Differential and Integral Equations, Volume 2, Number 2, April 1989, pp. 193-202.

MULTIPLICITY OF kT-PERIODIC SOLUTIONS NEAR A GIVEN T-PERIODIC SOLUTION FOR NONLINEAR HAMILTONIAN SYSTEMS*

MARIA LETIZIA BERTOTTI
Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento, Italy

(Submitted by: Luigi Salvadori)

1. Introduction and results. In this note we consider the question of existence and multiplicity of periodic solutions with minimal period kT ($k \in \mathbb{N}$) near an equilibrium ($z \equiv 0$) for systems of the form

$$\dot{z} = J\nabla H(t, z), \quad z \in \mathbb{R}^{2n}. \tag{1.1}$$

Here J is the skewsymmetric matrix

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \in L(\mathbb{R}^{2n}),$$

∇ stands for the gradient in the z-variable, $H \in C^2(\mathbb{R} \times \Omega, \mathbb{R})$ (with Ω a neighborhood of $0 \in \mathbb{R}^{2n}$) is the Hamiltonian function, which is assumed to depend periodically on time

$$H(t+T, z) = H(t, z) \quad T > 0 \ \forall t \in \mathbb{R}, \ \forall z \in \mathbb{R}^{2n},$$

$$H(t, 0) \equiv 0 \quad \forall t \in \mathbb{R}$$

and

$$\nabla H(t, 0) \equiv 0 \quad \forall t \in \mathbb{R}.$$

Such a problem occures, for example, if we know a T-periodic solution $\bar{x}$ of an Hamiltonian system whose Hamiltonian does not depend explicitly on time and if we look for periodic solutions nearby, having period kT.

The existence of these solutions requires assumptions on the linearized part of the equation at 0 and also on the nonlinear part. Let the Taylor expansion of the function H at 0 be

$$H(t, z) = \frac{1}{2} \langle Qz, z \rangle + \hat{H}(t, z) \tag{1.2}$$

where $\hat{H}(t, z) = o(|z|^2)$ uniformly in t. We shall assume for the linearized system

$$\dot{z} = JQz$$

Received June 12, 1988.
*Partially supported by G.N.F.M. - C.N.R.
AMS Subject Classifications: 58F22, 58F05, 34C25.

that the symmetric matrix Q is independent on time, that JQ is diagonalizable and that it has purely imaginary eigenvalues only. Moreover, we shall assume that this linear equation does not admit any periodic solution of period $T > 0$ except the trivial solution $z \equiv 0$. This assumption implies, in particular, that $z \equiv 0$ is an isolated T-periodic solution of the system (1.1). We emphasize that Q is not required to be definite in sign.

Concerning the nonlinear part, we shall make the following assumption:

$$\begin{cases} \text{there exist two constants } p > 2 \text{ and } c > 0 \text{ and a small } \epsilon^* > 0 \text{ such that} \\ c|z|^p \leq p\hat{H}(t,z) \leq \left\langle \nabla\hat{H}(t,z)\,,\, z \right\rangle \quad \forall z \in \mathbb{R}^{2n},\ \ |z| \leq \epsilon^*. \end{cases} \tag{1.3}$$

Finally, to simplify the notations, we may assume that $T = 1$.

We will prove the following.

Theorem 1.1. *For every small $\epsilon > 0$ and every positive integer j, there is a constant $k^* = k^*(\epsilon, j)$ with the following property: for every prime integer $k \geq k^*$ the Hamiltonian system (1.1) possesses at least nj periodic solutions having minimal period k, which are contained in the prescribed neighborhood of the origin:*

$$\sup_t |z(t)| \leq \epsilon.$$

Talking of minimal period here, we simply require that the k-periodic solution is not a k-times iterated 1-periodic solution.

