrix of $\mathcal{B}$.

Conversely assume $\mathcal{B}$ to be invertible with inverse $\mathcal{B}^{-1} = (J_{ik}) \in \mathcal{L}(\mathcal{E})$. First we show that all entries of $\mathcal{B}$ and $\mathcal{B}^{-1}$ commute. Take $j, l \in \{1, \dots, n\}$ and define $\mathcal{D}_1 := \mathrm{diag}(B_{jl}, \dots, B_{jl}) \in \mathcal{L}(\mathcal{E})$. Since the entries of $\mathcal{B}$ commute we have $\mathcal{D}_1\mathcal{B} = \mathcal{B}\mathcal{D}_1$. This implies $\mathcal{B}^{-1}\mathcal{D}_1 = \mathcal{D}_1\mathcal{B}^{-1}$, i.e., every entry of $\mathcal{B}^{-1}$ commutes with $B_{jl}$. Thus every entry of $\mathcal{B}$ commutes with all entries of $\mathcal{B}^{-1}$. Using this we obtain $\mathcal{D}_2\mathcal{B} = \mathcal{B}\mathcal{D}_2$ where $\mathcal{D}_2 := \mathrm{diag}(J_{jl}, \dots, J_{jl})$ for some fixed $j, l \in \{1, \dots, n\}$. This yields $\mathcal{B}^{-1}\mathcal{D}_2 = \mathcal{D}_2\mathcal{B}^{-1}$, i.e., the entries of $\mathcal{B}^{-1}$ commute. Now by [8], Proposition 7, p.334 we conclude $Id = \det(\mathcal{B})\det(\mathcal{B}^{-1}) = \det(\mathcal{B}^{-1})\det(\mathcal{B})$, i.e., $\det\mathcal{B}$ is invertible with inverse $\det(\mathcal{B}^{-1})$. ∎

For $A$ unbounded the situation is more complicated. Take, for example, $M = \begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{A} = \begin{pmatrix} A & Id \\ Id & 0 \end{pmatrix}$. Here $\Delta(\mathcal{A}) = -Id$ is invertible while the "formal" inverse $\begin{pmatrix} 0 & Id \\ Id & -A \end{pmatrix}$ is unbounded and hence $\mathcal{A}$ does not have a bounded inverse. The following theorem shows how to deal with this situation.

**Theorem 2.3.** *Let $A$ be a linear unbounded operator on $E$. For $M \in M_n(\mathcal{G})$ and $\mathcal{A} := M(A)$ the following assertions are equivalent.*

(a) *$\mathcal{A}$ is invertible in $\mathcal{L}(\mathcal{E})$*

(b) *$M$ is invertible in $M_n(\mathcal{N})$*

(c) *(i) $\frac{1}{\det M} \in \mathcal{N}$ and*
*(ii) $\lim_{\alpha\to\infty} M^{-1}(\alpha)$ exists in $M_n(\mathbb{C})$.*

**Proof:** (a) $\iff$ (b): Let $SMT = D+N$ be a decomposition as in Lemma 1.5. Since $S$ and $T$ are invertible in $M_n(\mathcal{N})$ the bounded operator matrices $S(A)$ and $T(A)$ are also invertible with bounded inverse. From this and Proposition 1.4 we deduce that $\mathcal{A}$ is invertible if and only if $S(A)\mathcal{A}T(A) = (SMT)(A)$ is invertible in $(\mathcal{E})$. Therefore we may restrict ourselves to consider $\mathcal{A} = M(A)$ where $M$ is given by a sum $M = D + N$. Here $N$ is contained in $M_n(\mathcal{N})$ and $D = (d_{ik}) \in M_n(\mathbb{C}_0[x])$ is diagonal.

