Differential and Integral Equations, Volume 2, Number 2, April 1989, pp. 132-143.

STABILIZING SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns the stabilization of a second differential controlled equation in R^n, $x'' + \partial\phi(x) \ni u$, $\|u(t)\| \leq 1$ by a feedback law of the form $u = \psi(x')$. Applications to the stabilizations of the movement of an elastic string with a discrete distribution of masses and limited from below by a rigid obstacle is given.

1. Introduction. This work is concerned with the controlled second order differential equation

$$\frac{d^2x}{dt^2} + \partial\phi(x) \ni u \quad \text{in } R^+ \tag{1.1}$$

where $x : [0,T] \to R^n$, $\|u(t)\| \leq 1$ a.e., $t > 0$, $x'' = d^2x/dt^2$ and $\partial\phi : R^n \to 2^{R^n}$ is the subdifferential of a lower semicontinuous convex function $\phi : R^n \to (-\infty, +\infty]$; i.e., (see e.g. [2])

$$\partial\phi(x) = \{y \in R^n;\ \phi(x) - \phi(u) \leq \langle y, x-u\rangle \ \ \forall u \in R^n\}.$$

We have denoted by $\langle\cdot,\cdot\rangle$ the usual scalar product in R^n and by $\|\cdot\|$ the Euclidean norm of R^n.

The main result, Theorem 2 below, amounts to saying that the above equation can be stabilized by a nonlinear feedback law $u = \psi(dx/dt)$.

By solution to the Cauchy problem

$$\frac{d^2x}{dt^2} + \partial\phi(x) \ni \psi(\frac{dx}{dt}) \quad \text{in } (0,T) \tag{1.2}$$
$$x(0) = x_0, \ \ \frac{dx}{dt}(0) = x,$$

we mean a function $x \in W^{1,\infty}([0,T]; R^n)$ such that (see [3])

(a) $d^2x/dt^2 + \mu - \psi(dx/dt) = 0$ in the sense of distribution where μ is a bounded measure on $[0,T]$ such that

$$\int_0^T (\phi(v(s)) - \phi(x(s)))\, ds \geq \mu(v - x) \tag{1.3}$$

Received March 10, 1988.
AMS Subject Classifications: 34D05, 34C10.

for any continuous function $v \in C([0,T]; R^n)$ such that $\phi(v) \in L^1(0,T)$;

(b) $dx/dt \in BV([0,T]; R^n)$ and the energy is conserved:

$$\begin{aligned}\frac{1}{2}\Big\|\frac{d^+x}{dt}(t)\Big\|^2 + \phi(x(t)) &= \frac{1}{2}\Big\|\frac{d^-x}{dt}(t)\Big\|^2 + \phi(x(t)) \\ &= \frac{1}{2}\|x_1\|^2 + \phi(x_0) + \int_0^t \left\langle \frac{dx}{ds}(s),\, \psi(\frac{dx}{ds}(s))\right\rangle ds\end{aligned} \tag{1.4}$$

for all $t \in [0,T]$.

(c) $x(0) = x_0$ and $x_1 - (dx^+/dt)(0) + \partial\psi_{K_0}(x_0) \ni 0$ where $K_0 = \overline{\operatorname{dom}\phi}$ and ψ_{K_0} is the indicator function of K_0.

We have denoted by $\operatorname{dom}\phi$ the effective domain of ϕ, i.e., the set $\{x \in R^n;\ \phi(x) < \infty\}$ and by $BV([0,T]; R^n)$ the space of functions $x : [0,T] \to R^n$ with bounded variation; $W^{1,\infty}([0,T]; R^n)$ is the space $x : [0,T] \to R^n$ absolutely continuous with $\frac{dx}{dt} \in L^\infty(0,T; R^n)$.

For the sake of simplicity, we will use in the following, the notations, $d^2x/dt^2 = x''$ and $dx/dt = x'$. Throughout in the sequel, $\psi : R^n \to R^n$ is locally Lipschitz.

We will prove first the following existence result for problem (1.2).

Theorem 1. *Suppose that $\phi : R^n \to (-\infty, +\infty)$ is a lower semicontinuous convex function such that $\operatorname{int}(\operatorname{dom}\phi) \neq \emptyset$. Then, for $x_0 \in \operatorname{dom}\phi$ and $x_1 \in R^n$, the problem (1.2) has at least one solution on the real positive semiaxis R^+.*

A result of this type has been previously proved by M. Schatzman [4] for a related equation.

Theorem 2. *Let ϕ be such that $\inf\phi > -\infty$ and $(\partial\phi)^{-1}(0) = 0$. If $\psi : R^n \to R^n$ is Lipschitzian and*

$$\langle \psi(x), x\rangle < 0 \quad \forall x \neq 0, \tag{1.5}$$

then for every solution x of problem (1.2), we have

$$\lim_{t\to\infty} (x(t), x'(t)) = (0,0). \tag{1.6}$$

Corollary 1. *Under assumptions of Theorem 2, the feedback law $u = \psi(x')$ where*

$$\psi(y) = \begin{cases} -y & \text{if } \|y\| \le 1 \\ -y/\|y\| & \text{if } \|y\| > 1 \end{cases} \tag{1.7}$$

stabilizes the control system (1.1) with the control constraint $\|u(t)\| \le 1$.

