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QUALITATIVE SIMULATION OF DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the QSIM algorithm introduced by Kuipers (see [9])to track down the changes of monotonicity properties of solutions to a differential equation or other observations of the solutions. It introduces the concept of "qualitative cells" where the monotonicity properties of the states of the system remain the same. It provides sufficient conditions for the non emptiness of such cells, for their singularities, for the transition from one cell to another, and characterizes also the qualitative equilibria and repellors of the associated qualitative system.

Introduction. The purpose of this paper is to revisit the QSIM algorithm introduced by Kuipers in [9] for studying the qualitative evolution of solutions to a differential equation

$$x'(t) = f(x(t)) \tag{1}$$

where the state x ranges over a closed subset K of a finite dimensional vector-space $X := \mathbb{R}^n$.

We recall that the *contingent cone* $T_K(x)$ to a subset K at $x \in K$ is the closed cone of elements v satisfying

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} = 0$$

and that K is a viability domain if and only if

$$\forall\, x \in K, \quad f(x) \in T_K(x).$$

We posit the assumptions of Nagumo Theorem:

Theorem. [Nagumo]. Under assumptions

$$\begin{cases} i) & F \text{is continuous with linear growth} \\ ii) & K \text{is a closed viability domain,} \end{cases} \tag{2}$$

a closed subset K is a viability domain if and only if it enjoys the viability property: for any initial state $x_0 \in K$, there exists a solution $x(\cdot)$ to the differential equation (1), which is *viable* in the sense that $x(t)$ remains in K for all $t \geq 0$. [1, Chapter 4].

The *qualitative state* of a solution to the differential equation (1) at a given time t is the knowledge of the monotonicity property of each component $x_i(t)$ of a solution $x(\cdot)$ to

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this differential equation; i.e., the knowledge of the sign of the derivatives $x_i'(t)$. Hence, the *qualitative behavior* is the evolution of the qualitative states of the solution; i.e., the evolution of the vector of signs of the components of $x'(t) = f(x(t))$, which must be determined without solving the differential equation.

In order to denote the qualitative states and track down their evolution, we introduce (see [5]) the strict and large confluence frames $(\mathcal{R}^n, Q_n)$ and $(\mathcal{R}^n, \overline{Q}_n)$ of $\mathbb{R}^n$ where $\mathcal{R}^n$ denotes the n-dimensional *confluence space* defined by

$$\mathcal{R}^n := \{ -, 0, + \}^n$$

and Q_n is the set-valued map from $\mathcal{R}^n$ to $\mathbb{R}^n$ associating with every $a \in \mathcal{R}^n$ the convex cone

$$Q_n(a) := \mathbb{R}^n_a := \{v \in \mathbb{R}^n \mid \text{ sign of } (v_i) = a_i\}$$

and

$$\overline{Q}_n(a) := a\mathbb{R}^n_+ := \{v \in \mathbb{R}^n \mid \text{ sign of } (v_i) = a_i \text{ or } 0\}\,.$$

We observe that the inverse of the set-valued map Q_n is the single-valued map s_n from $\mathbb{R}^n$ to $\mathcal{R}^n$ defined by:

$$\forall\, i \in \{1, \dots, n\}, \quad s_n(x)_i := \text{sign of } x_i\,.$$

For studying the qualitative behavior of the differential equation (1); i.e., the evolution of the functions $t \mapsto s_n(x'(t))$ associated to solutions $x(\cdot)$ of the differential equation, we split the viability domain K of the differential equation into 3^n *"qualitative cells"* K_a and *"large qualitative cells"* $\overline{K}_a$ defined by

$$K_a := \{x \in K \mid f(x) \in \mathbb{R}^n_a\} \ \text{ and } \ \overline{K}_a := \{x \in K \mid f(x) \in a\mathbb{R}^n\}\,. \tag{3}$$

Indeed, the quantitative states $x(\cdot)$ evolving in a given qualitative cell K_a share the same monotonicity properties because, as long as $x(t)$ remains in K_a,

$$\forall\, i = 1, \dots, n, \ \text{sign of } \frac{dx_i(t)}{dt} = a_i\,.$$

The qualitative cell K_0 is then the set of equilibria. Such an equilibrium does exist whenever the viability domain K is convex and compact of the system, because $K_0 = \{x \in K \mid f(x) = 0\}$. (see Theorem 6.4.11 of [2]).