Take some fixed number $\lambda \in \rho(A)$. Then $U := \mathrm{diag}(\xi_\lambda^{\nu_1}, \dots, \xi_\lambda^{\nu_n})$ where $\nu_k := [d_{kk}]$ for $k = 1, \dots, n$ is invertible in $M_n(\mathcal{N})$. By definition of $U$ we have $B := MU^{-1} \in M_n(\mathcal{N})$ and $D(U(A)) = D(\mathcal{A})$. Hence, $\mathcal{A} = B(A)U(A)$ by Proposition 1.4 (iii). Since $U(A)$ is invertible in $\mathcal{L}(\mathcal{E})$ this implies that $\mathcal{A}$ is invertible if and only if $B(A)$ is invertible. Using the fact that all entries of $B(A)$ are bounded and commute we obtain from Lemma 2.2, that $B(A)$ is invertible if and only if $\Delta(B(A)) = (\det(B))(A)$ is invertible in $\mathcal{L}(E)$. By Theorem A.2 (v) this is equivalent to $\frac{1}{\det(B)} \in \mathcal{N}$. Thus the inversion formula for matrices implies that $B(A)$ is invertible in $\mathcal{L}(\mathcal{E})$ if and only if $B$ is invertible in $M_n(\mathcal{N})$. The only thing that remains to be proved is that $B$ is invertible in $M_n(\mathcal{N})$ if and only if $M = BU$ is invertible in $M_n(\mathcal{N})$.

Let $B^{-1} \in M_n(\mathcal{N})$. Then $M^{-1} = U^{-1}B^{-1} \in M_n(\mathcal{N})$. For the converse implication note that $[m_{kk}] = \nu_k$ where $m_{kk}$ denotes the $k$-th diagonal entry of $M$. If $M^{-1}$ exists it is given by $M^{-1} = (\det M)^{-1} \cdot (\tilde{m}_{ik})$ where $(\tilde{m}_{ik}) = \mathrm{Adj}(M)$. Here $\mathrm{Adj}(M)$ denotes the co-factor matrix of $M$ (see[8], p.334). Since $[\det M] = \sum_{k=1}^n \nu_k$ and $[\tilde{m}_{ik}] \le \sum_{j\neq i,k} \nu_j$ we conclude $B^{-1} = U \cdot M^{-1} \in M_n(\mathcal{N})$.

(b) $\iff$ (c): First assume (b). The assertion (c.i) follows from the multiplicativity of the determinant while (c.ii) is clear by the definition of $\mathcal{N}$. If (c) is true, then $M$ is invertible in $M_n(\mathcal{G})$ by (c.i). The second hypothesis now implies $M^{-1} \in M_n(\mathcal{N})$. ∎

From the proof of Theorem 2.3 we also obtain an explicit matrix representation for the resolvent $R(\mu_0, \mathcal{A})$.

**Corollary 2.4.** *For $\mu_0 \in \rho(\mathcal{A})$ the resolvent $R(\mu_0, M)$ is an element of $M_n(\mathcal{N})$ and*

$$R(\mu_0, \mathcal{A}) = R(\mu_0, M)(A).$$

In the above counter-example $\mathcal{A} = \begin{pmatrix} A & Id \\ Id & 0 \end{pmatrix}$ we have $\det M = -1$ which is invertible in $\mathcal{N}$ but $M^{-1}(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix}$ has no limit for $\alpha \to \infty$. Still, condition 2.3(c) seems to be artificial and not very useful. In order to arrive at a concrete description of the spectrum $\sigma(\mathcal{A})$ we first state some consequences of the above considerations.

**Corollary 2.5.** *Let $A$ be a linear unbounded operator on $E$. For $M \in M_n(\mathcal{G})$ and $\mathcal{A} := M(A)$ the following assertions are equivalent.*

(a) $\mu_0 \in \rho(\mathcal{A})$
(b) *(i)* $\Delta(\mu_0 - \mathcal{A})$ *is invertible in* $\mathcal{L}(\mathcal{E})$ *and*
*(ii)* $Q := \lim_{\alpha\to\infty} R(\mu_0, M)(\alpha)$ *exists in* $M_n(\mathbb{C})$.