A related feedback control has been used by M. Slemrod [5] to stabilize the hyperbolic beam equation with boundary control.

In §3, we will give an application to the stabilization of the elastic string with a discrete distribution of masses and limited from below by a rigid obstacle.

2. Proofs.

Proof of Theorem 1: Since the proof is similar to that given in [4], it will be outlined only. We use the following notations

$$\phi_\lambda(x) = \inf\left\{\frac{1}{2\lambda}\|x-y\|^2 + \phi(y);\ y \in H\right\} \tag{2.1}$$

and recall that ϕ_λ is continuously differentiable, convex and

$$\partial\phi_x(x) = \lambda^{-1}(I - J_\lambda)(x), \quad J_\lambda = (I + \lambda\partial\phi_\lambda)^{-1}, \quad \forall\lambda > 0 \tag{2.2}$$

$$\partial\phi_\lambda(x) \in \partial\phi(J_\lambda(x)), \quad \phi_\lambda(x) = \phi(J_\lambda x) + \frac{1}{2\lambda}\|x - J_\lambda(x)\|^2 \tag{2.3}$$

$$\lim_{\lambda\to 0} \phi_\lambda(x) = \phi(x), \quad x \in R^n . \tag{2.4}$$

Consider the approximating problem

$$\begin{aligned} &u''_\lambda + \partial\phi_\lambda(u_\lambda) = \psi(u'_\lambda), \quad t \in [0,T] \\ &u_\lambda(0) = u_0, \quad u'_\lambda(0) = u_1 \end{aligned} \tag{2.5}$$

which has a unique solution $u \in C^2([0,T]; R^n)$ because $\partial\phi_\lambda$ and ψ are Lipschitzian. Moreover, we have

$$\frac{1}{2}\|u'_\lambda(t)\|^2 + \phi_\lambda(u_\lambda(t)) = \frac{1}{2}\|u_1\|^2 + \phi_\lambda(u_0) + \int_0^t \langle u'_\lambda(s), \psi(u'_\lambda(s))\rangle\, ds. \tag{2.6}$$

Since ϕ is bounded from below by an affine function, it follows by (2.1), (2.3) that there exist $a \in R^n$ and $b \in R$ independent of λ such that

$$\phi_\lambda(y) \geq (a, y) + b, \quad \forall\, y \in R^n, \ \lambda > 0. \tag{2.7}$$

Substituting the latter in (2.6) and using the obvious inequality

$$\|\psi(y)\| \leq \|y\|, \quad \forall y \in R^n \tag{2.8}$$

we get, after some calculation

$$\|u'_\lambda(t)\|^2 \leq \|u_1\|^2 + 2\phi(u_0) + C_1 \int_0^T \|u'_\lambda(s)\|^2\, ds + C_2, \quad \forall t \in [0,T]$$

and by Gronwall's lemma

$$\|u'_\lambda(t)\|^2 \leq \left(\|u_1\|^2 + 2\phi(u_0)\right) e^{Ct}, \quad \forall t \in [0,T] \tag{2.9}$$

where C is a positive constant independent of λ.

In the sequel, we shall use the notations: $C^0 = C([0,T]; R^n)$, $L^p = L^p([0,T]; R^n)$ $(1 \leq p \leq \infty)$, $BV = BV([0,T]; R^n)$, $W^{-1,\infty}([0,T]; R^n) = W^{-1,\infty}$ and denote by $\mathcal{M}$ the space of all bounded R^n-valued measures on $[0,T]$; i.e., the dual of C^0.

By estimates (2.6), (2.9) and Arzelà lemma, we may extract a subsequence again denoted λ, such that

$$u_\lambda \to x \quad \text{in } C^0, \tag{2.10}$$

$$u'_\lambda \to T' \quad \text{weakly} * \text{ in } L^\infty, \tag{2.11}$$

$$\phi_\lambda(u_\lambda) \to \chi \quad \text{weakly} * \text{ in } L^\infty. \tag{2.12}$$

By (2.3), we see that

$$\|u_\lambda(t) - J_\lambda(u_\lambda(t))\|^2 \leq C\lambda, \quad \forall\lambda > 0, \ t \in [0,T].$$

Since $J_\lambda(u_\lambda(t))$, $D(\partial\phi) \subset \operatorname{dom}\phi$, we infer that $x(t) \in \overline{\operatorname{dom}\phi}$ for all $t \in [0,T]$.

