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AN EXACT FORMULA FOR THE BRANCH OF PERIOD-4-SOLUTIONS OF $\dot{x} = -\lambda f(x(t-1))$ WHICH BIFURCATES AT $\lambda = \pi/2$

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Introduction. In this paper, we construct an exact formula for the periods of solutions of the following system of ordinary differential equations

$$\begin{cases} \dot{x} = -\lambda f(y(t)) \\ \dot{y} = \lambda f(x(t)). \end{cases} \tag{1}$$

Throughout the paper, we assume that f is a function which satisfies the following conditions: $f(x)x > 0$ for $x \neq 0$, f is odd and differentiable, $f'(0) = 1$. The formula we obtain for the periods is a function $T(x_0, \lambda)$ of λ and $x_0 > 0$ where $(x_0, 0)$ is any initial data for system (1). It is known (see [4]) that period-4-solutions of (1) yield period-4-solutions of

$$\dot{x} = -\lambda f(x(t-1)). \tag{2}$$

For these solutions, $T(x_0, \lambda)$ yields a function $\lambda(x_0)$ which shows that a branch of period-4-solutions of (2) bifurcates at $\lambda = \pi/2$. Using $\lambda(x_0)$, we show that the higher derivatives of f at 0 determine the behaviour of the branch near $\pi/2$.

Section 1. In this section, we construct the formula for periods of the solutions of system (1) with initial value of the form $(x(0), y(0)) = (x_0, 0)$, where $x_0 > 0$. We assume that there exists a real interval I containing 0 and that f verifies the following hypothesis on I :

$$\{f \text{ is odd and } f \in C^1(I),\ f(x)x > 0 \text{ for } x \neq 0, \text{ and } f'(0) = 1\}. \tag{H1}$$

Proposition 1.1. *Assume that f satisfies the hypothesis (H1) and consider the following initial value problem*

$$\begin{cases} \dfrac{dx}{dt} = -\lambda f(y(t)) \\ \dfrac{dy}{dt} = \lambda f(x(t)) \\ (x(0), y(0)) = (x_0, 0) \quad \textit{where } x_0 \in I. \end{cases} \tag{1}$$

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Set $F(y) = \int_0^y f(u)\,du$. *Then, the period of the corresponding solution is the function* $T(x_0, \lambda)$ *defined by*

$$T(x_0, \lambda) = \frac{4}{\lambda} \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))}.$$

Period-4-solutions yield

$$\lambda(x_0) = \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))}.$$

Proof: First suppose $\lambda = 1$. Given $(x_0, 0)$, system (1) has one solution $(x(t), y(t))$ with $x(0) = x_0$ and $y(0) = 0$ such that there exists a time T with $x(T) = 0$ and $y(T) > 0$, (for more details see [4]).

Integrating from 0 to T yields $T = \int_0^{x_0} (1/f(y))\,dx$. Now, observing that $F(x) + F(y)$ is constant along the solutions of (1), we find easily that $y = F^{-1}(F(x_0) - F(x))$. The period of $(x(t), y(t))$ is $4T$, hence

$$T(x_0, 1) = 4 \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))}.$$

For $\lambda \neq 1$, we observe that if $z = \lambda F(x)$, then $x = (\lambda F)^{-1}(z)$ and $x = F^{-1}(z/\lambda)$; hence, $(\lambda F)^{-1}(\cdot) = F^{-1}(\cdot/\lambda)$. Now we have

$$T(x_0, \lambda) = 4 \int_0^{x_0} \frac{dx}{(\lambda f)((\lambda F)^{-1}(\lambda F(x_0) - \lambda F(x)))} = \frac{4}{\lambda} \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))}.$$

For $T(x_0, \lambda) = 4$, we get

$$\lambda(x_0) = \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))}. \tag{1.2}$$

This completes the proof of Proposition 1.1.

Formula (1.2) is a parameterization of the branch of period-4-solutions of the differential delay equation (2).

Corollary 1.2. *Assume that f satisfies the hypothesis (H1). Then as λ varies, the only bifurcation point of a branch of period-4-solutions of*

$$\dot{x} = -\lambda f(x(t-1))$$

from the trivial solution is $(0, \pi/2)$.