There are, of course, many results, which, under appropriate conditions on the linearized part and the nonlinearity near the equilibrium point, guarantee an abundance of "small" periodic solutions having large periods. Here, we mention the classical Birkhoff and Lewis Theorem ([1], [2]; see also for a stronger version of that Theorem, Moser [3] and [4]) and the results contained in the papers [5] by Harris, [6] by Rabinowitz, [7] by Conley and Zehnder, [8] by Benci and Fortunato, and [9] by the author. For some comments about these results, we refer to §1 in [9].

The purpose of this note is to establish a *multiplicity* theorem for subharmonic solutions, i.e., solutions having period k, for *prescribed* k in a *prescribed* neighborhood of 0.

We point out that in contrast to [9], where an analogous multiplicity result is proved, we do not assume here any symmetry condition for H. The periodic solutions of (1.1) are obtained by means of a variational approach, as critical points of a suitable functional. Now, generally, it is for functionals which are invariant under a certain group action that one can find multiple critical points. While in [9] the invariance of the functional is a consequence of a particular symmetry (evenness in z) of the Hamiltonian function, in the present note we just take advantage of a $\mathbb{Z}_k$-symmetry which is intrinsc in the problem of the search for k-periodic solutions. Problems with $\mathbb{Z}_k$-invariance have been recently treated by Tarantello in [10] and Michalek and Tarantello in [11]. Also in those papers, multiple subharmonic solutions are found but in a global rather than local setting as considered here.

We point out here the differences in the assumptions between this note and [9]. First of all, in [9] we directly treated the case in which the matrix $Q = Q(t)$ depends on time. But in view of Floquet's theory (see e.g., [12], p. 192), the assumption that Q is not a constant matrix is not very restrictive. Indeed, whenever the monodromy matrix of a linear equation with periodic coefficients,

$$\dot{x} = JS(t)x, \qquad (S(t) = S(t+T) = S(t)^T), \tag{1.5}$$

is diagonal with eigenvalues $\mu_s = \exp(T\lambda_s)$, then Floquet's theory guarantees the existence of a canonical change of variables, which reduces the original linear equation to an equation with constant coefficients,

$$\dot{y} = JQy, \tag{1.6}$$

where JQ is diagonal with eigenvalues λ_s. Therefore, if in our problem, the Hamiltonian is

$$K(t,x) = \frac{1}{2}\langle S(t)x\,,\,x\rangle + \hat{K}(t,x)$$

with $S(t) = S(t+T)$ and $\hat{K}(t,x) = o(|x|^2)$ and the above diagonality hypothesis is satisfied, a canonical change of variables exists such that the transformed problem has a time-independent linear part. Moreover, it is not difficult to see that if $\hat{K}(t,x)$ satisfies assumption (1.4), then also the transformed superquadratic term $\hat{H}(t,y)$ in the Taylor expansion does.

In [9] we assume only one Floquet multiplier to lie on the unit circle. But in addition, we need this Floquet multiplier to be nondegenerate, a requirement which does not appear here.

The requirement that 0 is an isolated 1-periodic solution of (1.1) amounts in the present situation to assume $(1/2\pi)\omega_j \notin \mathbb{Z}$, where $\lambda_j = i\omega_j$ $(1 \le j \le n)$ and $\lambda_{j+n} = \lambda_j^-$ $(1 \le j \le n)$ are the eigenvalues of $-JQ$. But, of course, the main difference between [9] and this note consists in having now dropped the evenness assumption for H.

2. Proof of Theorem 1.1. Since we are looking for solutions of (1.1) in a neighborhood of the origin, we can extend the Hamiltonian H to all of $z \in \mathbb{R}^{2n}$ in a suitable way: we may assume that H satisfies for all $z \in \mathbb{R}^{2n}$

$$\beta|z|^p \le p\hat{H}(t,z) \le \left\langle \nabla\hat{H}(t,z)\,,\,z\right\rangle, \tag{2.1}$$

and

$$\frac{1}{\alpha}|z|^p \le \hat{H} \le \alpha|z|^p \tag{2.2}$$

for two positive constants α, β and for $p > 2$, and

$$\hat{H}(t,z) = \alpha|z|^p \quad \text{if } |z| \ge R \text{ for some } R > 1. \tag{2.3}$$