The sequence $\{\partial\phi_\lambda(u_\lambda)\}$ is bounded in L^1. Indeed, let $a \in \overset{0}{\operatorname{dom}}\,\phi$, and $\rho \in R_+$, $B(a,\rho) \subset \overset{0}{\operatorname{dom}}\,\phi$. If $v \in C^0$ and $\|v\|_\infty \leq 1$, we have:

$$\langle \partial\phi_\lambda(u_\lambda(t))),\ a + \rho v(t) - u_\lambda(t)\rangle \leq \phi_\lambda(a + \rho v(t)) - \phi_\lambda(u_\lambda). \tag{2.13}$$

Let $c = \sup_{x \in B(a,\rho)} \phi(x)$ $(< \infty)$ and M such that $\phi_\lambda(u_\lambda) \leq M$. Then, (2.13) implies that

$$\begin{aligned}
&\rho\Big|\int_0^T \langle \partial\phi_\lambda(u_\lambda(t)),\ v(t)\rangle\, dt\Big| \\
&\leq \Big|\int_0^T \phi_\lambda(a + \rho v(t))\, dt\Big| + \Big|\int_0^T \phi_\lambda(u_\lambda(t))\, dt\Big| + \Big|\int_0^T \langle \partial\phi_\lambda(u_\lambda(t)),\ a - u_\lambda(t)\rangle\, dt\Big| \\
&\leq cT + MT + \Big|\int_0^T \langle \psi(u'_\lambda) - u''_\lambda,\ a - u_\lambda\rangle\, dt\Big| \\
&\leq cT + MT + \int_0^T \|\psi(u'_\lambda)\|\, |a - u_\lambda|\, dt + \Big|\int_0^T \langle u''_\lambda,\ a - u_\lambda\rangle\, dt\Big| \\
&\leq cT + MT + Tc_1 + |\, \langle u'_\lambda,\ a - u_\lambda\rangle \big]_0^T | + \Big|\int_0^T \langle u'_\lambda,\ a - u'_\lambda\rangle\, \Big| \leq C,
\end{aligned}$$

(we shall denote by C several positive constants independent of λ).

If we take now $v(t) = \big(\partial\phi_\lambda(u(t))\big)/\big(\|\partial\phi_\lambda(u(t))\|\big)$, we obtain:

$$\int_0^T \|\partial\phi_\lambda(u_\lambda(t))\|\, dt \leq \frac{k}{\rho}.$$

Hence, we can extract a new subsequence such that

$$\partial\phi_\lambda(u_\lambda) \to \mu \quad \text{weakly} * \text{in } \mathcal{M}. \tag{2.14}$$

Because the identity map $i : \mathcal{M} \to W^{-1,\infty}$ is compact, it follows that

$$\partial\phi_\lambda(u_\lambda) \to \mu \quad \text{in } W^{-1,\infty}. \tag{2.15}$$

Since $\{u'_\lambda\}$ is bounded in L^∞, so is $\{\psi(u'_\lambda)\}$. Thus, we can extract a generalized subsequence such that

$$\psi(u'_\lambda) \to \nu \quad \text{weakly} * \text{in } L^\infty \tag{2.16}$$

and therefore,

$$\psi(u'_\lambda) \to \nu \quad \text{in } W^{-1,\infty} \quad (\text{or weakly} * \text{in } \mathcal{M}). \tag{2.17}$$

Then, passing at limit in (2.5), it follows that there exists $w \in W^{-1,\infty}$ such that $u''_\lambda \to w$ in $W^{-1,\infty}$ (or $w \in \mathcal{M}$, $u''_\lambda \to w$ weakly $*$ in $\mathcal{M}$).

Now, (2.11) gives us $w = \chi''$ in the sense of distributions, so that by the Helly theorem, $u'_\lambda(t) \to x'(t)$, $(\forall t \in [0,T])$ and

$$u'_\lambda \to x' \quad \text{in } L^p \text{ for } p \in [1,\infty).$$

Then, $\psi(u'_\lambda) \to \psi(x')$ in L^p, $1 \le p < \infty$, and letting λ tend to zero in (2.5), it follows that $x'' = \mu = \psi(x')$ in the sense of distributions.

For that μ defined above, we shall prove (1.3). We will show first that:

$$\phi_\lambda(u_\lambda) \to \phi(x) \text{ in } L^1, \tag{2.18}$$

According to (2.4),

$$\phi_\lambda(u_\lambda) - \phi(J_\lambda u_\lambda) = \frac{1}{2\lambda}\|J_\lambda u_\lambda - u_\lambda\|^2.$$

Let us prove that

$$\lambda^{-1}\int_0^T \|J_\lambda u_\lambda - u_\lambda\|^2\,dt \to 0 \text{ as } \lambda \to 0. \tag{2.19}$$

We observe that

$$\frac{u_\lambda - J_\lambda u_\lambda}{\lambda} = \partial\phi_\lambda(u_\lambda) \to \mu \text{ weakly } * \text{ in } \mathcal{M}.$$