Proof: We show that $\lambda(x_0)$ approaches $\pi/2$ as x_0 approaches zero. Since f verifies the hypothesis (H1), we can write $f(x) = x + o(x)$ (as $x \to 0$). Hence,

$$f(F^{-1}(F(x_0) - F(x))) \equiv \sqrt{2(F(x_0) - F(x))} \quad \text{as} \quad x_0 \to 0, \;\; 0 \leq x \leq x_0.$$

Thus,

$$\lambda(x_0) = \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))} \equiv \int_0^{x_0} \frac{dx}{\sqrt{2(F(x_0) - F(x))}} \quad \text{as } x_0 \to 0.$$

Let us show that the last integral is convergent. Set $x = ux_0$, $0 \le u \le 1$. Since $f(x)$ is equivalent to x as x approaches zero, we can write

$$F(x_0) - F(x) = F(x_0) - F(ux_0) = \int_{ux_0}^{x_0} f(s)\,ds \ge c\frac{x_0^2}{2}(1-u^2)$$

where c is some positive constant and $0 < x < x_0$. Thus, we have

$$\int_{1-\epsilon}^{1} \frac{du}{\sqrt{2(F(x_0) - F(ux_0))}} \le \frac{1}{\sqrt{c}x_0}\int_{1-\epsilon}^{1}\frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{c}x_0}\sin^{-1} u\Big]_{1-\epsilon}^{1}$$

which approaches zero as ϵ approaches zero. Thus,

$$\int_0^{x_0} \frac{dx}{\sqrt{2(F(x_0) - F(x))}}$$

is convergent. We complete the proof by setting $x = ux_0$, $dx = x_0 du$ in

$$\int_0^{x_0} \frac{dx}{\sqrt{2(F(x_0) - F(x))}},$$

we obtain

$$\begin{aligned}\int_0^{x_0} \frac{dx}{\sqrt{2(F(x_0) - F(x))}} &= x_0\int_0^1 \frac{du}{\sqrt{2(F(x_0) - F(ux_0))}}\\ &= \frac{x_0}{\sqrt{2F(x_0)}}\int_0^1 \frac{du}{\sqrt{1-(F(ux_0)/F(x_0))}}\end{aligned}$$

and

$$\begin{aligned}\lambda(x_0) &\equiv \frac{x_0}{\sqrt{2F(x_0)}}\int_0^1 \frac{du}{\sqrt{1-(F(ux_0)/F(x_0))}}\\ &\equiv \frac{x_0}{\sqrt{2F(x_0)}}\int_0^1 \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{2}\frac{x_0}{\sqrt{2F(x_0)}}\end{aligned}$$

as $x_0 \to 0$. Since $F(x) \equiv x^2/2$, we have $x_0/\sqrt{2F(x_0)} \to 1$ and $\lambda(x_0) \to \pi/2$ as $x_0 \to 0$.

We end this section by the statement of some condition which insures that the branch is subcritical near $\pi/2$.

Proposition 1.3. *Assume that f verifies the hypothesis (H1) and that for x_0 close to 0 and $0 < x < x_0$, we have*

$$\int_x^{x_0} f(s)\,ds > \int_0^{\sqrt{x_0^2 - x^2}} f(s)\,ds.$$

Then the branch of period-4-solutions of equation (2) which bifurcates from the trivial solution at $\lambda = \pi/2$ is subcritical.

We need the following lemma in order to prove Proposition 1.3.

Lemma 1.4. *Under the hypotheses of Proposition 1.3, the following inequality*

$$f(F^{-1}(F(x_0) - F(x))) > \sqrt{x_0^2 - x^2}$$

holds.

Proof: For u small enough and $0 < \alpha < 1$ we can write the following inequality

$$\frac{1}{1-u} \int_{ux_0}^{x_0} f(s)\, ds > \frac{1}{1-u} \int_0^{x_0\sqrt{1-u^2}} f(s)\, ds.$$

But $f(s) \geq \alpha s$ for s small enough. These two inequalities yield $x_0 f(x_0) \geq \alpha x_0^2$, x_0 small enough. Hence, $f(x_0) \geq x_0$, $x_0 > 0$. On the other hand, the inequality

$$\int_x^{x_0} f(s)\, ds > \int_0^{\sqrt{x_0^2 - x^2}} f(s)\, ds$$

means that

$$F(x_0) - F(x) > F(\sqrt{x_0^2 - x^2});$$

that is,

$$F^{-1}(F(x_0) - F(x)) > \sqrt{x_0^2 - x^2}.$$

Finally,

$$f(F^{-1}(F(x_0) - F(x))) > \sqrt{x_0^2 - x^2},$$

which proves the lemma.