*If one of the assertions (a) or (b) is valid, then $Q = \lim_{n\to\infty} R(\mu_0, M)(\alpha_n)$ for every sequence $(\alpha_n) \subset \mathbb{C}$ with limit $\infty$.*

**Proof:** We have to show that $\Delta(\mu_0 - \mathcal{A})$ is invertible in $\mathcal{L}(E)$ if and only if $\frac{1}{\det(\mu_0 - M)} \in \mathcal{N}$. This is valid by the definition of the determinant $\Delta$ and Theorem A.2 (v). ∎

**Lemma 2.6.** *Let $S \in M_n(\mathbb{C}(x))$. Then the matrix $S$ is invertible in $M_n(\mathbb{C}(x))$ if and only if there exists some $\alpha \in \mathbb{C}$ such that the complex matrix $S(\alpha)$ is invertible in $M_n(\mathbb{C})$.*

**Proof:** Let $S$ be invertible, i.e., $\det(S) \neq 0$. Then there are infinitely many $\alpha$ such that $(\det(S))(\alpha) \neq 0$. Since $\det(S)$ has, as a non-zero rational function, only finitely many zeros and poles we may assume that $S(\alpha)$ is defined. Using $\det(S(\alpha)) = (\det(S))(\alpha) \neq 0$ we obtain the assertion. Conversely assume that there is some $\alpha \in \mathbb{C}$ such that the complex matrix $S(\alpha)$ is invertible. Then from $0 \neq \det(S(\alpha)) = (\det(S))(\alpha)$ we conclude that $\det(S) \neq 0$, hence $S$ is invertible in $M_n(\mathbb{C}(x))$. ∎

In order to handle condition 2.3(c.ii) we now give a different characterization of this property.

**Lemma 2.7.** *Let $\mu_0 \in \mathbb{C}$ such that $Q := \lim_{\alpha\to\infty} R(\mu_0, M)(\alpha)$ exists. For $\lambda_0 \in \mathbb{C}$ the following assertions are equivalent.*

(a) *(i)* $(\lambda_0 - M)$ *is invertible in* $M_n(\mathbb{C}(x))$ *and*
(ii) $\lim_{\alpha\to\infty} R(\lambda_0, M)(\alpha)$ *exists.*
(b) $\lambda_0 \notin \{\mu_0 - \lambda^{-1} : \lambda \in \sigma(Q),\ \lambda \neq 0\}$.

**Proof:** The lemma holds for $\lambda_0 = \mu_0$, hence assume $\lambda_0 \neq \mu_0$.
(a) ⇒ (b): We have to show $(\mu_0 - \lambda_0)^{-1} \notin \sigma(Q)$. Using the resolvent equation one easily verifies that $(\mu_0 - \lambda_0) + (\mu_0 - \lambda_0)^2 R(\lambda_0, M)$ gives the inverse of $(\mu_0 - \lambda_0)^{-1} - R(\mu_0, M)$. Hence assumption (a) implies that

$$\begin{aligned}(\mu_0 - \lambda_0) + (\mu_0 - \lambda_0)^2 \lim_{\alpha\to\infty} R(\lambda_0, M)(\alpha) &= \lim_{\alpha\to\infty} \left((\mu_0 - \lambda_0)^{-1} - R(\mu_0, M)(\alpha)\right)^{-1} \\ &= R\left((\mu_0 - \lambda_0)^{-1}, Q\right)\end{aligned}$$

exists. This shows $(\mu_0 - \lambda_0)^{-1} \notin \sigma(Q)$.
(b) $\Rightarrow$ (a): By hypothesis $\big((\mu_0 - \lambda_0)^{-1} - Q\big)$ is invertible, hence $(\mu_0 - \lambda_0)^{-1} - R(\mu_0, M(\alpha))$ is invertible in $M_n(\mathbb{C})$ for $\alpha$ sufficiently large. Now the identity

$$(\mu_0 - \lambda_0)^{-2} \left((\mu_0 - \lambda_0) - R(\mu_0, M)(\alpha)\right)^{-1} - (\mu_0 - \lambda_0) = R(\lambda_0, M(\alpha))$$

combined with Lemma 2.6 and the continuity of the inversion implies (a). ∎

The above lemma yields a characterization of $\sigma(\mathcal{A})$ in the form of a *spectral mapping theorem*. To be more precise, we show that $\sigma(\mathcal{A})$ consists of a set $\sigma_1(\mathcal{A})$ obtained via a "spectral mapping" from the spectra $\sigma(\mathcal{A})$ and $\sigma(M(\alpha))$ and an "exceptional" set.