Studying the qualitative evolution of the differential equation amounts to knowing the laws (if any) which govern the transition from one qualitative cell K_a to other cells *without solving the differential equation.*

But before proceeding further, we shall generalize our problem — free of any mathematical cost — to take care of physical considerations.

Instead of studying the monotonicity properties of each component $x_i(\cdot)$ of the state of the system under investigation, which can be too numerous, we shall only study the monotonicity properties of m functionals $V_j(x(\cdot))$ on the state (for instance, energy or entropy functionals in physics, observations in control theory, various economic indices in economics) which do matter.

The previous case is the particular case when we take the n functionals V_i defined by $V_i(x) := x_i$.

We shall assume for simplicity, that these functionals V_j are continuously differentiable around the viability domain K.

We denote by $\mathbb{V}$ the map from X to $Y := \mathbb{R}^m$ defined by

$$\mathbb{V}(x) := (V_1(x), \dots, V_m(x))$$

and we introduce the strict and large confluence frames $(\mathcal{R}^m, Q_m)$ and $(\mathcal{R}^m, \overline{Q}_m)$ of Y for studying the qualitative evolution of the observation $\mathbb{V}(x(\cdot))$.

Since the derivative of the observation $\mathbb{V}(x(\cdot))$ is equal to $\mathbb{V}'(x(\cdot))x'(\cdot) = \mathbb{V}'(x(\cdot))f(x(\cdot))$, it will be convenient to set

$$\forall x \in K, \quad g(x) := \mathbb{V}'(x)f(x). \tag{4}$$

Hence, we associate with each qualitative state a the qualitative cells K_a and the large qualitative cells $\overline{K}_a$ defined by

$$K_a := \{x \in K \mid g(x) \in \mathbb{R}^m_a\} \quad \text{and} \quad \overline{K}_a := \{x \in K \mid g(x) \in a\mathbb{R}^m\}. \tag{5}$$

In other words, as long as $x(t)$ remains in K_a,

$$\forall j = 1, \dots, m, \quad \text{sign of } \frac{d}{dt} V_j(x(t)) = a_j.$$

In particular, the m functions $V_j(x(t))$ remain constant while they evolve in the qualitative cell K_0.

By using observation functionals chosen in such a way that many qualitative cells are empty, the study of transitions may be drastically simplified; this is a second reason to carry our study in this more general setting. This is the case, for instance when the observation functionals are "Lyapunov functions" $V_j : K \mapsto \mathbb{R}$. We recall that V is a Lyapunov function if $< V'(x), f(x) > \leq 0$ for all $x \in K$, so that $V(x(\cdot))$ decreases along the solutions to the differential equation. Hence, if the observation functionals are Lyapunov functions, the qualitative cells K_a are empty whenever a component a_i is positive. In this case, we have at most 2^m non empty qualitative cells.

Here again, studying the qualitative evolution of the differential equation amounts to finding the laws which govern the transition from one qualitative cell K_a to other qualitative cells. Naturally, we would like to know this evolution directly without solving the differential equation, and therefore, without knowing the state of the system, but only some of its properties. In other words, the problem arises whether we can map the differential equation (1) to a discrete dynamical system $\Phi : \mathcal{R}^m \rightsquigarrow \mathcal{R}^m$ on the qualitative space $\mathcal{R}^m$. This is not always possible, and we have thus to define the class of differential equations which enjoy this property.