The existence proof is based on the well known variational principle, according to which the subharmonic (k-periodic) solutions are the critical points of the functional

$$\begin{aligned} f(z) = f_k(z) &= \int_0^k \left\{\frac{1}{2}(-J\dot{z},z) - H(t,z)\right\} dt \\ &= \int_0^k \left\{\frac{1}{2}\big(-J\dot{z}(t) - Qz(t), z(t)\big) - \hat{H}(t,z(t))\right\} dt \end{aligned} \tag{2.4}$$

with the boundary condition $z(k) = z(0)$.

If $A = A_k$ denotes the selfadjoint operator in $L^2(0,k)$, which on its domain

$$D(A) = \left\{z \in H^1([0,k],\ \mathbb{R}^{2n}) : z(k) = z(0)\right\}$$

is defined by

$$Az \doteq -J\dot{z} - Qz \tag{2.5}$$

and

$$\phi(z) \doteq \int_0^k \hat{H}(t,z)\,dz, \tag{2.6}$$

then the functional f can be expressed as

$$f(z) = \frac{1}{2}(Az, z) - \phi(z). \tag{2.7}$$

As a side remark, we observe that the spectrum of A is discrete; it is given by the eigenvalues

$$\mu_{mj} = \frac{2\pi}{k}m - \omega_j, \qquad 1 \le j \le n,\ m \in \mathbb{Z},$$

and 0 is in the resolvent set or it is an isolated eigenvalue with finite multiplicity. We will extend f to a functional on the Hilbert space $E = W^{1/2}$, which is defined as the space of all k-periodic functions $z \in L^2((0,k), \mathbb{R}^{2n})$

$$z(t) = \sum_{j\in\mathbb{Z}} \exp\big(\frac{2\pi}{k}tJ\big) z_j, \qquad z_j \in \mathbb{R}^{2n},$$

whose Fourier coefficients satisfy

$$\sum_{j\in\mathbb{Z}} |j|\,|z_j|^2 < +\infty.$$

The inner product of E is usually defined by

$$\langle z\,,\,w\rangle_E = \sum_{j\in\mathbb{Z}} \big(1 + \frac{2\pi}{k}|j|\big)\,\langle z_j\,,\,w_j\rangle_{\mathbb{R}^{2n}} \tag{2.8}$$

and the corresponding norm is

$$\| z \|_E^2 = \langle z\,,\,z\rangle_E\,, \qquad \forall z \in E.$$

However, we introduce another equivalent norm which is adapted to the problem. We normalize, first, the eigenfunctions ξ_j and $\xi_{j+n} = \bar{\xi_j}$ $(1 \le j \le n)$, of $-JQ$, corresponding to the eigenvalues λ_j, λ_{j+n} $(1 \le j \le n)$, such that

$$\begin{aligned} (Q\xi_j, \xi_l) &= \omega_l \delta_{j,l} \quad \text{for } 1 \le j,\ l \le n \\ (Q\xi_j, \bar{\xi_l}) &= 0 \qquad\quad \text{for } 1 \le j,\ l \le n. \end{aligned} \tag{2.9}$$

(Here, $(\cdot\,,\,\cdot)$ indicates the usual scalar product in $\mathbb{C}^{2n}$). Hence, we set

$$\phi_{jm}(t) = \sqrt{2}\exp\big(i\frac{2\pi}{k}mt\big)\xi_j\,, \qquad j = 1,\cdots,2n,\ m \in \mathbb{Z}. \tag{2.9}$$