Then, it would be enough to show that for $\lambda \to 0$

$$J_\lambda u_\lambda - u_\lambda \to 0 \text{ in } C^0. \tag{2.20}$$

Let $v_\lambda = J_\lambda u_\lambda$. Since $\lambda^{-1}(v_\lambda - u_\lambda) = \partial\phi_\lambda(v_\lambda)$ we get

$$M \ge \phi_\lambda(u_\lambda) - \phi_\lambda(v_\lambda) \ge \langle u_\lambda - v_\lambda, \partial\phi_\lambda(v_\lambda)\rangle = \frac{1}{\lambda}\|v_\lambda - u_\lambda\|^2.$$

Hence, $\|v_\lambda - u_\lambda\|_\infty \le \sqrt{2M\lambda} \to 0$ as $\lambda \to 0$, as claimed. This implies that

$$\phi_\lambda(u_\lambda) - \phi(J_\lambda u_\lambda) \to 0 \text{ in } L^1 \text{ as } \lambda \to 0. \tag{2.21}$$

Because $(J_\lambda u_\lambda - u_\lambda) \to 0$ in C^0, $u_\lambda - u \to 0$ in C^0, it follows that $J_\lambda u_\lambda \to u$ in C^0, and therefore,

$$\liminf_{\lambda\to 0}\int_0^T \phi_\lambda(u_\lambda(t))\,dt \ge \liminf\int_0^T \phi(J_\lambda(u_\lambda(t)))\,dt \ge \int_0^T \phi(x(t))\,dt \tag{2.22}$$

because the convex integrand $u \to \int_0^T \phi(u(t))\,dt$ is lower semicontinuous in L^1. Next, by the inequality

$$\phi_\lambda(u_\lambda) \le \phi_\lambda(x) + \langle\partial\phi_\lambda(u_\lambda), u_\lambda - x\rangle \le \phi(x) + \langle\partial\phi_\lambda(u_\lambda), u_\lambda - x\rangle$$

we find that

$$\limsup_{\lambda\to 0}\int_0^T \phi_\lambda(u_\lambda(t))\,dt \le \int_0^T \phi(x(t))\,dt$$

as claimed. Letting λ tend to zero in inequality

$$\int_0^T (\phi_\lambda(v(t)) - \phi_\lambda(u_\lambda(t)))\,dt \ge \int_0^T \langle\partial\phi_\lambda(u_\lambda(t)), v(t) - u_\lambda(t)\rangle\,dt,$$

we get (1.2); i.e.,

$$\int_0^T (\phi(v(t)) - \phi(x(t)))\,dt \ge \mu(v - x), \quad \forall v \in C^0. \tag{2.23}$$

Next, passing to limit in (2.6), we obtain the energy conservation relation (1.4).

Finally, relation (c) follows as in [3], thereby completing the proof.

Remark. If $x(t) \in \overset{0}{\mathrm{dom}}\,\phi$ for $t \in [t_1, t_2]$, then $x \in W^{2,\infty}([t_1,t_2], R^n)$ and $x'' + \partial\phi(x) \ni \psi(\chi')$ a.e. on $[t_1, t_2]$. Indeed, let $k = \{x(t)/t \in [t_1, t_2]\}$. Then,

$$\delta = d(k, \partial\,\mathrm{dom}\,\phi) > 0.$$

If $S = \{x \in R^n,\ d(x,k) \leq \delta/2\}$, then S is compact, $S \subset \overset{0}{\mathrm{dom}}\,\phi$. Because ϕ is continuous on $\overset{0}{\mathrm{dom}}\,\phi$, it follows:

$$|\phi(x)| \leq M, \quad \forall x \in S, \quad \text{for some } M \in R^+.$$

Let $A \subset [t_1, t_2]$, with the Lebesgue measure 0. For every $x \in R^n$, $\|x\| = 1$, we may define

$$v : [t_1, t_2] \to R^n, \quad v(t) = \begin{cases} x(t) & \text{if } t \notin A \\ x(t) + \frac{\delta}{2}x & \text{if } t \in A. \end{cases}$$

By (1.2), we have (we may assume that $\mu \in (L^\infty)^*_\cdot)$)

$$\int_{t_1}^{t_2} \phi(v(s)) - \phi(x(s))\,ds \geq \mu(|v - x|) = \frac{\delta}{2}\langle \mu(A), x\rangle\ .$$

Clearly, $\phi(v(\cdot)) \in L^1([t_1,t_2], R)$. But,

$$\left|\int_{t_1}^{t_2} \phi(v(s)) - \phi(x(s))\,ds\right| \leq \int_{t_1}^{t_2} |\phi(v(s)) - \phi(x(s))|\,ds = \int_A \phi(v(s)) - \phi(x(s))\,ds = 0,$$

and, therefore,

$$\langle \mu(A), x\rangle \leq 0, \quad \forall x \in R^n,\ \|x\| = 1,$$

that is, $\mu(A) = 0$. Then, by Radon-Nikodim theorem, $\mu \in L^1([t_1,t_2], R^n)$, $x'' \in L^1([t_1,t_2], R^n)$, and then, by (1.3), we see that $x'' + \partial\phi(x) \ni \psi(\chi')$ almost everywhere. Because the set $\{\partial\phi(x)/x \in k\}$ is bounded in R^n, we may also conclude that x'' is bounded $\Longrightarrow$ $x'' \in L^\infty([t_1,t_2], R^n)$, and hence, $x \in W^{2,\infty}([t_1,t_2], R^n)$.