But before doing so, we shall characterize the qualitative equilibria, which are the qualitative states a, such that the solutions which arrive in the qualitative cell $\overline{K}_a$ remain in this cell, as well as the qualitative repellors b, such that any solution which arrives in $\overline{K}_b$ must leave this cell in finite time. We shall finally provide conditions insuring that the qualitative cells are not empty or singular around $\bar{x} \in K_0$ in the sense that there is no other $x \in \overline{K}_a$ in a neighborhood of $\bar{x}$.

1. Transitions between qualitative cells. We shall assume from now on that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K.

Let us denote by $S : K \mapsto \mathcal{C}^1(0, \infty; X)$ the "solution map" associating with each initial state $x_0 \in K$ the solution $Sx_0(\cdot)$ to the differential equation (1) starting at x_0.

Definition 1.1. *Let us consider a map f from K to X and m observation functionals $V_j : K \mapsto \mathbb{R}$. We denote by $\mathcal{D}(f, \mathbb{V})$, the subset of qualitative states $a \in \mathcal{R}^n$ such that K_a is not empty.*

We shall say that $c \in \mathcal{D}(f, \mathbb{V})$ is a "successor" of $b \in \mathcal{D}(f, \mathbb{V})$ if, for all initial states $x_0 \in \overline{K}_b \cap \overline{K}_c$, there exists $\tau \in]0, +\infty]$ such that $Sx_0(s) \in K_c$ for all $s \in]0, \tau[$.

A qualitative state $a \in \mathcal{D}(f, \mathbb{V})$ is said to be a "qualitative equilibrium" if it is its own successor. It is said to be a "qualitative repellor" if for any initial state $x_0 \in \overline{K}_a$, there exists $t > 0$ such that $Sx_0(t) \notin \overline{K}_a$.

Our first objective is to express the fact that c is a successor of b through a set-valued map Φ. For that purpose, we shall set

$$h(x) := g'(x)f(x) = V''(x)(f(x), f(x)) + V'(x)f'(x)f(x).$$

We introduce the notation

$$\overline{K}_a^i := \{x \in \overline{K}_a \,|\, g(x)_i = 0\}$$

(naturally, $\overline{K}_a = \overline{K}_a^i$ whenever $a_i = 0$.) We shall denote by Γ the set-valued map from $\mathcal{R}^m$ to itself defined by

$$\forall\, a \in \mathcal{R}^m, \;\; (\Gamma(a))_i \text{ is the set of signs of } h_i(x) \text{ when } x \in \overline{K}_a^i\,. \tag{6}$$

We also set $I_0(x) := \{i = 1, \dots m \,|\, g(x)_i = 0\}$ and

$$\mathbb{R}_+^{I_0(x)} := \{v \in \mathbb{R}^m | v_i \geq 0 \;\; \forall\, i \in I_0(x)\}\,.$$

We introduce the operations $\wedge$ on $\mathcal{R}^m$ defined by

$$(b \wedge c)_i := \begin{cases} b_i & \text{if } b_i = c_i \\ 0 & \text{if } b_i \neq c_i \end{cases}$$

and the set-valued operation $\vee$ where $b \vee c$ is the subset of qualitative states a such that

$$a_i := b_i \;\text{ or }\; c_i\,.$$

We set

$$a \# b \Longleftrightarrow \forall\, i = 1, \dots, m, \;\; a_i \neq b_i\,.$$

Proposition 1.1. *The set-valued map Γ satisfies the consistency property*

$$\Gamma(a \vee 0) \subset \Gamma(a) \tag{7}$$

and thus,

$$\Gamma(b \wedge c) \subset \Gamma(b) \cap \Gamma(c).$$

Proof: To say that $\overline{K}_b$ is contained in $\overline{K}_a$ amounts to saying that b belongs to $a \vee 0$. When this is the case, we deduce that for all $i = 1, \dots, m$, $\overline{K}_b^i \subset \overline{K}_a^i$, so that the signs taken by $h(x)_i$ when x ranges over $\overline{K}_b^i$ belong to the set of $\Gamma(a)_i$ of signs taken by the same function over $\overline{K}_a^i$. Therefore, $\Gamma(b)$ is contained in $\Gamma(a)$. Since $b \wedge c$ belongs to both $b \wedge 0$ and $c \vee 0$, we deduce that $\Gamma(b \wedge c)$ is contained in both $\Gamma(b)$ and $\Gamma(c)$. ∎

