# LINEARIZED OSCILLATIONS FOR EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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**Abstract.** Let $[\cdot]$ denote the greatest-integer function and consider the nonlinear equation with piecewise constant arguments

$$\dot{y}(t) + \sum_{i=1}^{m} P_i(t) f_i(y[t - k_i])) = 0, \quad t \geq 0. \tag{1}$$

We obtained sufficient and also necessary and sufficient conditions for the oscillation of all solutions of Eq. (1) in terms of the oscillation of all solutions of an associated linear equation with piecewise constant arguments.

**1. Introduction.** Consider the equation with piecewise constant arguments (EPCA for short)

$$\dot{y}(t) + \sum_{i=1}^{m} P_i(t) f_i(y[t - k_i])) = 0, \quad t \geq 0 \tag{1}$$

where for $i = 1, 2, \cdots, m$

$$P_i \in C[\mathbb{R}^+, \mathbb{R}], \quad f_i \in C[\mathbb{R}, \mathbb{R}], \quad k_i \in \mathbb{N}$$

$\mathbb{N}$ is the set of natural numbers $0, 1, 2, \cdots$, and $[\cdot]$ denotes the greatest-integer function.

Our aim in this paper is to obtain sufficient and also necessary and sufficient conditions for the oscillation of all solutions of Eq. (1) in terms of the oscillation of all solutions of an associated linear EPCA with constant coefficients of the form

$$\dot{x}(t) + \sum_{i=1}^{m} p_i x([t - k_i]) = 0, \quad t \geq 0. \tag{2}$$

In the process, we will obtain some interesting results connecting the oscillation of all solutions of Eq. (2) to its characteristic equation

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{3}$$

and to the existence of positive solutions for certain linear difference inequalities. In obtaining our clear-cut results, it is important to note that Eq. (1) does not contain terms in $y(t)$ or $f(y(t))$.

Some of the linearized oscillation results in this paper were motivated by the recent work in [3], [4] and [5] about delay differential equations.

Throughout this paper we will use the notation

$$k = \max_{1 \le i \le m} k_i.$$

Clearly, $k \ge 0$.

By a solution of Eq. (1) we mean a function $y$ which is defined on the set

$$\big\{ -k, -k+1, \cdots, -1, 0 \big\} \cup (0, \infty)$$

and which satisfies the following properties:

  (i) $y$ is continuous on $[0, \infty)$.
  (ii) The derivative $\dot{y}(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of the points $t \in \mathbb{N}$ where one-sided derivatives exist.
  (iii) Eq. (1) is satisfied on each interval $[n, n+1)$ for $n \in \mathbb{N}$.

Let $A_{-k}, A_{-k+1}, \cdots, A_{-1}, A_0$ be any given real numbers. Then, as we will show in Lemma 1, Eq. (1) has a unique solution $y$ satisfying the conditions

$$y(-k) = A_{-k}, \quad y(-k+1) = A_{-k+1}, \cdots, \quad y(0) = A_0. \tag{4}$$

As is customary, a solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. For the case of difference equations, we will say that a solution $\{A_n\}$ is oscillatory if the terms $A_n$ of the sequence are not eventually positive or eventually negative. Otherwise, the solution $\{A_n\}$ is called nonoscillatory.

Recently, there has been a lot of activity concerning the oscillations of linear EPCA. See, for example, Aftabizadeh, Wiener and Xu [1], Cooke and Wiener [2] and the references cited therein. As mentioned in these references, EPCA represent a hybrid of continuous and discrete dynamical systems and combine properties of both differential and difference equations. Within intervals of certain length, they have the structure of continuous dynamical systems while continuity of the solution at the points which join consecutive intervals leads to difference equations.

**2.** In this section, we will establish necessary and sufficient conditions for the oscillation of all solutions of Eq. (1) in terms of the oscillation of all solutions of an associated nonlinear difference equation. First, we need the following result:

**Lemma 1.** *Let $A_{-k}, A_{-k+1}, \cdots, A_{-1}, A_0$ be given. Then (1) and (4) has a unique solution $y$ given by*

$$y(t) = A_n - \sum_{i=1}^{m} \Big[ \int_n^t P_i(s)\,ds \Big] f_i(A_{n-k_i}) \quad \text{for } n \le t < n+1 \text{ and } n \in \mathbb{N} \tag{5}$$