Proof of Theorem 2: By the energy conservation relation

$$\frac{\|x'(t)\|}{2} + \phi(x(t)) \leq \frac{\|x'(0)\|}{2} + \phi(x(0)) = E.$$

Then,

$$\|x'(t)\| \leq \sqrt{2E}, \quad \forall t \in R^+. \tag{2.24}$$

Note that every level set $\{u;\ \phi(u) \leq M\}$ is bounded. Indeed, assume by contradiction that there exists $\|x_n\| \to \infty$, $\psi(x_n) \leq M$. Then, we may extract a subsequence such that $(x_n/\|x_n\|) \to \alpha$, $\|\alpha\| = 1$. Since ϕ is convex, we have:

$$\phi(\frac{x_n}{\|x_n\|}) \leq \phi(0) \cdot \frac{\|x_n\|_{-1}}{\|x_n\|} + \phi(x_n)\frac{1}{\|x_n\|}$$

and passing at limit $n \to \infty$ we get

$$\phi(\alpha) \leq \phi(0).$$

But $0 \in \partial\phi(0) \Longrightarrow \phi(x) \geq \phi(0)$, $\forall x \in R^n$, so $\phi(x) \geq \phi(\alpha)$, $\forall x \in R^n \Longrightarrow 0 \in \partial\phi(\alpha) \Longrightarrow \alpha = 0$, in contradiction with $\|\alpha\| = 1$.

Thus, it follows that $\|x(t)\| \leq M$, $\forall t \in R^+$, for some $M \in R^+$.

Let us prove first that

$$\lim_{t\to\infty} x(t) = 0. \tag{2.25}$$

Assume by contradiction that there exists $\epsilon > 0$ and $t_n \to \infty$, such that $\|x(t_n)\| > \epsilon$. Let $T > 0$, and $x_n : [0,T] \to R^n$, $x_n(t) = x(t_n + t)$. Because x_n, x'_n are uniformly bounded, it follows that $\partial\phi(x_n)$ are uniformly bounded measures on $[0,T]$. Then, we can extract a subsequence such that

$$x_n \to v \quad \text{in } C^0 \tag{2.26}$$

$$x'_n \to v' \quad \text{in } L^p,\ \ \forall 1 \leq p < \infty \tag{2.27}$$

$$x''_n \to v'' \quad \text{weakly} * \text{ in } \mathcal{M} \tag{2.28}$$

Because $t_n \to \infty$, and $\int_0^\infty \langle x', \psi(x')\rangle\, dt$ is convergent, it follows that

$$\lim_{n\to\infty} \int_{t_n}^{t_n+T} \langle x', \psi(x')\rangle\, dt = 0.$$

Hence,

$$\lim_{n\to\infty} \int_0^T \langle x'_n, \psi(x'_n)\rangle\, dt = 0$$

and therefore,

$$\int_0^T \langle v', \psi(v')\rangle\, dt = 0.$$

Inasmuch as $\langle v', \psi(v')\rangle \leq 0$, we conclude that $\langle v', \psi(v')\rangle = 0$ almost everywhere. Hence, $v' = 0$, and $v = c$, $v'' = 0$ almost everywhere. Since $x''_n \to 0$ weakly $*$ in $\mathcal{M}$, and $\psi(\chi'_n) \to 0$ in L^p, for all $p \in [1, \infty)$, it follows that $\mu_n \to 0$ weakly $*$ in $\mathcal{M}$. This yields

$$\lim_{n\to\infty} \mu_n(c - x_n) = 0 \tag{2.29}$$

and therefore,

$$\lim_{n\to\infty} \mu_n(y - x_n) = 0, \quad \text{for every } y \in R^n. \tag{2.30}$$

According to (1.3), we have

$$\int_0^T (\phi(y) - \phi(x_n)))\, dt \geq \mu_n(y - \chi_n),$$

and passing at limit, we get

$$0 \leq \underline{\lim}_{n\to\infty} \int_0^T (\phi(y) - \phi(\chi_n(t)))\, dt \leq T(\phi(y) - \phi(c))$$

(because ϕ is l.s.c). Hence, $\phi(y) \geq \phi(c)$, $\forall y \in$domϕ which implies that $0 \in \partial\phi(c) \Longrightarrow c = 0$. Since $\chi_n \to 0$ in C^0, we arrived at a contradiction, because $\|x(t_n)\| > \epsilon$. This proves that $\lim_{t\to\infty} x(t) = 0$.