These functions span E so that $\forall z \in E$ we can write

$$\begin{aligned} z(t) &= \sum_{j=1}^{n}\sum_{m=-\infty}^{+\infty} \operatorname{Re}\big(z_{jm}\phi_{jm}(t)\big) \\ &= \sum_{j=1}^{n}\sum_{m=-\infty}^{+\infty} \frac{1}{2}\big(z_{jm}\phi_{jm}(t) + \bar{z}_{jm}\bar{\phi}_{jm}(t)\big) \qquad z_{jm} \in \mathbb{C}. \end{aligned}$$

Moreover, they satisfy

$$J\frac{d}{dt}(\operatorname{Re}\phi_{jm}(t)) = \frac{2\pi}{k}\frac{m}{\omega_j}Q(\operatorname{Re}\phi_{jm}(t)). \tag{2.10}$$

In view of (2.9) and (2.10), we get by an easy computation

$$\mathcal{A}(z) \doteq \int_0^k (-J\dot{z}(t) - Qz(t), z(t))\,dt = k\sum_{j=1}^{n}\sum_{m=-\infty}^{+\infty}\Big(\frac{2\pi}{k}m - \omega_j\Big)|z_{jm}|^2. \tag{2.11}$$

Set

$$E^+ = \Big\{z \in E : z(t) = \sum_{j=1}^{n}\sum_{m>(k/2\pi)\omega_j} \operatorname{Re}(z_{jm}\phi_{jm}(t))\Big\}$$

$$E^0 = \Big\{z \in E : z(t) = \sum_{j=1}^{n}\sum_{m=(k/2\pi)\omega_j} \operatorname{Re}(z_{jm}\phi_{jm}(t))\Big\}$$

$$E^- = \Big\{z \in E : z(t) = \sum_{j=1}^{n}\sum_{m<(k/2\pi)\omega_j} \operatorname{Re}(z_{jm}\phi_{jm}(t))\Big\}.$$

Clearly, $\dim E^+ = \dim E^- = +\infty$ and $\dim E^0 < +\infty$ (E^0 could be $= \{0\}$). If we write for $z \in E : z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+$, we will define as our norm in E

$$\| z \|^2 \doteq \mathcal{A}(z^+) - \mathcal{A}(z^-) + \| z^0 \|^2_{L^2} \tag{2.12}$$

and we shall denote by $\langle\cdot\,,\,\cdot\rangle$ the corresponding scalar product. The norm (2.12) is indeed equivalent to the norm (2.8) (see [9]).

We define the bounded selfadjoint operator $L \in \mathcal{L}(E)$ by extending

$$\langle Lz\,,\, z\rangle = (Az, z), \tag{2.13}$$

where on the right-hand side there is the L^2 scalar product. The functional f on E is, therefore, expressed by

$$f(z) = \frac{1}{2}\langle Lz\,,\, z\rangle - \phi(z). \tag{2.14}$$

It can be checked similarly as in [13] or [14] that f is of class C^1 on E, that $\nabla\phi$, the gradient of ϕ is compact, where

$$\langle\nabla\phi(z)\,,\, w\rangle \doteq \int_0^k (\hat{H}(t,z), w)\,dt$$

and that f satisfies the Palais-Smale condition (recall that f is said to satisfy the (PS) condition in the interval $[a,b]$ if any sequence satisfying

$$f(u_n) \underset{n\to\infty}{\longrightarrow} c \in [a,b] \quad \text{and} \quad \| f'(u_n) \|_{E^*} \underset{n\to\infty}{\longrightarrow} 0$$

admits a convergent subsequence). The spectrum of L is a point spectrum and to any negative (resp. positive, resp. null) eigenvalue of A there corresponds a negative (resp. positive, resp. null) eigenvalue of L. Moreover, the eigenvectors of every such pair coincide.