Proof of Proposition 1.3. We prove Proposition 1.3 by showing that $\lambda(x_0) \leq \pi/2$ for x_0 close to zero. We set $x = ux_0$; then using Lemma 1.4, we get

$$\lambda(x_0) = \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))} \leq \int_0^{x_0} \frac{dx}{\sqrt{x_0^2 - x^2}} = \int_0^1 \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{2}.$$

An example of function f verifying the hypotheses of Proposition 1.3 is $f(s) = \tan s$ with $I = (-\pi/2, \pi/2)$. With $0 \leq x \leq x_0 < \pi/2$ and $f(s) = \tan s$, the inequality $\int_x^{x_0} f(s)\, ds > \int_0^{\sqrt{x_0^2 - x^2}} f(s)\, ds$ means

$$\ln \frac{\cos x}{\cos x_0} > \ln \frac{1}{\cos \sqrt{x_0^2 - x^2}} \quad \text{or} \quad \frac{\cos x}{\cos x_0} > \frac{1}{\cos \sqrt{x_0^2 - x^2}}.$$

Thus, we have to prove that $\cos x \cos \sqrt{x_0^2 - x^2} - \cos x_0$ is positive for $0 \leq x \leq x_0$. We set $x = ux_0$ and $\phi(u) = \cos ux_0 \cos(x_0\sqrt{1-u^2}) - \cos x_0$ where $0 \leq u \leq 1$. We observe that $\phi(0) = \phi(1) = 0$ and that the derivative of ϕ is

$$\begin{aligned}
\phi'(u) &= -x_0 \sin(ux_0) \cos(x_0\sqrt{1-u^2} + ux_0 \cos ux_0 \sin(x_0\sqrt{1-u^2}) \frac{1}{\sqrt{1-u^2}} \\
&= ux_0^2 \cos(x_0\sqrt{1-u^2}) \Big[\frac{\tan x_0\sqrt{1-u^2}}{x_0\sqrt{1-u^2}} - \frac{\tan ux_0}{ux_0} \Big].
\end{aligned}$$

Thus the derivative ϕ' of ϕ has the same sign as

$$\frac{\tan x_0\sqrt{1-u^2}}{x_0\sqrt{1-u^2}} - \frac{\tan ux_0}{ux_0}\,.$$

We study the variations of $\psi(X) = \tan X/X$. $\psi'(X) = \big(X(1+\tan^2 X) - \tan X\big)/X^2$ which is of the same sign as $K(X) = X(1+\tan^2 X) - \tan X$. But $K'(X) = 1+\tan^2 X + 2\tan X(1+\tan^2 X) - (1+\tan^2 X) = 2\tan X(1+\tan^2 X)$, which is positive for $X \geq 0$. So K is increasing and $K(0) = 0$, hence K is positive for $X \geq 0$. This proves that $\psi(X)$ is positive increasing.

Finally, $\phi'(u)$ is positive for $\sqrt{1-u^2} \geq u$, that is, for $0 \leq u \leq \sqrt{2}/2$, and $\phi'(u)$ is positive for $\sqrt{2}/2 \leq u \leq 1$. Hence, $\phi(u)$ increases from zero to $\phi(\sqrt{2}/2)$, then decreases to $\phi(1) = 0$. So $\phi(u)$ is positive. This proves that $\tan x$ verifies the hypotheses of proposition 1.1.

Section 2. In this section we show that the higher derivatives of f at 0 determine the behaviour of the branch of period-4-solutions of equation (2) which bifurcates from the trivial solution. We rewrite the formula of $\lambda(x_0)$ in a more convenient form by setting $y = F(x)$, or $x = F^{-1}(y)$ and $dx = (F^{-1})'(y)dy$. Then we get

$$\lambda(x_0) = \int_0^{x_0} \frac{dx}{f(F^{-1}(F(x_0) - F(x)))} = \int_0^{F(x_0)} (F^{-1})'(F(x_0) - y)(F^{-1})'(y)\,dy\,.$$