Definition 1.2. *We shall associate with the system $(f, \mathbf{V})$ the discrete dynamical system on the confluence set $\mathcal{R}^m$ defined by the set-valued map $\Phi : \mathcal{R}^m \rightsquigarrow \mathcal{R}^m$ associating with any qualitative state b the subset*

$$\Phi(b) := \{c \in \mathcal{D}(f, \mathbf{V}) \mid \Gamma(b \wedge c) \subset c \vee 0\}. \tag{8}$$

We begin with necessary conditions for a qualitative state c to be a successor of b.

Proposition 1.2. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. If $c \in \mathcal{D}(f, \mathbf{V})$ is a successor of b, then c belongs to $\Phi(b)$.*

For proving this proposition, we need to compute the contingent cone to the cell K_a, which is the intersection of K and of the inverse image of a cone. We need for that formulas which require the "Clarke tangent cone" $C_K(x)$ to K at $x \in K$: it is the set of $v \in X$ such that

$$\lim_{h \to 0+, K \ni y \to x} \frac{d(y + hv, K)}{h} = 0.$$

Lemma 1.1. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. If v belongs to the contingent cone to $\overline{K}_a$ at x, then condition*

$$v \in T_K(x) \text{ and } \forall i \in I_0(x), \text{ sign of } (g'(x)v)_i = a_i \text{ or } 0 \tag{9}$$

is satisfied. The converse is true if we posit the transversality assumption:

$$\forall x \in \overline{K}_a, \quad g'(x)C_K(x) - a\mathbb{R}_+^{I_0(x)} = \mathbb{R}^m.$$

Proof: Since the large qualitative cell $\overline{K}_a$ is the intersection of K with the inverse image by g of the convex cone $a\mathbb{R}_+^m$, we know (see [2 Theorem 7.3]) the contingent cone to $\overline{K}_a$ at some $x \in \overline{K}_a$ is contained in

$$T_K(x) \cap g'(x)^{-1} T_{a\mathbb{R}_+^m}(g(x))$$

and is equal to this intersection provided that the "transversality assumption"

$$g'(x)C_K(x) - C_{a\mathbb{R}_+^m}(g(x)) = \mathbb{R}^m$$

is satisfied. On the other hand, we know that $a\mathbb{R}_+^m$ being convex,

$$C_{a\mathbb{R}_+^m}(y) = T_{a\mathbb{R}_+^m}(y) = aT_{\mathbb{R}_+^m(ay)} \supset a\mathbb{R}_+^m$$

and that $v \in T_{\mathbb{R}_+^m}(z)$ if and only if

$$\text{whenever } z_j = 0, \text{ then } v_j \geq 0.$$

Consequently, $v \in T_{a\mathbb{R}_+^m}(g(x))$ if and only if

$$\text{whenever } g(x)_j = 0, \text{ then sign of } v_j = a_j \text{ or } 0;$$

i.e., $T_{a\mathbb{R}^m_+}(g(x)) = a\mathbb{R}_+^{I_0(x)}$. Hence, v belongs to the contingent cone to $\overline{K}_a$ at x if and only if v belongs to $T_K(x)$ and $g'(x)v$ belongs to $T_{a\mathbb{R}^m_+}(g(x))$; i.e., the sign of $(g'(x)v)_j$ is equal to a_j or 0 whenever j belongs to $I_0(x)$. ∎