**Theorem 2.8.** *Let $M \in M_n(\mathcal{G})$ and $\mathcal{A} := M(A)$. If for some $\mu_0 \in \mathbb{C}$ the limit $Q := \lim_{\alpha\to\infty} R(\mu_0, M(\alpha)) \in M_n(\mathbb{C})$ exists (e.g., $\mu_0 \in \rho(\mathcal{A})$), then the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is given by*

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A}),$$

*where*

$$\sigma_1(\mathcal{A}) := \bigcup_{\mu\in\sigma(A)} \sigma(M(\mu)) = \sigma(M(\sigma(A))$$

*and*

$$\sigma_2(\mathcal{A}) := \left\{\mu_0 - \lambda^{-1} : \lambda \in \sigma(Q),\, \lambda \neq 0\right\}.$$

**Proof:** The existence of $Q$ for $\mu_0 \in \rho(\mathcal{A})$ was already shown in Corollary 2.5. Lemma 2.7 in combination with Corollary 2.5 shows that $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda \in \sigma_2(\mathcal{A})$ or $\Delta(\lambda - \mathcal{A})$ is not invertible in $\mathcal{L}(E)$. This last property is equivalent to $0 \in \sigma(p_\lambda(A))$ where $p_\lambda := \det(\lambda - M) \in \mathcal{G}$. By the spectral mapping theorem for elements in $\mathcal{G}$ (see Theorem A.2 (vi)) we obtain

$$\sigma\left(p_\lambda(A)\right) = p_\lambda(\sigma(A)) = \left\{\det\left(\lambda - M(\mu)\right) : \mu \in \sigma(A)\right\}.$$

But $0 \in \bigcup_{\mu\in\sigma(A)} \sigma(\lambda - M(\mu))$ is equivalent to $\lambda \in \bigcup_{\mu\in\sigma(A)} \sigma(M(\mu)) = \sigma_1(\mathcal{A})$. ∎

As a first observation we state that the spectrum of $\mathcal{A}$ consists either of the whole complex plane $\mathbb{C}$ or of the set $\sigma_1(\mathcal{A})$ and at most finitely many "exceptional" points in $\sigma_2(\mathcal{A})$. In fact, $|\sigma_2(\mathcal{A})| \leq n - 1$ since $Q$ is never invertible. In the following corollaries we give more precise information on the location of $\sigma_2(\mathcal{A})$.

**Corollary 2.9.** *If the principal coefficient matrix $M_m$ of the unbounded part of $M$ is invertible, then $\sigma_2(\mathcal{A}) = \emptyset$ and therefore the following spectral mapping formula holds:*

$$\sigma(\mathcal{A}) = \sigma\left(M\left(\sigma(A)\right)\right).$$

**Proof:** It suffices to show that $\lim_{\alpha\to\infty} R(0, M)(\alpha) = 0$. Using simple matrix theory it follows from the hypothesis that $[\det M] = n \cdot m$. On the other hand, every element of $\operatorname{Adj}(M)$, the co-factor matrix of $M$ (see [8]), has degree at most $n \cdot (m-1)$ and we conclude that

$$\lim_{\alpha\to\infty} R(0, M)(\alpha) = \Big((\det M)^{-1} \operatorname{Adj}(M)\Big)(\alpha) = 0. \quad ∎$$

**Corollary 2.10.** *If $\sigma(A)$ is bounded and $\lim_{\alpha\to\infty} R(\mu_0, M(\alpha))$ exists for some $\mu_0 \in \mathbb{C}$, then $\sigma(\mathcal{A})$ is also bounded. More precisely, $\sigma(\mathcal{A})$ is the union of $\sigma(M(\sigma(A)))$ and finitely many isolated points.*