An important role is now played by the $\mathbb{Z}_k$-symmetry of the functional, namely the invariance of f with respect to j ($j = 1, 2, \cdots, k$)-time shifts:

$$f(z(t+j)) = f(z(t)).$$

We also emphasize here that the fixed point set of the $\mathbb{Z}_k$ action

$$F \doteq \operatorname{Fix} \mathbb{Z}_k = \{z : gz = z, \ g \in \mathbb{Z}_k\}$$

consists of all the 1-periodic functions in E. These two facts (the invariance of f and the size of F) suggest one to try and apply an abstract critical point theorem due to Tarantello [10], which, for the convenience of the reader, will be stated below. Other results (e.g., Theorem 2.9 in [18]; more generally see [13]-[18]) could be used to deduce the existence or critical values for our functional. But in our situation, they would not be sufficient to guarantee the multiplicity of the solutions (see below).

We recall a crucial definition introduced in [10]:

Let $T : E \to E$ denote a norm preserving (complex) operator that generates a $\mathbb{Z}_k$-group action on E, i.e., $T^k = Id$.

Definition 2.1. *A subspace D of a real Hilbert space E is called j-nice if it satisfies the following properties:*

(a) *$\dim_{\mathbb{R}} D = 2j$ and D is T-invariant;*

(b) *there exists an homeomorphism $\psi = \psi_j : D \to \mathbb{C}^j$ such that $T_j : \mathbb{C}^j \to \mathbb{C}^j$ with $T_j = \psi \cdot T \cdot \psi^{-1}$ is a unitary representation of the $\mathbb{Z}_k$ group action on $\mathbb{C}^j$ given by*

$$T_j(x_1, \cdots, x_j) = \left(e^{i(2\pi/k)m_{j,1}} x_1, \cdots, e^{i(2\pi/k)m_{j,j}} x_j\right), \qquad x_l \in \mathbb{C}, \ l = 1, \cdots, j$$

for some integer $m_{j,s} \neq 0$ relatively prime to k, $s = 1, \cdots, j$.

Theorem 2.1. (Tarantello, [10]). *Let $f \in C^1(X, R)$ be a T-invariant functional satisfying*

$$f(u) = \frac{1}{2} \langle Lu, u \rangle - \phi(u), \qquad u \in X$$

where L is a bounded selfadjoint T-equivariant operator and $\nabla \phi : X \to X$ is compact. Assume that there exist invariant subspaces X^- and X_{j+} satisfying: $X_{j+} \subset X^+ = (X^-)^\perp$ is j-nice, $LX^- = X^-$, and such that for some constants $c_0 < c_\infty$ and $\rho > 0$, we have

(f_1) *$f(u) < c_\infty \quad \forall u \in X^- \oplus X_{j+}$*

(f_2) *$f(u) \geq c_0 \quad \forall u \in X^+ \cap S_\rho$*

(f_3) *$f^{-1}([c_0 - \epsilon_0, c_\infty)) \cap F^- = \emptyset$ for some $\epsilon_0 > 0$ with $F^- = X^- \cap F$*

(f_4) *$f^{-1}([c_0, c_\infty)) \cap F_0 = \emptyset$ where $F_0 = F \cap \{u : f'(u) = 0\}$*

If f satisfies (PS) in $[c_0, c_\infty)$, then there exist at least $j\mathbb{Z}_k$-distinct critical points $u_1, \cdots, u_j$ such that

$$c_0 \leq f(u_h) \leq c_\infty, \quad \forall h = 1, \cdots, j.$$

By saying that two points u and v are $\mathbb{Z}_k$-distinct, we mean that $T^j u \neq v \ \forall j = 0, 1, \cdots, k-1$. In order to apply this abstract Theorem, we have to find subspaces X^- and X_{j+} for our problem.