Let us prove now that $x'(t) \to 0$ for $t \to \infty$. Let $a \in \overset{0}{\mathrm{dom}}\,\phi$, $M > \phi(a)$. Because ϕ is continuous on $\overset{0}{\mathrm{dom}}\phi$, there exists $\delta > 0$ such that $\phi(u) \leq M$, $\forall u \in B(a,\delta) \subset \overset{0}{\mathrm{dom}}\phi$. Let $k = \mathrm{dom}\,\phi \cap B(0,\delta)$. Because $\lim_{t\to\infty} x(t) = 0$, it follows that $x(t) \in k$ for $t \geq t_0$. Therefore, we may assume without any loss of generality, that $x(t) \in k$, for all $t \geq 0$. According to (1.3), we have

$$\int_t^{t+\rho} (\phi(x(t)+a) - \phi(x(t)))\, dt \geq \mu(a)\,. \tag{2.31}$$

Then,

$$x'_+(t+\rho) - x'_-(t) = x''([t,t+\rho]) = -\mu([t,t+\rho]) + \int_t^{t+\rho} \psi(x'(s))\, ds,$$

and therefore,

$$\langle x'_+(t+\rho), a\rangle - \langle x'_-(t), a\rangle \geq -\int_t^{t+\rho} \phi(x(t)+a) - \phi(x(s)) + \langle \psi(x'(s)), a\rangle\; ds.$$

In other words,

$$\langle x'_+(t+\rho), a\rangle - \langle x'_-(t), a\rangle \geq -\rho C \quad \text{for } 0 < \rho \leq T \tag{2.32}$$

where C is independent of t and λ.

Let us prove that $\lim_{t\to\infty} \langle x'(t), a\rangle = 0$ (we shall assign to x', the value 0 in the discontinuity points). By contradiction, assume there exists $t_n \to \infty$ such that $|\langle x'(t_n), a\rangle| > \epsilon > 0$, ϵ fixed. If there exists t_n such that $\langle x'(t_n), a\rangle \geq \epsilon$, then for $\rho \leq \epsilon/2C$, we have:

$$\langle x'_+(t_n+\rho), a\rangle \geq \langle x'(t_n), a\rangle - \rho C \geq \frac{\epsilon}{2},$$

that is, $\langle x'(s), a\rangle \geq \epsilon/2$, almost everywhere on $[t_n, t_n + \epsilon/2C]$. Then,

$$\|x'(s)\| \geq \frac{\epsilon}{2\|a\|}\,, \quad \text{a.e. on } [t_n, t_n + \frac{\epsilon}{2C}]. \tag{2.33}$$

Let $A = \inf\{-\langle x, \psi(x)\rangle\,;\ \epsilon/2\|a\| \leq \|x\| \leq \sqrt{2E}\}$ and note that, by hypothesis, $A > 0$. Then,

$$-\langle x'(s), \psi(x'(s))\rangle \geq A > 0, \quad \text{a.e. on } [t_n, t_n + \frac{\epsilon}{2C}]$$

Then the energy conservation relation yields

$$\int_0^\infty -\langle x'(t), \psi(x'(t))\rangle\; dt < \infty\,.$$

Therefore, because $t_n \to \infty$, we have

$$\lim_{n\to\infty} \int_{t_n}^{t_n+\epsilon/2C} -\langle \chi'(t),\ \psi(\chi'(t))\rangle\; dt = 0$$

which leads to a contradiction because

$$\int_{t_n}^{t_n+\epsilon/2C} -\langle u'(t),\ \psi(u'(t))\rangle\; dt \geq \frac{\epsilon A}{2C} > 0,$$

If there exists t_n such that $\langle \chi'(t_n), a\rangle \leq -\epsilon$, $t_n \to \infty$, we use the same argument on $[t_n - \epsilon/2C, t_n]$. Hence, $\lim \langle \chi'(t), a\rangle = 0$, for all $a \in \overset{0}{\mathrm{dom}}\phi$. Because $\overset{0}{\mathrm{dom}}\phi \neq \emptyset$, this implies $\lim_{t\to\infty} \chi'(t) = 0$, thereby completing the proof.

3. Stabilization of the elastic string with obstacle. Consider an elastic string with discrete distribution of masses which is clamped at $z = 0$, $z = L$ and limited from below by a rigid obstacle.