Using the last expression of $\lambda(x_0)$, we can prove the following relations between the higher derivatives of $\lambda(x_0)$ and those of f at zero: we assume that there exists a real interval I containing zero and that f verifies the following hypothesis on I :

$$\{f \text{ is odd, } f(x)x > 0;\ f \text{ is analytic and } f'(0) = 1\}\,. \tag{H2}$$

Proposition 2.1. *Assume that f satisfies the hypotheses (H2). Then we have the following equivalence:*

$$\{\lambda'(0) = \cdots = \lambda^{(2n-1)}(0) = 0,\ \lambda^{(2n)}(0) \neq 0\} \iff$$
$$\{f''(0) = \cdots = f^{(2n)}(0) = 0,\ f^{(2n+1)}(0) \neq 0\} \quad \text{for } n \geq 1.$$

Proof: This proposition is a direct consequence of Proposition 2.2, 2.3 and 2.4 below. First, we give a new expression of $\lambda(x_0)$ using the hypothesese on f. We can write $F(x) = (x^2/2)\Phi(x)$, where Φ is some function such that $\Phi(0) = 1$. Thus, $x = F^{-1}(y) = \sqrt{2F(x)/\Phi(x)} = \sqrt{2y}\Psi(y)$, where Ψ is some function such that $\Psi(0) = 1$. This yields the next expression of $\lambda(x_0)$:

$$\lambda(x_0) = \int_0^{F(x_0)} \Big[\frac{\Psi(F(x_0) - y)}{\sqrt{2(F(x_0) - y)}} + \sqrt{2F(x_0) - y)}\Psi'(F(x_0) - y)\Big]\Big[\frac{\Psi(y)}{\sqrt{2y}} + \sqrt{2y}\Psi'(y)\Big]\,dy$$

because $(F^{-1})'(y) = \frac{\Psi(y)}{\sqrt{2y}} + \sqrt{2y}\Psi'(y)$.

Proposition 2.2. *Assume that the hypothesis of Proposition 2.1 is satisfied and set*

$$F(y) = \int_0^y f(u)\,du; \qquad H(u) = \int_0^u (F^{-1})'(u-y)(F^{-1})'(y)\,dy.$$

Then we have the following equivalence:

$$\{\lambda''(0) = \cdots = \lambda^{2(n-1)}(0) = 0,\ \lambda^{(2n)}(0) \neq 0\} \iff$$
$$\{H'(0) = \cdots = H^{(n-1)}(0) = 0,\ H^{(n-1)}(0) \neq 0\}, \quad n \geq 2.$$

Proof: $\lambda(\cdot) = H(F(\cdot))$ is an even function since F is even. By a lemma proved by Whitney (see [3] page 248), there exists some function $\mu(\cdot)$ such that $\lambda(u) = \mu(u^2)$. Comparing the expansions of λ and μ near zero, we get

$$\{\lambda''(0) = \cdots = \lambda^{2(n-1)}(0) = 0,\ \lambda^{(2n)}(0) \neq 0\} \iff$$
$$\{\mu'(0) = \cdots = \mu^{(n-1)}(0) = 0,\ \mu^{(n)}(0) \neq 0\}.$$

On the other hand,

$$H(y) = \lambda(F^{-1}(y)) = \lambda(\sqrt{2y}\Psi(y)) = \mu(2y(\Psi(y))^2).$$

Hence, $H^{(k)}(y)$ is a combination of derivatives of μ of order between 1 and k evaluated at the point $2y(\Psi(y))^2$. Also $H^{(n)}(0) = 2^n\mu^{(n)}(0)$. Thus,

$$\{H^{(k)}(0) = 0 \ \text{ for } \ 1 \leq k \leq n-1\} \iff \{\mu^{(k)}(0) = 0 \ \text{ for } \ 1 \leq k \leq n-1\}$$

and

$$H^{(n)}(0) \neq 0 \iff \mu^{(n)}(0) \neq 0.$$

Finally, we have

$$\{\lambda''(0) = \cdots = \lambda^{2(n-1)}(0) = 0,\ \lambda^{(2n)}(0) \neq 0\} \iff$$
$$\{H'(0) = \cdots = H^{(n-1)}(0) = 0,\ H^{(n)}(0) \neq 0\}$$

which ends the proof.

Proposition 2.3 below relates the derivatives of H to those of the function Ψ introduced before.