**Proof:** This follows from the observation that $\sigma_1(\mathcal{A})$ is compact whenever $\sigma(A)$ is bounded.

**Corollary 2.11.** *If $\sigma(A)$ is unbounded and $Q := \lim_{\alpha\to\infty} R(\mu_0, M(\alpha))$ exists for some $\mu_0 \in \mathbb{C}$, then*

$$\sigma(\mathcal{A}) = \overline{\sigma_1(\mathcal{A})}.$$

**Proof:** Let $\lambda_0 \in \sigma_2(\mathcal{A})$, i.e., $(\mu_0 - \lambda_0)^{-1} \in \sigma(Q)$. Since $\sigma(A)$ is unbounded there exists a sequence $(\lambda_k) \subset \sigma(A)$ converging to $\infty$. Then $Q_k := R(\mu_0, M)(\lambda_k)$ exists for all except finitely many $k$ and converges to $Q$. By Theorem 5.1, p.107 [7] we can find a sequence $(\mu_k) \subset \mathbb{C}$ such that $\mu_k \in \sigma(Q_k) = \sigma(\mu_0 - M(\lambda_k))^{-1}$ and $\mu_k \to (\mu_0 - \lambda_0)^{-1}$. Therefore $(\mu_0 - \mu_k^{-1}) \in \sigma(M(\lambda_k)) \subset \sigma_1(\mathcal{A})$ and $(\mu_0 - \mu_k^{-1})$ converges to $\lambda_0$ for $k \to \infty$. ∎

This last corollary is particularly useful since it allows us to determine the spectrum of $\mathcal{A}$ by calculating the eigenvalues of the matrices $M(\mu)$, $\mu \in \sigma(A)$, without worrying about the "exceptional" spectral values in $\sigma_2(\mathcal{A})$.

Next we consider some typical examples. For $M \in M_n(\mathbb{C}[x])$ we use as usual the notation $\mathcal{A} = M(A)$. Moreover we assume that $A$ is unbounded.

**Examples.** (1) Let

$$M = \begin{pmatrix} \xi^2 & \xi^3 \\ 1 & \xi \end{pmatrix} \in M_2(\mathbb{C}[x]).$$

Since

$$R(\lambda, M) = \frac{1}{\lambda^2 - \lambda(\xi^2 + \xi)} \begin{pmatrix} \xi & -\xi^3 \\ -1 & \xi^2 \end{pmatrix}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ there is no $\lambda \in \mathbb{C}$ such that $\lim_{\alpha\to\infty} R(\lambda, M)(\alpha)$ exists. Using Corollary 2.5 we conclude that $\sigma(\mathcal{A}) = \mathbb{C}$.

(2) Let

$$M = \begin{pmatrix} \xi^3 & \xi^2 \\ \xi & -\xi^3 \end{pmatrix} \in M_2(\mathbb{C}[x]).$$

Since the principal coefficient matrix

$$M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible we obtain from Theorem 2.8 that

$$\sigma(\mathcal{A}) = \left\{\pm\sqrt{\mu^6 + \mu^3} : \mu \in \sigma(A)\right\}.$$

(3) Let $q \in \mathbb{C}[x]$, $q(x) = \sum_{k=0}^{n} a_k x^k$. For $\epsilon \in \mathbb{C}$ define

$$M_\epsilon := \begin{pmatrix} 0 & 1 \\ q & \epsilon\xi^n \end{pmatrix} \quad \text{and} \quad \mathcal{A}_\epsilon := M_\epsilon(A).$$

Then for $\epsilon \neq 0$ we have

$$\sigma_1(\mathcal{A}_\epsilon) = \left\{\frac{\epsilon\mu^n \pm \sqrt{\epsilon^2\mu^{2n} + 4q(\mu)}}{2} : \mu \in \sigma(A)\right\} \quad \text{and} \quad \sigma_2(\mathcal{A}_\epsilon) = \left\{-\tfrac{a_n}{\epsilon}\right\}.$$

Note that for unbounded $\sigma(A)$ we have $\frac{-a_n}{\epsilon} \in \overline{\sigma_1(\mathcal{A}_\epsilon)}$. This can be shown by using the binomial series expansion of $\sqrt{1+\lambda}$ for $|\lambda| < 1$. If $\epsilon = 0$, then

$$\sigma(\mathcal{A}_0) = \sigma_1(\mathcal{A}_0) = \left\{ \pm\sqrt{q(\mu)} : \mu \in \sigma(A) \right\}.$$

For applications to stability theory the *spectral bound*

$$\mathrm{s}(\mathcal{A}) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$$

is of special interest (see [10]). Matrix techniques in combination with the above corollary yield an interesting and simple upper estimate for $\mathrm{s}(\mathcal{A})$.