Proof of Proposition 1.2: Let c be a successor of b. Take any initial state x_0 in $\overline{K}_b \cap \overline{K}_c$ and set $x(t) := Sx_0(t)$. We observe that the intersection of two qualitative cells $\overline{K}_b$ and $\overline{K}_c$ is equal to

$$\overline{K}_b \cap \overline{K}_c := \overline{K}_{b \wedge c}\,.$$

Since the solution $x(t)$ to the differential equation crosses the intersection $\overline{K}_{b\wedge c}$ towards $\overline{K}_c$, $f(x_0)$ belongs to the contingent cone $T_{\overline{K}_c}(x_0)$ because

$$\liminf_{h\to 0+} \frac{d(x_0 + hf(x_0), K_c)}{h} \le \liminf_{h\to 0+} \left\|x'(0) - \frac{x(h) - x_0}{h}\right\| = 0\,.$$

By Lemma 1.1, this implies that

$$\forall\, x_0 \in \overline{K}_{b\wedge c},\ \ \forall\, i \in I_0(x_0),\ \ \text{sign of } h_i(x_0) = c_i \text{ or } 0$$

or, equivalently, that

$$\Gamma(b \wedge c) \subset c \vee 0\,.$$

Hence, c belongs to $\Phi(b)$, as it was stated.

2. Qualitative equilibrium and repellor. We can characterize the qualitative equilibria of the differential equation (1).

Theorem 2.1. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. We posit the transversality assumption*

$$\forall\, x \in \overline{K}_a,\ \ g'(x)C_K(x) - a\mathbb{R}_+^{I_0(x)} = \mathbb{R}^m\,. \tag{10}$$

Then a is a qualitative equilibrium if and only if a belongs to $\Phi(a)$.

Proof: We already know that if a is a qualitative equilibrium, then a belongs to $\Phi(a)$. We shall prove the converse statement, and, for that purpose, observe that saying that a is a qualitative equilibrium amounts to saying that $\overline{K}_a$ enjoys the viability property (or is invariant by f). By the Nagumo Theorem, this is equivalent to saying that $\overline{K}_a$ is a viability domain; i.e., that

$$\forall\, x \in \overline{K}_a,\ \ f(x) \in T_{\overline{K}_a}(x)\,.$$

By Lemma 1.1, we know that $f(x)$ belongs to the contingent cone $T_K(x)$ by assumption; this amounts to saying that

$$\forall\, x \in \overline{K}_a,\ \ \forall\, i \in I_0(x),\ \ \text{sign of } (g'(x)f(x))_i = a_i \text{ or } 0;$$

i.e., that $\Gamma(a \wedge a) = \Gamma(a) \subset a \vee 0$. Hence, a is a fixed point of Φ. ∎

What happens if a large qualitative cell $\overline{K}_a$ is not a viability domain of f? We recall (see [5]) that there exists a "viability kernel" Viab(M) of a closed subset $M \subset K$ of a closed subset K of the domain of f, which is the largest closed viability domain of the restriction $f|_M$ of f. It is the subset of elements $x \in M$ such that $Sx(t) \in M$ for all $t \ge 0$. It may naturally be empty.

We infer from the definition of the viability kernel that

Proposition 2.1. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. We posit the transversality assumption*

$$\forall x \in \overline{K}_a, \quad g'(x)C_K(x) - a\mathbb{R}_+^{I_0(x)} = \mathbb{R}^m \,. \tag{11}$$

Then a is a qualitative repellor if and only if the Viab($\overline{K}_a$) is empty. If, for some $b \in a \vee 0$, the qualitative cell $\overline{K}_b$ is contained in Viab($\overline{K}_a$), *then a is the only successor of b.*

Proof: To say that some $x_0 \in \overline{K}_a$ does not belong to the viability kernel of $\overline{K}_a$ means that for some $t > 0, Sx_0(t) \notin \overline{K}_a$. If this happens for all $x_0 \in \overline{K}_a$, then obviously, a is a qualitative repellor.