Proposition 2.3. *Assume that the expansions of H and Ψ are given by*

$$\Psi(z) = \sum_{j\geq 0} a_j z^j \quad \textit{and} \quad H(u) = \sum_{k\geq 0} H_k u^k.$$

Then we have the following equivalence:

$$\{H_1 = \cdots = H_{n-1} = 0,\ H_n \neq 0\} \iff \{a_1 = \cdots = a_{n-1} = 0,\ a_n \neq 0\}.$$

Proof: In terms of Ψ, $H(u)$ reads

$$H(u) = \int_0^u \left[\frac{\Psi(u-y)}{\sqrt{2(u-y)}} + \sqrt{2(u-y)}\Psi'(u-y)\right]\left[\frac{\Psi(y)}{\sqrt{2y}} + \sqrt{2y}\Psi'(y)\right]dy.$$

Set $y = uz$ in $H(u)$ and we get

$$H(u) = \frac{1}{2}\int_0^1 \frac{\Psi(u(1-z))\cdot\Psi(uz)}{\sqrt{z(1-z)}}\,dz + 2u\int_0^1 \Psi(u(1-z))\cdot\Psi'(uz)\sqrt{\frac{z}{1-z}}\,dz + 2u^2\int_0^1 \Psi'(uz)\Psi'(u(1-z))\sqrt{z(1-z)}\,dz.$$

Since $\Psi'(z) = \sum_{j\geq 1} ja_j z^{j-1}$, the coefficient of the term of order k in the expansion of H is

$$\frac{1}{2}\sum_{j=0}^{k} a_j a_{k-j} B(j+\frac{1}{2}\,,\,k-j+\frac{1}{2}) + 2\sum_{j=0}^{k-1} a_j a_{k-j}(k-j)B(j+\frac{1}{2}\,,\,k-j+\frac{1}{2})$$
$$+\,2\sum_{j=0}^{k-2} a_{j+1}a_{k-j-1}(j+1)(k-j-1)B(j+\frac{3}{2}\,,\,k-j-\frac{1}{2}),$$

where

$$B(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1}\,dz$$

is the beta function. Thus,

$$H_k = ca_k B(\frac{1}{2}\,,\,k+\frac{1}{2}) + \Theta_k(a_1,\cdots,a_{k-1})$$

where Θ_k is bilinear with respect to the a_j's and c is a constant, $c\neq 0$. Hence, ($a_j = 0$, for $j = 1,\cdots,n-1$) $\iff H_n = ca_nB(\frac{1}{2}\,,\,n+\frac{1}{2})$. This ends the proof, since $B(\frac{1}{2}\,,\,n+\frac{1}{2})\neq 0$. ∎

Here are some explicit values of coefficients H_k :
$H_0 = \frac{1}{2}a_0^2B(\frac{1}{2}\,,\,\frac{1}{2}) = \frac{\pi}{2}$ since $a_0 = \Psi(0) = 1$. H_0 corresponds to the bifurcation point.
$H_1 = 3a_1B(\frac{1}{2}\,,\,\frac{3}{2})$, $H_2 = 5B(\frac{1}{2}\,,\,\frac{5}{2}) + \frac{9}{2}B(\frac{3}{2}\,,\,\frac{3}{2})a_1^2\cdots$.
The next proposition relates the coefficients a_j to the derivatives of f.

Proposition 2.2. *Assume that f satisfies the hypothesis of Proposition 2.1 and that the expansions of f and Ψ are given respectively by*

$$f(x) = \sum_{j\geq 0} b_j x^j \quad\text{and}\quad \Psi(y) = \sum_{i\geq 0} a_i y^i\,.$$

Then the following equivalence holds:

$$\{a_1 = \cdots = a_{n-1} = 0,\ a_n\neq 0\} \iff \{f'''(0) = \cdots = f^{(2n-1)}(0) = 0,\ f^{(2n+1)}(0)\neq 0\}.$$

Proof: Since $\Psi(0) = 1$ we may assume that near zero we have

$$\Psi(y) = 1 + a_n y^n,\quad f(x) = x + \sum_{j>1} b_j x^{2j+1}\ \text{ and }\ F(x) = \frac{x^2}{2} + \sum_{j\geq 1} b'_j x^{2(j+1)}\,.$$