Assume that, at the points $z_1 = a_1$, $z_2 = a_1 + a_2, \ldots$, $z_n = a_1 + a_2 + \cdots + a_n = L$, of the (mass less) string, act gravitational force fields with mass m. If we denote by $x_i(t)$ the displacement of the string in the point z_i, we have (see e.g. [1], p. 87-100):

$$mx_1'' = -c\left(\frac{x_1}{a_2} + \frac{x_1 - x_2}{a_2}\right)$$

$$mx_2'' = -c\left(\frac{x_2 - x_1}{a_2} + \frac{x_2 - x_3}{a_3}\right);$$

$$\cdots\cdots\cdots$$

i.e.,

$$x'' + Ax = 0 \quad \text{in } R^+ \tag{3.1}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and A is the symmetric matrix

$$\begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2} & -\frac{1}{a_2} & \cdots & 0 & 0 \\ -\frac{1}{a_2} & \frac{1}{a_2} + \frac{1}{a_3} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & -\frac{1}{a_n} & \frac{1}{a_n} + \frac{1}{a_{n+1}} \end{pmatrix}.$$

If the string is limited from below by the obstacle $z_i = b_i$, i.e., $x_i \geq b_i$ for all $i = 1, \cdots, n$ then the dynamic is described by the variational inequality

$$x'' + Ax + \partial I_C(x) \ni 0 \quad \text{in } R^+ \tag{3.2}$$

where I_C is the indicator function of the convex set $C = \prod_{i=1}^n [b_i, \infty)$; in other words

$$\langle x'' + Ax\,,\, x - y\rangle \leq 0, \quad \forall\, y \in C,$$

or equivalently,

$$x_i'' + (Ax)_i = 0 \quad \text{in } [t;\, x_i(t) > b_i]$$

$$x_i'' + (Ax)_i \geq 0, \quad x_i \geq b_i \quad \text{in } R^+.$$

Here, we will consider the corresponding controlled equation

$$x'' + Ax + \partial I_C(x) \ni u \quad \text{in } R^+ \tag{3.3}$$

where the control function $u(t)$ is subject to constraints $\|u(t)\| \leq 1\ \forall t \geq 0$.

System (3.1) can be written in the form (1.1) if we set $\phi(x) = \frac{1}{2}\langle Ax\,,\, x\rangle + I_C(x)$. Since the matrix A is positive definite, Corollary 1 is applicable here to conclude that

Corollary 2. *The feedback control $u = \psi(x')$ where*

$$\psi(y) = \begin{cases} -y & \text{if } \|y\| \leq 1 \\ \dfrac{-y}{\|y\|} & \text{if } \|y\| > 1 \end{cases} \tag{3.4}$$

stabilizes system (3.3).

In what follows, we shall consider that the obstacle is convex; i.e.,

$$b_i(a_i + a_{i+1}) \leq a_{i+1}b_{i-1} + a_i b_{i+1}\,, \quad \text{for } i = 1, \cdots, 2n \tag{3.5}$$

and $b_0 = b_{n+1} = 0$. Our problem is to control the system, acting only on a part of the components of the field. More precisely, for $\mathcal{A} \subset \{1, \cdots, n\}$ we shall define

$$\tilde{x}_i = \begin{cases} x_i & \text{if } i \in \mathcal{A} \\ 0 & \text{if } i \in \mathcal{A}. \end{cases}$$

Corollary 3. *Assume that the unique solution x to the equation*

$$x'' + \partial\phi(x) = \psi(\tilde{x}') \tag{3.6}$$

which satisfies $\tilde{x} = 0$, is $x = 0$. Then, every solution (x, x') of (3.6) approaches zero as $t \to \infty$.

Proof: The proof is identical to that of Theorem 2, except for several points that will be explained below.

The convergence of the integral $\int_0^\infty \langle x'\,,\, \psi(\tilde{x}')\rangle\; dt$ will give us:

$$\begin{aligned} \tilde{x}_n' &\to 0 \quad \text{in } L^p,\ \ \forall p \in [1, \infty) \\ \tilde{x}_n &\to 0 \quad \text{in } C^0. \end{aligned} \tag{3.7}$$

Then, it is easy to show that the function v which we obtain as a limit of a subsequence of $\{x_n\}$ is also a solution for (3.6). (3.7) shows us that $v = 0$, and thus our hypothesis asserts that $v = 0$.

In order to show that $\langle x'(t)\,,\, a\rangle \to 0$ as $t \to \infty$, by the preceding argument, x_n' (defined in the same way), should tend to 0 in L^p, $1 \leq p < \infty$. Then, the inequality (2.32) will lead us to a contradiction with the presumption that $|\,\langle \chi(t_n)\,,\, a\rangle\,| \geq \epsilon > 0$.

Now, we want to find a sufficient condition in order that the hypothesis of Corollary 3 be fulfilled.

We shall suppose that at least one of the inequalities (3.5) is strictly satisfied. That is, $b_i < 0$, $\forall\, i = 1, \cdots, n$. Let x be a solution of the equation

$$x'' + \partial\phi(x) = 0 \tag{3.8}$$

and let k be such that (3.5) is strictly satisfied for $i = k$. Therefore, we want to find sufficient conditions such that $\tilde{x} = 0 \Longrightarrow x = 0$.