**Proposition 2.12.** *Let $M_0, \dots, M_m \in M_n(\mathbb{C})$ be normal matrices. If $M = \sum_{k=0}^m \xi^k M_k$ and $\mathcal{A} := M(A)$, then*

$$\mathrm{s}(\mathcal{A}) \leq \sup_{\mu\in\sigma(A)} \sum_{k=0}^{m} \sup_{\alpha_k\in\sigma(M_k)} Re\left(\mu^k\alpha_k\right) \leq \sup_{\alpha_k\in\sigma(M_k)} \sum_{k=0}^{m} \mathrm{s}(\alpha_k A^k). \tag{2.1}$$

**Proof:** For the numerical range $V(S) := \{(Sy, y) : \|y\|_2 = 1\}$ of a matrix $S \in M_n(\mathbb{C})$ it is known that $\sigma(S) \subset V(S)$. But for normal matrices $S$ one has $\mathrm{co}(\sigma(S)) = V(S)$, where "co(.)" denotes the convex hull (see [9]). From these facts we obtain

$$\sigma\left(M(\mu)\right) \subset V\left(M(\mu)\right) \subset \sum_{k=0}^{m} \mu^k V(M_k) = \sum_{k=0}^{m} \mu^k \mathrm{co}\left(\sigma(M_k)\right).$$

This implies that

$$\begin{aligned}
\sup_{\mu\in\sigma(A)} \operatorname{Re}\left(\sigma(M(\mu))\right) &\leq \sup_{\mu\in\sigma(A)} \sum_{k=0}^{m} \sup\operatorname{Re}\left(\mu^k\mathrm{co}(\sigma(M_k))\right) \\
&= \sup_{\mu\in\sigma(A)} \sum_{k=0}^{m} \sup\operatorname{Re}\left(\mu^k(\sigma(M_k))\right) \\
&= \sup_{\mu\in\sigma(A)} \sum_{k=0}^{m} \sup_{\alpha_k\in\sigma(M_k)} \operatorname{Re}\left(\mu^k\alpha_k\right) \\
&\leq \sum_{k=0}^{m} \sup_{\alpha_k\in\sigma(M_k)} \mathrm{s}(\alpha_k A^k).
\end{aligned}$$

We have used here the abbreviations $K+L = \{z+w : z \in K, w \in L\}$ and $\operatorname{Re}(K) = \{\operatorname{Re}(z) : z \in K\}$ for any two subsets $K$, $L$ of $\mathbb{C}$. ∎

**Appendix: The functional calculus.** In the following we will provide some results concerning the functional calculus. Let $A$ be a densely defined operator on some Banach space $E$ with non-empty resolvent set $\rho(A)$. We recall the definitions from part 1:

$$\mathcal{G} = \mathcal{G}(A) := \{f \in \mathbb{C}(x) : f \text{ is holomorphic in a neighborhood of } \sigma(A)\},$$

$$\mathcal{N} = \mathcal{N}(A) := \{f \in \mathcal{G}(A) : [f] \leq 0\} \quad \text{and}$$

$$\mathbb{C}_0[x] := \{f \in \mathbb{C}[x] : f(0) = 0\}.$$

Since $\mathcal{G} = \mathbb{C}_0[x] \oplus \mathcal{N}$ it is possible to decompose every $f \in \mathcal{G}$ into a unique sum $f = p + q$ where $p \in \mathbb{C}_0[x]$ and $q \in \mathcal{N}$. For polynomials there is a well known functional calculus $\Phi_1$ specified by

$$\Phi_1 : p \mapsto \Phi_1(p) = p(A), \quad p \in \mathbb{C}[x],$$

where, by definition, $D(p(A)) := D(A^m)$, with $m = [p]$. Similarly, there is a functional calculus $\Phi_2$ for functions in $\mathcal{N}$ where $\Phi_2 : \mathcal{N} \to \mathcal{L}(E)$ is the homomorphism given by

$$\Phi_2 : q \mapsto \Phi_2(q) = q(A) = r(A)s^{-1}(A), \quad q = \frac{r}{s} \in \mathcal{N}.$$

For details we refer to [4].