*where the sequence $\{A_n\}$ satisfies the difference equation*

$$A_{n+1} - A_n + \sum_{i=1}^{m} \Big[ \int_n^{n+1} P_i(s)\,ds \Big] f_i(A_{n-k_i}) = 0, \quad n = 0, 1, 2, \cdots. \tag{6}$$

**Proof:** Let $y(t)$ be a solution of (1) and (4). Then in the interval $n \leq t < n+1$ for any $n \in \mathbb{N}$, Eq. (1) can be written in the form

$$\dot{y}(t) = -\sum_{i=1}^{m} P_i(t) f_i(A_{n-k_i}), \quad n \leq t < n+1 \tag{7}$$

where we use the notation

$$A_n = y(n) \quad \text{for } n \in \{-k, \cdots, -1, 0, 1, \cdots\}.$$

By integrating (7) from $n$ to $t$, we obtain (5) and by continuity, as $t \to n+1$, (5) yields (6). Conversely, let $\{A_n\}$ be the solution of Eq. (6) and define $y$ on $\{-k, -k+1, \cdots, -1, 0\} \cup (0, \infty)$ by (4) and (5). Then, clearly, for every $n \in \mathbb{N}$ and $n \leq t < n+1$, (5) implies (7) and, in turn, (7) is equivalent to Eq. (1) in the interval $n \leq t < n+1$. The proof is complete.

The main result in this section is the following.

**Theorem 1.** *Assume that there exists a sequence of functions*

$$\beta_n \in C\big[[n, n+1), [0, 1]\big], \quad n = 0, 1, 2, \cdots$$

*such that for $n$ sufficiently large*

$$\int_n^t P_i(s)\, ds \leq \beta_n(t) \int_n^{n+1} P_i(s)\, ds \quad \text{for } 1 \leq i \leq m \text{ and } n \leq t < n+1, \tag{8}$$

*and suppose that the function $f_i$ satisfy the condition*

$$u f_i(u) \geq 0 \quad \text{for } 1 \leq i \leq m \text{ and } u \in \mathbb{R}. \tag{9}$$

*Then Eq. (1) has a nonoscillatory solution if and only if the difference equation (6) has a nonoscillatory solution.*

**Proof:** Assume that $y(t)$ is a nonoscillatory solution of Eq. (1). Then

$$A_n = y(n) \quad \text{for} \quad n \geq -k$$

is a nonoscillatory solution of Eq. (6). Conversely, assume that $\{A_n\}$ is a nonoscillatory solution of Eq. (6). We will assume that the sequence $\{A_n\}$ is eventually positive. The case where $\{A_n\}$ is eventually negative is similar and will be omitted. Then, for $n$ sufficiently large, by using (5), (8), (9) and (6), we obtain for $n \leq t < n+1$,

$$y(t) \geq A_n - \sum_{i=1}^{m} \Big[\beta_n(t) \int_n^{n+1} P_i(s)\, ds\Big] f_i(A_{n-k_i})$$

$$= A_n + \beta_n(t)(A_{n+1} - A_n) = [1 - \beta_n(t)]A_n + \beta_n(t)A_{n+1} > 0$$

which shows that $y(t)$ is a nonoscillatory solution of Eq. (1). The proof is complete.

**Remark 1.** Condition (8) of Theorem 1 is satisfied, for example, when the functions $P_i(t)$ are identically equal to constants $p_i \in \mathbb{R}$ (by taking $\beta_n(t) = t - n$), or when *the functions $P_i(t)$ are nonnegative* (by taking $\beta_n(t) = 1$), or a combination of the above. On the other hand, Condition (9) is automatically satisfied when $f_i(u) = u$, that is for linear EPCA.

As a consequence of Theorem 1 and Remark 1, we have the following corollaries.

**Corollary 1.** *Consider the EPCA*

$$\dot{y}(t) + \sum_{i=1}^{m} p_i f_i(y([t - k_i])) = 0, \quad t \geq 0 \tag{10}$$

*where for $i = 1, 2, \cdots, m$, $p_i \in \mathbb{R}$, $k_i \in \mathbb{N}$ and $f_i \in C[\mathbb{R}, \mathbb{R}]$ satisfies (9). Then every solution of Eq. (10) is oscillatory, if and only if every solution of the associated difference equation*

$$A_{n+1} - A_n + \sum_{i=1}^{m} p_i f_i(A_{n-k_1}) = 0, \quad n = 0, 1, 2, \cdots \tag{11}$$