Let $[t_1, t_2] \subset [0, T]$. Then,

$$x_k'' = -(\partial\phi)_k(x) \geq -\lambda\Big(\frac{x_k - x_{k-1}}{a_k} + \frac{x_k - x_{k+1}}{a_{k+1}}\Big)$$
$$= \lambda(b_k - x_k)\Big(\frac{1}{a_k} + \frac{1}{a_{k+1}}\Big) + \frac{a_{k+1}b_{k-1} + a_k b_{k+1} - b_k(a_k + a_{k+1})}{a_k a_{k+1}}.$$

Hence, if $x_k(t_0) = b_k$, then we have $x_k'' > 0$ on $V(t_0)$, and thus $x_k(t) > t_0$ for $t \in V(t_0)/\{t_0\}$. Then, we can find $[t_3, t_4] \subset [t_1, t_2]$, such that

$$x_k(t) \geq b_k + \epsilon, \quad \forall\, t \in [t_3, t_4], \quad \text{for some } \epsilon > 0.$$

Therefore, by induction, it is easy to prove that there exists $[a, b] \subset [t_1, t_2]$, such that

$$x_i(t) > b_i + \epsilon, \quad \forall\, t \in [a, b], \quad i = 1, \cdots, n, \quad \text{for some } \epsilon > 0,$$

(we have to use induction for $i = k$ to 1, and also for $i = k$ to n). Then, $x(t) \in \overset{0}{\mathrm{dom}}\phi$ for $t \in [a, b]$, and (Remark, Theorem 2) hence, on $[a, b]$, x is a solution in the classical sense for $x'' + Ax = 0$.

Let now $\mathcal{A} = \{n_1, \cdots, n_k\}$, $0 < n_1 < \cdots < n_k < n+1$, $\tilde{x} = 0 \Longrightarrow x_{n_1}, \cdots, x_{n_k} = 0$. Then, we may interpret the system as being composed of $k+1$ independent systems,

$$S_1 = \{x_1, \cdots, x_{n_1 - 1}\}, \cdots, \text{ etc.}$$

Each of these systems has a solution of the form

$$\sum_{i=1}^{\alpha_i} v_i \sin\sqrt{\lambda_i}t + w_i \cos\sqrt{\lambda_i}t,$$

where λ_i are the proper values of the matrix obtained taking the corresponding lines and columns from matrix A. Since

$$x_{n_i} = 0, \quad x_{n_i}'' = 0, \quad \frac{x_{n_i} - 1}{a_i} + \frac{x_{n_i} - 1}{a_{j+1}} = 0,$$

every two neighbour systems must have the same frequencies of oscillations, because the functions $e^{i\lambda_i t}$ are linearly independent if λ_i are all distinct, real numbers. Therefore, for the existence of a solution $x \neq 0$, $\tilde{x} = 0$, it is necessary that all systems have a common frequency.

The matrix corresponding to each system is:

$$A_i = \begin{pmatrix} \frac{1}{a_{n_i+1}} + \frac{1}{a_{n_i+2}} & -\frac{1}{a_{n_i+2}} & \cdots \\ -\frac{1}{a_{n_i+2}} & \frac{1}{a_{n_i+2}} + \frac{1}{a_{n_i+3}} & \cdots \\ \vdots & & \\ \cdots & \cdots & \frac{1}{a_{n_i-1}} + \frac{1}{a_{n_i+1}} \end{pmatrix}.$$

If $n_i + 1 = n_{i+1}$, the matrix A_i is not well-defined, but in this case, we can immediately conclude that:

$$x_{n_i} = x_{n_i+1} = 0 \Longrightarrow x_i = 0, \quad \forall\, i = 1, \cdots, n,$$

therefore, we may set the corresponding proper values $\Gamma_i = \emptyset$. Otherwise, we shall denote

$$\Gamma_i = \{\lambda \mid \lambda \text{ is a proper value for } A_i\}.$$

Hence, we have proved the following

Corollary 4. *If* $\bigcap_{i=1}^{n} \Gamma_i = \emptyset$, *then the hypothesis of Corollary 2 is fulfilled.*

Remarks.

i) If, for some k, we have $k,\ k+1 \in \mathcal{A}$, the preceding condition is satisfied.

ii) If $a_1 = \cdots = a_{n+1}$, this condition is satisfied if and only if $(n_1, \cdots, n_k, n+1) = 1$;

iii) System (3.1) can be viewed as the finite difference approximation of the one dimensional wave equation with obstacle

$$\begin{aligned}
&y_{tt} - \gamma y_{xx} = u \quad \text{in } [y > \psi] \\
&y_{tt} - \gamma y_{xx} \geq u \quad \text{in } [0,1] \times R^+ \\
&y(t,x) \geq \psi(x) \quad \forall\, t \geq 0,\ x \in [0,1] \\
&y(t_0) = y(t,1) = 0 \\
&y(0,x) = y_0(x) \\
&y_t(0,x) = y_1(x)
\end{aligned}$$

(see [3] for a direct treatment of this problem). Formally, Corollary 1 amounts to saying that the feedback law $u = -\psi(y_t)$ stabilizes the system.

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