Next we combine these two maps in order to obtain a functional calculus $\Phi$ on $\mathcal{G}$.

**Definition A.1.** *Let $f = p + q \in \mathcal{G}$ where $p \in \mathbb{C}_0[x]$ and $q \in \mathcal{N}$. Then we define*

$$\Phi(f) := \Phi_1(p) + \Phi_2(q) = f(A)$$

*with domain $D(f(A)) = D(A^m)$ where $m = \max\{[f], 0\}$.*

**Theorem A.2.** *The map*

$$\Phi : f \mapsto \Phi(f) = f(A), \quad f \in \mathcal{G}$$

*is an "algebra homomorphism" in the the following sense.*

(i) $\Phi(f + g) = \overline{\Phi(f) + \Phi(g)}$ *for all $f, g \in \mathcal{G}$, in particular,*
$\Phi(f + g) = \Phi(f) + \Phi(g)$ *if $g \in \mathcal{N}$,*

(ii) $\Phi(fg) = \Phi(f)\Phi(g) = \overline{\Phi(g)\Phi(f)}$ *for all $f, g \in \mathcal{G}$ if $[g] \leq [f]$.*

*Moreover, if $A$ is unbounded, then*

(iii) $\Phi(f) \in \mathcal{L}(E) \iff f \in \mathcal{N}$,

(iv) $\Phi(f) = 0 \iff f = 0$,

(v) $\Phi(f)$ *is invertible in $\mathcal{L}(E)$* $\iff \frac{1}{f} \in \mathcal{N}$,
*in particular, $\Phi(f)^{-1} \in \Phi(\mathcal{N})$,*

(vi) $\sigma(\Phi(f)) = f(\sigma(A)) \cup \sigma_f$ *where*

$$\sigma_f = \begin{cases} \emptyset & \text{if } [f] > 0 \\ \{f(\infty)\} & \text{if } [f] \leq 0 \end{cases}.$$

The routine proof can be found in [5]. For the proof of Lemma 1.3 the following result is needed.

**Proposition A.3.** *Let $b, c \in \mathbb{C}[x]$ and $[b] \geq [c]$. Then there exists $f \in \mathcal{N}$ such that $bf + c \in \mathcal{N}$.*

**Proof:** To simplify the notation we assume that $0 \in \rho(A)$, i.e., that $j \in \mathcal{N}$ where $j(x) := 1/x$. If $b = 0$, then $c = 0$ and we may choose $f = 0$. Therefore assume $b \neq 0$.

Define $\tilde{f} := -\frac{c}{b} \in \mathbb{C}(x)$. Since $[b] \geq [c]$ we have $[\tilde{f}] \leq 0$. For $C > 0$ big enough $\tilde{f}$ has no poles in $\{x \in \mathbb{C} : |x| \geq C\}$ and we can expand $\tilde{f}$ into a Laurent series $\tilde{f}(x) = \sum_{k=0}^{\infty} \alpha_k x^{-k}$

for $|x| \geq C$. Let $f$ be the expansion of $\tilde{f}$ up to order $m := [b] - 1$, i.e., $f := \sum_{k=0}^{m} \alpha_k j^k$ and $d := \tilde{f} - f$. Then $f \in \mathcal{N}$ and $[d] \leq -(m+1) = -[b]$. Moreover we have

$$bf + c = b(\tilde{f} - d) + c = b\tilde{f} + c \ - bd = -bd.$$

Since $[bd] \leq 0$ and $bf + c \in \mathcal{G}$ we conclude that $bf + c \in \mathcal{N}$. ∎

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