*is oscillatory.*

**Corollary 2.** *Consider the linear EPCA*

$$\dot{y}(t) + \sum_{i=1}^{m} p_i y([t - k_i]) = 0 \tag{12}$$

*where $p_i \in \mathbb{R}$ and $k_i \in \mathbb{N}$ for $i = 1, 2, \cdots, m$. Then, every solution of Eq. (12) is oscillatory if and only if every solution of the associated difference equation*

$$A_{n+1} - A_n + \sum_{i=1}^{m} p_i A_{n-k_i} = 0, \quad n = 0, 1, 2, \cdots \tag{13}$$

*is oscillatory.*

**Corollary 3.** *Consider the linear EPCA*

$$\dot{y}(t) + \sum_{i=1}^{m} P_i(t) y([t - k_i]) = 0, \quad t \geq 0 \tag{14}$$

*where $P_i \in C\big[[0, \infty), \mathbb{R}^+\big]$ and $k_i \in \mathbb{N}$ for $i = 1, 2, \cdots, m$. Then every solution of Eq. (14) is oscillatory if and only if every solution of the associated difference equation*

$$A_{n+1} - A_n + \sum_{i=1}^{m} \Big[ \int_n^{n+1} P_i(s)\, ds \Big] A_{n-k_i} = 0, \quad n = 0, 1, 2, \cdots \tag{15}$$

*is oscillatory.*

**3. The Characteristics equation and difference inequalities.** Consider the linear EPCA

$$\dot{y}(t) + \sum_{i=1}^{m} p_i y([t - k_i]) = 0 \tag{16}$$

where

$$p_i \in \mathbb{R} \quad \text{and } k_i \in \mathbb{N} \text{ for } i = 1, 2, \cdots, m. \tag{17}$$

With Eq. (16), we associate the difference equation

$$A_{n+1} - A_n + \sum_{i=1}^{m} p_i A_{n-k_i} = 0, \quad n = 0, 1, 2, \cdots \tag{18}$$

and its characteristic equation

$$F(\lambda) \equiv \lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0. \tag{19}$$

It follows from the elements of the theory of linear difference equations and by an argument similar to that in Partheniadis [6] that every solution of Eq. (18) is oscillatory if and only if its characteristic equation (19) has no positive roots. On the other hand, from Corollary 2, we know that every solution of Eq. (16) is oscillatory if and only if every solution of Eq. (18) is oscillatory. Therefore, we have the following result.

**Theorem 2.** *Assume that (17) holds. Then the following statements are equivalent.*

  (a) *Every solution of Eq. (16) is oscillatory.*

  (b) *The characteristic equation (19) has no positive roots.*

The following result shows that, under some conditions, if a linear difference inequality has an eventually positive solution, so does the corresponding "limiting" equation.

**Lemma 2.** *For each $i = 1, 2, \cdots, m$ let $k_i \in \mathbb{N}$, $p_i \in (0, \infty)$ with $\sum_{i=1}^{m}(p_i + k_i) \neq 1$ and let $\{p_i(n)\}$ be sequences of real numbers such that*

$$\liminf_{n \to \infty} p_i(n) \geq p_i > 0, \quad i = 1, 2, \cdots, m. \tag{19}$$

*Suppose that the linear difference inequality*

$$B_{n+1} - B_n + \sum_{i=1}^{m} p_i(n) B_{n-k_i} \leq 0, \quad n = 0, 1, 2, \cdots \tag{20}$$

*has an eventually positive solution $\{B_n\}$. Then the characteristic equation (19) of the corresponding limiting equation (18) has a positive root.*

**Proof:** The case where $k = \max_{1 \leq i \leq m} k_i = 0$ is simple and will be omitted. So, assume $k > 0$. For $n$ sufficiently large set

$$\beta_n = \frac{B_n}{B_{n-1}}.$$

Then (20) yields

$$\beta_{n+1} - 1 + \sum_{i=1}^{m} p_i(n) \Big[ \prod_{j=0}^{k_i-1} \frac{1}{\beta_{n-j}} \Big] \leq 0. \tag{21}$$

Set

$$\beta = \limsup_{n \to \infty} \beta_n.$$

It follows from (21) that $0 < \beta < 1$ and that

$$\beta - 1 + \sum_{i=1}^{m} p_i \beta^{-k_i} \leq 0.$$

By using the notation in (19), we see that $F(\beta) \leq 0$ while $F(0^+) = \infty$. Thus Eq. (19) has a positive root in $(0, \beta)$. The proof is complete.

The following lemma is a useful comparison result for difference inequalities.