If $\overline{K}_b \subset \text{Viab}\,(\overline{K}_a)$, then for all $x_0 \in \overline{K}_b, Sx_0(t) \in \overline{K}_a$ for all $t \geq 0$. Hence, a is the only successor of b. ▮

3. The QSIM algorithm. We shall now distinguish the 2^n "full qualitative states" $a\#0$ from the other qualitative states, the "transition states."

When I is a non empty subset of $N := \{1, \ldots, m\}$, we associate with a full state $a\#0$ the transition state a^I defined by

$$a_i^I := \begin{cases} 0 & \text{if } i \in I \\ a_i & \text{if } i \notin I \,. \end{cases}$$

Lemma 3.1. *Let $a\#0$ be a qualitative state which is not a qualitative equilibrium. Then there exists a solution starting at some $x \in K_a$ and some $t_1 > 0$ such that $x(t) \in K_a$ for $t \in [0, t_1[$ and $x(t_1) \in K_{a^I}$ for some non empty subset $I \subset N$:*

$$\forall i \in I, \quad x(t_1) \in K_a^i.$$

Proof: Let us choose $x \in K_a$ and set $x(t) := Sx(t)$, $x(0) = x$.

Either $x(t)$ remains in $\overline{K}_a$ for all t, or there exists $\tau > 0$ such that $x(\tau) \notin \overline{K}_a$. Since a is not a qualitative equilibrium, the latter happens for at least one initial state x. Let $J_a := \{t > 0 \,|\, x(t) \notin \overline{K}_a\}$ and $t_1 := \inf_{J_a}$. Since $a\#0$ is a full state and since the initial point x belongs to the cell $\mathbb{R}_a^m$, which is open, then $x(t)$ remains in $\mathbb{R}_a^m$ for $t \in]0, \eta[$ for some $\eta > 0$, and thus, $t_1 > 0$. Since $x(t) \in K_a$ for all $t < t_1$, we deduce that $x(t_1)$ belongs to a transition cell $\overline{K}_a$.

By definition of t_1, there exists a sequence $t_n > t_1$ converging to t_1 such that $x(t_n) \notin \overline{K}_a$; i.e., such that $x(t_n) \in K$ and $g(x(t_n)) \notin a\mathbb{R}_+^m$. This means that there exists a non empty subset $I \subset N$ such that $g(x(t_1))_i = 0$ for all $i \in I$. ▮

We now face two types of problems:

1– What are the transition states $a^I \in a \vee 0$ such that the cell K_{a^I} is reached in finite time by at least one solution starting in K_a?

This problem is closely related to the "target problem" and other controllability issues in control theory, which received only partial solutions. We shall not attempt to answer this question in this paper.

2– What are the successors, if any, of a given transition state a^I?

The second question does not always receive an answer, since, starting from some initial state $x \in K_a^I$, there may exist two sequences $t_n > 0$ and $s_n > 0$ converging to 0+ such that $x(t_n) \in \overline{K}_a$ and $x(s_n) \notin \overline{K}_a$.

We can exclude this pathological phenomenon in two instances. One obviously happens when either a or the transition state a^I is an equilibrium; i.e., when

$$\Gamma(a)_i = 0 \text{ for } i \in I \text{ and } \Gamma(a)_i \subset \{a_i, 0\} \text{ for } i \notin I .$$

This also happens in the following situation:

Lemma 3.2. *Let $a \# 0$ be a full transition state. If $\Gamma(a) \# 0$ (and, thus, is reduced to a point) then for all transition states a^I, there exists a unique successor $b := \Phi(a^I) \# 0$; i.e., for all initial states x in the transition cell K_{a^I}, there exists $t_2 > 0$ such that, for all $t \in]0, t_2[$, the solution $x(t)$ remains in the full qualitative cell K_b.*