First, since 0 is a regular value or an isolated eigenvalue, there is a $\lambda > 0$ such that

$$\frac{1}{2}\langle Lz\,,\,z\rangle \geq \lambda \parallel z \parallel^2, \qquad \forall z \in E^+.$$

Using the embedding (see [13]) (here and henceforth, C will denote various positive constants; $\|\cdot\|_q$ denotes the norm in $L^q((0,k),\mathbb{R}^{2n})$)

$$\parallel z \parallel_p \leq C \parallel z \parallel, \qquad \forall z \in E \tag{2.15}$$

we estimate

$$\begin{aligned} f_k(z) &\geq \lambda \parallel z \parallel^2 - \int_0^k \hat{H}(t,z(t))\,dt \geq \lambda \parallel z \parallel^2 - C \parallel z \parallel_p^p \\ &\geq \lambda \parallel z \parallel^2 - C \parallel z \parallel^p = \parallel z \parallel^2 (\lambda - C \parallel z \parallel^{p-2}), \qquad \forall z \in E^+. \end{aligned}$$

Since $p > 2$, then for $\rho > 0$ sufficiently small,

$$z \in E^+, \ \parallel z \parallel = \rho \Longrightarrow f_k(z) \geq \frac{\lambda}{2} > 0. \tag{2.16}$$

On the other hand, f is bounded above on $E^- \oplus E^0 \oplus E_{jn^+}$. Here E_{jn^+} is defined as

$$E_{jn^+} = \Big\{ z(t) = \sum_{l=1}^{n} \sum_{m=m_l^k+1}^{m_l^k+j} \mathrm{Re}\,(z_{lm}\phi_{lm}(t)) \Big\} \ \text{ with } m_l^k \doteq [\frac{k}{2\pi}\omega_l]. \tag{2.17}$$

Indeed, $\forall z \in E^- \oplus E^0 \oplus E_{jn^+}$

$$\begin{aligned} f_k(z) &= \frac{1}{2}k \sum_{l=1}^{n} \sum_{m=-\infty}^{m_l^k+j} (\frac{2\pi}{k}m - \omega_l)|z_{lm}|^2 - \int_0^k \hat{H}(t,z(t))\,dt \\ &\leq \frac{1}{2}k \sum_{l=1}^{n} \sum_{m=-\infty}^{m_l^k+j} \frac{2\pi}{k}(m - \frac{k}{2\pi}\omega_l)|z_{lm}|^2 - \frac{1}{\alpha} \parallel z \parallel_p^p \\ &\leq \frac{1}{2}\frac{2\pi}{k} j \Big[k \sum_{l=m}^{n} \sum_{m=-\infty}^{m_l^k+j} |z_{lm}|^2 \Big] - \frac{1}{\alpha} \parallel z \parallel_p^p \\ &\leq \frac{\pi}{k} jC \parallel z \parallel_2^2 - \frac{1}{\alpha} \parallel z \parallel_p^p \leq \frac{\pi}{k} jC \parallel z \parallel_2^2 - \frac{1}{\alpha}\frac{1}{k^{(p-2)/2}} \parallel z \parallel_2^p \end{aligned}$$

and hence, $\forall z \in E^- \oplus E^0 \oplus E_{jn^+}$

$$f_k(z) \leq \big(\frac{2\pi j}{kp}\alpha C\big)^{2/(p-2)} \frac{p-2}{p} \pi j C \doteq \frac{C(j)}{k^{2/p-2}}. \tag{2.18}$$

We now can set $X^-(= (X^+)^\perp) \doteq E^- \oplus E^0 (= (E^+)^\perp)$ and $X_{jn^+} \doteq E_{jn^+}$ and we claim that E_{jn^+} is jn-nice provided $k > j/\underline{r}$. Here, $\underline{r}$ is defined as follows:

if $r_l = \omega_l/2\pi - [\omega_l/2\pi]\ \forall\, l = 1, \cdots, n$ (and hence $0 \leq r_l < 1$), then $\underline{r} = \min(1 - r_l)$. Clearly, $0 < \underline{r} \leq 1$. It is clear that $\dim E_{jn^+} = 2jn$ and E_{jn^+} is T-invariant.