**Lemma 3.** *Let $p_i \in (0, \infty)$, $k_i \in \mathbb{N}$ with $\sum_{i=1}^{m}(p_i + k_i) \neq 1$ and let $\lambda_0$ be a root of Eq. (19). Let $N_1 \in \mathbb{N}$, $N_1 \geq 1$ and let $\theta \in (0, \infty)$. Assume that $\{C_n\}$ is a solution of the difference inequality*

$$C_{n+1} - C_n + \sum_{i=1}^{m} p_i C_{n-k_i} \geq 0, \quad n = 0, 1, 2, \cdots, N_1 - 1 \tag{22}$$

*with initial conditions*

$$C_n = \theta \lambda_0^n, \quad n = -k, \cdots, -1, 0.$$

*Then,*

$$C_n \geq \theta \lambda_0^n, \quad n = 1, 2, \cdots, N_1. \tag{23}$$

**Proof:** The case where $k = 0$ is simple and will be omitted. So assume that $k > 0$. Set

$$\gamma_n = \frac{C_n}{C_{n-1}} \quad \text{for } n = -k+1, \cdots, -1, 0, 1, \cdots$$

provided that $C_{n-1} \neq 0$. Then, from (22), we see that

$$0 \leq \gamma_1 - 1 + \sum_{i=1}^{m} p_i \Big[ \prod_{j=0}^{k_i - 1} \frac{1}{\gamma_{-j}} \Big] = \gamma_1 - 1 + \sum_{i=1}^{m} p_i \lambda_0^{-k_i} = \gamma_1 - \lambda_0$$

and so $\gamma_1 \geq \lambda_0$ or equivalently $c_1 \geq \theta \lambda_0$. In a similar manner, (12) yields

$$0 \leq \gamma_2 - 1 + \sum_{i=1}^{m} p_i \Big[ \prod_{j=0}^{k_i - 1} \frac{1}{\gamma_{1-j}} \Big] \leq \gamma_2 - 1 + \sum_{i=1}^{m} p_i \lambda_0^{-k_i} = \gamma_2 - \lambda_0$$

and so $\gamma_i \geq \lambda$ or $c_2 \geq \theta \lambda_0^2$. The proof follows by induction and will be omitted

## 4. Linearized oscillations.

Consider the nonlinear EPCA

$$\dot{y}(t) + \sum_{i=1}^{m} P_i(t) f_i(y[t - k_i]) = 0, \quad t \geq 0 \tag{1}$$

where

$$P_i \in C[\mathbb{R}^+, \mathbb{R}], \quad f_i \in C[\mathbb{R}, \mathbb{R}] \quad \text{and } k_i \in \mathbb{N} \text{ for } i = 1, 2, \cdots, m. \tag{24}$$

Set

$$p_i(n) \equiv \int_n^{n+1} P_i(s) \, ds \quad \text{for } i = 1, 2, \cdots, m.$$

Throughout this section we will assume that

$$\int_n^t P_i(s) \, ds \leq p_i(n) \quad \text{for } n \leq t < n+1 \text{ and } i = 1, 2, \cdots, m. \tag{25}$$

Condition (25) is satisfied, for example, when $P_i(t) \geq 0$. It should also be noticed that when (25) is satisfied then (8) is also satisfied (by taking $\beta_n(t) = 1$). We will also assume throughout this section that

$$u f_i(u) > 0 \quad \text{for } u \neq 0 \text{ and } i = 1, 2, \cdots, m. \tag{26}$$

The following theorem is one of the main results in this paper.

**Theorem 3.** *In addition to Conditions (24), (25) and (26), assume that the following three conditions are satisfied:*

*(i)*
$$\liminf_{u \to 0} \frac{f_i(u)}{u} = 1, \quad i = 1, 2, \cdots, m. \tag{27}$$

*(ii)   There exist positive numbers $p_i$ such that $\sum_{i=1}^{m}(p_i + k_i) \neq 1$ and*

$$\liminf_{n \to \infty} p_i(n) \geq p_i > 0, \quad i = 1, 2, \cdots, m. \tag{28}$$