Proof: We consider an initial state $x \in K_{a^I}$. If $i \notin I$, then the sign of $g(x)_i$ is equal to $a_i \neq 0$, and thus, there exists $\eta_i > 0$ such that the sign of $g(x(t)_i)$ remains equal to $a_i^I = a_i$ when $t \in [0, \eta_i[$. If $i \in I$, then $g(x)_i = 0$, and we know that the sign of the derivative $\frac{d}{dt} g_i(x(t))_{|t=0} = h_i(x)$ is equal to $\Gamma(a)_i$ and is different from 0. Hence, there exists $\eta_i > 0$ such that the sign of $h(x(t))_i$ remains equal to b_i when $t \in]0, \eta_i[$, so that the sign of

$$g_i(x(t)) = \int_{t_1}^{t} h_i(x(\tau))\, d\tau$$

remains equal to $\Gamma(a)_i$ on the interval $]0, \eta_i[$.

Hence, we have proved that there exists some $\eta > 0$ such that $x(t) \in K_b$ for $t \in]0, t_2[$ where

$$b_i := \begin{cases} \Gamma(a)_i & \text{when } i \in I \\ a_i & \text{when } i \notin I \end{cases}$$

and where $t_2 := \min_i \eta_i > 0$. ∎

Definition 3.1. *We shall say that the system $(f, \mathbb{V})$ is "strictly filterable" if and only if for all full states $a \in \mathcal{D}(f, \mathbb{V}) \# 0$, either $\Gamma(a) \# 0$ or a is a qualitative equilibrium or all the transition states a^I $(I \neq \emptyset)$ are qualitative equilibria.*

We deduce from Definition 3.1 and the above observations the following consequence:

Theorem 3.1. *Let us assume that f is continuously differentiable, that the m functions V_j are twice continuously differentiable around the viability domain K and that the system $(f, \mathbb{V})$ is "strictly filterable." Let $a \in \mathcal{R}^m$ be an initial full qualitative state. Then, for any initial state $x \in K_a$, the sign vector $a_x(t) := s_m(\frac{d}{dt}(\mathbb{V}(Sx(t))))$ is a solution to the QSIM algorithm defined in the following way:*

There exists a sequence of qualitative states a_k satisfying

$$a_0 := a \text{ and } a_{k+1} \in \Phi(a_k \vee 0) \tag{12}$$

and a sequence $t_0 := 0 < t_1 < \ldots < t_n < \ldots$ such that

$$\begin{cases} \forall\, t \in]t_k, t_{k+1}[, \quad a(t) = a_k \\ a(t_{k+1}) = a_k \wedge a_{k+1} . \end{cases} \tag{13}$$

In other words, we know that the vector signs of the variations of the observations of the solutions to (1) evolve according the set-valued dynamical system (12) and stop when a_k is either a qualitative equilibrium or all its transition states a_k^I are qualitative equilibria.

Remark. The solutions to the QSIM algorithm (12) do not necessarily represent the evolution of the vector signs of the variations of the observations of a solution to the differential equation.

Further studies must bring answers allowing to delete impossible transitions from one full qualitative cell $\overline{K}_a$ to some of its transition cells K_{a^I}. This is the case of a qualitative equilibrium, for instance, since a is the only successor of itself. Therefore, the QSIM algorithm requires the definition of the set-valued map $\Gamma : \mathcal{R}^m \rightsquigarrow \mathcal{R}^m$ by computing the signs of the m functions $h_i(\cdot)$ on the qualitative cells K_a^i for all $i \in N$ and $a \in \mathcal{D}(f, \mathbf{V})\#0$. If by doing so, we observe that the system is strictly filterable, then we know that the set-valued dynamical system (12) contains the evolutions of the vector signs of the m observations of solutions to the differential equation (1).

4. Nonemptiness and singularity of qualitative cells. The question we answer now is whether these qualitative cells are non empty.

Theorem 4.1. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. Let $\bar{x}$ belong to the qualitative cell K_0. We posit the transversality condition:*

$$g'(\bar{x})C_K(\bar{x}) - a\mathbb{R}_+^m = \mathbb{R}^m \,. \tag{14}$$

Then the qualitative cell K_a is nonempty and $\bar{x}$ belongs to its closure. In particular, if

$$g'(\bar{x})C_K(\bar{x}) = \mathbb{R}^m,$$

then the 3^m qualitative cells K_a are nonempty. (We have a chaotic situation since every qualitative behavior can be implemented as a initial qualitative state.)