The part b) of Definition 2.1 can be verified by taking, for $\psi = \psi_{jn}$, the map defined by

$$\psi : \sum_{l=1}^{n} \sum_{m=m_l^k+1}^{m_l^k+j} \mathrm{Re}\,(z_{lm}\phi_{lm}(t)) \longrightarrow (z_{1,m_l^k+1}, \cdots, z_{1,m_l^k+j}, \cdots, z_{n,m_l^k+j}) \in \mathbb{C}^{jn}$$

(for details, see [10]). The assumptions (f_1) and (f_2) of Theorem 2.1 are, therefore, verified and (f_3) is an immediate consequence of the fact that in our case $c_0 > 0$ (see (2.16)). Indeed because of our assumptions, $f_k(z) \leq 0$ if $z \in E^- \oplus E^0$. At this point, the existence of at least jn critical values c_l of f_k

$$c_0 \leq c_l \leq c_\infty$$

can be deduced provided k is a prime number satisfying $k > j/\underline{r} \doteq \underline{k}(j)$. These critical values are not necessarily distinct; in order to find at least jn distinct critical points, assumption (f_4) has to be verified. This amounts to showing that none of the found critical points is in the fixed point set F or equivalently, that the critical points $z_k \in f^{-1}(c_0, c_\infty)$ are not 1-periodic functions.

To prove (f_4) we need an a priori estimate, which is a posteriori used also to evaluate the amplitude of the periodic orbits found.

Lemma 2.1. *$\forall j \in \mathbb{N}$, and $\forall \epsilon > 0$, there exists a $k^* = k^*(\epsilon, j) \in \mathbb{N}$ with the following property: for every prime integer $k \geq k^*(\epsilon, j) \geq \underline{k}(j)$, it follows that for every critical point z_k of f_k, $z_k \in f^{-1}$ (c_0, c_∞)*

$$|z_k(t)|_{L^\infty} < \epsilon. \tag{2.19}$$

Postponing the proof of this Lemma, we want now to prove (f_4). This is a consequence of the fact that 0 is, by assumption, an isolated 1-periodic solution of (1.1) in $C^1(\mathbb{R}, \mathbb{R}^{2n})$, while the considered functions z_k tend to 0 as $k \to \infty$ uniformly in C^1 (this follows by (2.19)). If the z_k's would have period 1, we would get a contradiction.

We remark that in this way, we also have the minimality of the period k for the solutions z_k.

Proof of Lemma 2.1: We show first that, for the found critical points of f_k, we have

$$\| z \|_p^p \leq C f_k(z) \tag{2.20}$$

with a constant C independent on k. Indeed, since z satisfies

$$\dot{z} = J\nabla H(t, z)$$

we have

$$-\frac{1}{2}(Az, z) + \frac{1}{2}\int_0^k (\nabla \hat{H}(t, z), z)\, dt = 0,$$

so that in view of (2.7), (2.1) and (2.2),

$$f_k(z) = \frac{1}{2}\int_0^k (\nabla \hat{H}(t, z), z)\, dt - \int_0^k \hat{H}(t, z)\, dt \geq (\frac{p}{2} - 1)\int_0^k \hat{H}(t, z)\, dt \geq C \| z \|_p^p$$

proving the claim (2.20). Hence, from (2.20) and (2.18) and by the way the critical points are found (namely as minimax; see [10]), we conclude for the critical points z_k of f_k found above that

$$\| z_k \|_p^p \leq C(j)(\frac{1}{k})^{2/p-2}$$

and hence,

$$\| z_k \|_p \longrightarrow 0 \text{ as } k \to \infty. \tag{2.21}$$

It is now possible to obtain for k large, for a critical point z_k

$$\sup |z_k(t)| \le R + C \| z_k \|_p^p \le 2R, \tag{2.22}$$

(see indeed [9]). Consequently, also

$$|\dot{z}(t)| \text{ is bounded (uniformly in } k), \tag{2.23}$$

so that in view of (2.21),

$$\lim_{k\to\infty} \sup_t |z(t)| = 0.$$

Specifically, by easy calculations, it is possible to see by (2.21) and (2.23) that the following is true: $\forall \epsilon > 0$, $\forall j \in \mathbb{N}$, there is a $k^*(\epsilon, j) \in \mathbb{Z}$ such that if $k \ge k^*$ for the critical points z_k above considered, it is