Now,

$$y^2 = F(\sqrt{2}\Psi(y^2)y) = \frac{1}{2}(\sqrt{2}y + \sqrt{2}a_n y^{2n+1})^2 + \sum_{j>1} b_j(\sqrt{2}y + \sqrt{2}a_n y^{2n+1})^{2(j+1)}.$$

Identifying the exponents of y in both sides shows that $\{a_i = 0 \text{ for } i = 1, \cdots, n-1\}$ is equivalent to $\{b_i = 0 \text{ for } i = 3, \cdots, (2n-1)\}$. Moreover, we must have $2a_n + 2^{n+1}b'_n = 0$. Thus, $\{a_n \neq 0 \Longrightarrow b'_n \neq 0\} \Longrightarrow b_n \neq 0$. The same reasoning shows that $\{b_3 = \cdots = b_{2n-1} = 0 \text{ and } b_{2n+1} \neq 0\}$ implies that $\{a_1 = \cdots = a_{n-1} = 0,\ a_n \neq 0.\}$ This ends the proof. ∎

Proposition 2.1 follows from Propositions 2.2, 2.3 and 2.4. Let us end this paper by an explicit calculation of $\lambda''(0)$ in terms of $f'''(0)$. Since $\lambda(x_0) = H(F(x_0))$, $F' = f$, $f(0) = 0$ and $f'(0) = 1$ one finds $\lambda''(0) = H'(0)$. But

$$H(u) = \frac{1}{2}\int_0^1 \frac{\psi(u(1-z))\psi(uz)}{\sqrt{z(1-z)}}\,dz + 2u\int_0^1 \psi(u(1-z))\psi'(uz)\sqrt{\frac{z}{1-z}}\,dz$$
$$+2u^2\int_0^1 \psi'(uz)\psi'(u(1-z))\sqrt{z(1-z)}\,dz.$$

Differentiating H at zero yields

$$H'(0) = \left[\frac{1}{2}\int_0^1 \frac{dz}{\sqrt{z(1-z)}} + 2\int_0^1 \sqrt{\frac{z}{1-z}}\,dz\right]\psi(0)\psi'(0) = \frac{3\pi}{2}\psi(0)\psi'(0).$$

So we have to calculate $\psi(0)$ and $\psi'(0)$. Successive differentiations of the equality $y^2 = F(\sqrt{2}y\psi(y^2))$ yield

$$0 = f''(\sqrt{2}y\psi(y^2))\left(\sqrt{2}\psi(y^2) + 2\sqrt{2}y^2\psi'(y^2)\right)$$
$$+ 3f'(\sqrt{2}y\psi(y^2))\left(\sqrt{2}\psi(y^2) + 2\sqrt{2}y^2\psi'(y^2)\right)\left(6\sqrt{2}y\psi'(y^2) + 4\sqrt{2}y^3\psi''(y^2)\right)$$
$$+ f(\sqrt{2}y\psi(y^2))\left(6\sqrt{2}\psi'(y^2) + 24\sqrt{2}y^2\psi''(y^2) + 8\sqrt{2}y^4\psi'''(y^2)\right).$$

Since $f''(u) \approx f'''(0)u$ and $f(u) \approx f'(0)u$ as $u \to 0$, we can neglect the terms of order 1 in the last identities; then we get

$$0 = (\sqrt{2}\psi(0))^4 f'''(0) + 36f'(0)\psi(0)\psi'(0) + 12f'(0)\psi'(0) = 48\psi'(0) + 4f'''(0).$$

Hence, $\psi'(0) = -\frac{1}{12}f'''(0)$. Finally, $\lambda''(0) = H'(0) = Af'''(0)$ where

$$A = -\frac{1}{24}\int_0^1 \frac{dz}{\sqrt{z(1-z)}} - \frac{1}{6}\int_0^1 \sqrt{\frac{z}{1-z}}\,dz = -\frac{\pi}{8}.$$

Thus, $\lambda''(0) = H'(0) = -\frac{\pi}{8}f'''(0)$ and the sign of $\lambda''(0)$ is completely determined by the one of $f'''(0)$. Hence, the bifurcating branch is super or subcritical depending on whether $f'''(0)$ is positive or negative.

Remark. In Proposition 2.1, we can assume that $f \in C^\infty(I)$ instead of f analytic, since the lemma of Whitney is proved with C^∞-hypothesis. See [5] and [6].

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