Proof: We apply the Constrained Inverse Function Theorem: if X is a Banach space, Y a finite dimensional vector-space and f a continuous map from a neighborhood of K to Y, continuously differentiable around some point $x_0 \in K$ and if

$$f'(x_0)C_K(x_0) = Y,$$

then there exists a constant $l > 0$ such that, for all $y \in Y$ close enough to $f(x_0)$, there exists a solution $x \in K$ to the equation $f(x) = y$ satisfying $\|x - x_0\| \leq l\|y - f(x_0)\|$. (see [2], [3]). We take here the map $(x, y) \mapsto g(x) - y$ from $X \times Y$ to Y restricted to the closed subset $K \times a\mathbb{R}_+^m$ at the point $(\bar{x}, 0)$. Its Clarke tangent cone is equal to the product $C_K(\bar{x}) \times a\mathbb{R}_+^m$ since

$$C_{a\mathbb{R}_+^m}(0) = a\mathbb{R}_+^m \,.$$

Therefore, we know that there exists $\epsilon > 0$ such that, for all $z \in \epsilon[-1, +1]^m$, there exists an element $x \in K$ and an element $y \in a\mathbb{R}_+^m$ satisfying $g(x) - y = z$ and $\|x - \bar{x}\| + \|y\| \leq l\|z\|$. Taking, in particular, $z_i = a_i\epsilon$, we see that $g(x)_i = a_i\epsilon + y_i$ and thus, that the sign of $g(x)_i$ is equal to a_i for all $i = 1, \ldots, m$. Hence, x belongs to K_a and $\|x - \bar{x}\| \leq l\epsilon$. ∎

Let $\bar{x}$ belong to K_0. We shall say that the qualitative cell $\overline{K}_a$is "singular" at $\bar{x}$ if $\bar{x}$ is locally the only point of the qualitative cell $\overline{K}_a$; i.e., if there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$\forall x \in N(\bar{x}) \cap K, \quad x \neq \bar{x}, \quad g(x) \notin a\mathbb{R}_+^m \,.$$

Theorem 4.2. *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. Let $\bar{x}$ belong to the qualitative cell K_0. We posit the following assumption:*

$$T_K(\bar{x}) \cap (g'(\bar{x})^{-1}(a\mathbb{R}^m_+)) = 0. \tag{15}$$

Then the qualitative cell $\overline{K}_a$ is singular at $\bar{x}$.

Proof: We follow the same arguments as in [3]. Assume the contrary: for all $n > 0$, there exists $x_n \in K$, $x_n \neq \bar{x}$ such that $g(x_n)$ does belong to $a\mathbb{R}^m_+$. Let us set $h_n := \|x_n - \bar{x}\| > 0$ converge to 0 and $v_n := \|(x_n - \bar{x})/h_n\|$. Since v_n belongs to the unit ball, which is compact, a subsequence (again denoted) v_n converges to some element v of the unit ball. This limit v belongs also to the contingent cone $T_K(\bar{x})$ because, for all $n > 0$, $\bar{x} + h_n v_n = x_n$ belongs to K.

Finally, since $g(\bar{x} + h_n v_n) = g(x_n) \in a\mathbb{R}^m_+$ for all $n > 0$ and $g(\bar{x}) = 0$, we infer that the limit $g'(\bar{x})v$ of the difference quotients $\big((g(\bar{x} + h_n v_n) - g(\bar{x}))/h_n\big) \in a\mathbb{R}^m_+$ belongs to $a\mathbb{R}^m_+$. Hence, we have proved the existence of a non zero element

$$v \in T_K(\bar{x}) \cap (g'(\bar{x}))^{-1}(a\mathbb{R}^m_+)$$

a contradiction of the assumption. ▮

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