$$|z_k(t)|_{L^\infty} < \epsilon.$$

∎

Having proved (f_4) by Theorem 2.1 and by Lemma 2.1, we can finally conclude that $\forall \epsilon > 0$, $\forall j \in \mathbb{N}$, there are at least jn distinct critical points z_k of the functional f_k, such that

$$|z_k(t)|_{L^\infty} < \epsilon,$$

provided k is prime and large enough (i.e., $k \ge k^*(\epsilon, j) \ge \underline{k}(j)$).

We finally recall that the original Hamiltonian H (for which we only postulated local assumptions in a neighborhood of 0) was extended to all of $\mathbb{R}^{2n}$ by requiring the growth assumptions (2.1)-(2.3). For the easy details concerning a way this extension can be carried out, we refer, e.g., to [8]. What is important here is that clearly, the results which can be obtained which are relative to a neighborhood of 0 are not affected by the behavior of the system outside this neighborhood.

REFERENCES

[1] G.D. Birkhoff, "Dynamical Systems," A.M.S. Colloquium Publications, Vol. IX, 1927.

[2] G.D. Birkhoff and D.C. Lewis, *On the periodic motions near a given periodic motion of a dynamical system,* Ann. Mat. Pura Appl., 12 (1934), 117-133.

[3] J.K. Moser, *Proof of a generalized form of a fixed point theorem due to* G.D. Birkhoff, Lecture Notes in Math., 597, Springer Verlag (1977), 464-494.

[4] J.K. Moser, *The Birkhoff-Lewis fixed point theorem,* Appendix in Klingenberg: "Lecture on closed geodesics," Springer Verlag (1978), 115-121.

[5] T.C. Harris, *Periodic solutions of arbitrary long periods in Hamiltonian systems,* Jour. Diff. Eq., 4 (1968), 131-141.

[6] P.H. Rabinowitz, *On Subharmonic solutions of Hamiltonian systems,* Comm. Pure Appl. Math., 33 (1980), 609-633.

[7] C.C. Conley and E. Zehnder, *An index theory for periodic solutions of an Hamiltonian system,* Lecture Notes in Math., 1007, Springer Verlag (1983), 132-145.

[8] V. Benci and D. Fortunato, *A Birkhoff-Lewis type result for a class of Hamiltonian systems,* Manus. Math., 59 (1987), 441-456.

[9] M.L. Bertotti, *Multiplicity of subharmonic solutions of forced Hamiltonian systems near an equilibrium,* preprint.

[10] G. Tarantello, *Subharmonic solutions for Hamiltonian systems via a $\mathbb{Z}_p$ pseudoindex theory,* preprint.

[11] R. Michalek and G. Tarantello, *Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems,* preprint.

[12] V.I. Arnold, "Geometrical Methods in the Theory of Ordinary Differential Equations," Springer Verlag, 1983.
[13] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* Conference Series in Mathematics A.M.S., 65, 1986.
[14] V. Benci and P.H. Rabinowitz, *Critical point theorems for indefinite functionals,* Inv. Math., 52 (1979), 241-273.
[15] P.H. Rabinowitz, *Mimimax methods for indefinite functionals,* Proc. Symposia in Pure Math., A.M.S., 45, Part 2 (1986), 287-306.
[16] V. Benci, *A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations,* Comm. Pure Appl. Math., 34 (1981), 393-432.
[17] V. Benci, A. Capozzi, and D. Fortunato, *Periodic solutions of Hamiltonian systems with superquadratic potential,* Ann. Math. Pura Appl., 143 (1986), 1-46.
[18] P. Bartolo, V. Benci,and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity,* Nonl. Anal., T.M.A., 7 (1983), 981-1012.