*(iii)   Every solution of the linearized equation*

$$\dot{x}(t) + \sum_{i=1}^{m} p_i x([t \dot{-} k_i]) = 0 \tag{29}$$

*is oscillatory.*
*Then every solution of Eq. (1) is oscillatory.*

**Proof:** Assume, for the sake of contradiction, that Eq. (1) has a nonoscillatory solution. Then by Theorem 1, the difference equation (6) has a nonoscillatory solution $\{A_n\}$. We will assume that $\{A_n\}$ is eventually positive. The case where $\{A_n\}$ is eventually negative is similar and will be omitted. From Eq. (6), we see that eventually $A_n > A_{n+1}$ and so $L \equiv \lim_{n \to \infty} A_n$ exists and is a nonnegative number. We claim that $L = 0$. Otherwise, $L > 0$ and taking limit inferiors in (6) as $n \to \infty$ leads to the contradiction

$$L - L + \sum_{i=1}^{m} p_i f_i(L) \leq 0.$$

Now rewrite Eq. (6) in the form

$$A_{n+1} - A_n + \sum_{i=1}^{m} Q_i(n) A_{n-k_i} = 0$$

where
$$Q_i(n) = p_i(n) \frac{f_i(A_{n-k_i})}{A_{n-k_1})}, \quad i = 1, 2, \cdots, m.$$

As
$$\liminf_{n \to \infty} Q_i(n) \geq p_i > 0, \quad i = 1, 2, \cdots, m,$$

it follows, by Lemma 2, that the characteristic equation (19) has a positive root. This, in view of Theorem 2, contradicts the hypothesis that every solution of Eq. (29) is oscillatory and completes the proof.

The following result is a partial converse of Theorem 3.

**Theorem 4.** *In addition to Conditions (24), (25) and (26), assume that the following three conditions are satisfied:*

*(i)   There exists a positive number $\delta$ such that*

$$f_i(u) \leq u \quad \text{for } 0 \leq u \leq \delta \text{ and } i = 1, 2, \cdots, m. \tag{30}$$

*(ii)   There exist positive numbers $p_i$ such that $\sum_{i=1}^{m}(p_i + k_i) \neq 1$ and*

$$0 \leq p_i(n) \leq p_i, \quad i = 1, 2, \cdots, m. \tag{31}$$

*(iii)   Every solution of Eq. (1) is oscillatory.*

*Then every solution of Eq. (29) is oscillatory.*

**Proof:** Assume, for the sake of contradition, that Eq. (29) has a nonoscillatory solution. Then, by Theorem 2, Eq. (19) has a positive root $\lambda_0$. Set

$$\theta = \delta\lambda_0^k \quad \text{where } k = \max_{1 \le i \le m} k_i$$

and consider the solution $\{A_n\}$ of Eq. (6) with initial conditions

$$A_n = \theta\lambda_0^n, \quad n = -k, \cdots, -1, 0.$$

We now claim that

$$A_n > 0 \quad \text{for } n = 1, 2, \cdots. \tag{32}$$

Otherwise, there exists an integer $N_1 \ge 1$ such that

$$A_n > 0 \quad \text{for } -k \le n < N_1 \text{ and } A_{N_1} \le 0.$$

By using Eq. (6), we see that

$$A_{n+1} < A_n \quad \text{for } 0 \le n < N_1$$

and, in particular,

$$0 < A_n < A_0 = \theta = \delta\lambda_0^k < \delta \quad \text{for } n = 1, 2, \cdots, N_1 - 1.$$

In view of (30) and (31), Eq. (6) yields

$$A_{n+1} - A_n + \sum_{i=1}^m p_i A_{n-k_i} \ge 0 \quad \text{for } 1 \le n \le N_1 - 1.$$

Next, Lemma 3 implies that $A_{N_1} \ge \theta\lambda_0^{N_1} > 0$ and this contradiction established our claim (32).

Finally, in view of Theorem 1, (32) implies that Eq. (1) has a nonoscillatory solution. This contradiction completes the proof of the theorem.

**Remark 2.** The conclusion of Theorem 4 remains true if conditions (i) and (ii) are replaced by the following:

(i') There exists a positive number $\delta$ such that

$$f_i(u) \ge u \quad \text{for } -\delta \le u \le 0 \text{ and } i = 1, 2, \cdots, m. \tag{30'}$$

(ii') There exist positive numbers $p_i$ such that

$$p_i(n) \ge p_i, \quad i = 1, 2, \cdots, m.$$

By utilizing Theorems 3 and 4 and Remark 2, we obtain the following powerful linearized oscillation result.

**Corollary 4.** *Consider the nonlinear EPCA*

$$\dot{y}(t) + \sum_{i=1}^{m} p_i f_i(y([t - k_i])) = 0, \quad t \geq 0 \tag{33}$$

*where for $i = 1, 2, \cdots, m$, the following three conditions hold:*

(i) *$p_i > 0$, $k_i \in \mathbb{N}$ and $\sum_{i=1}^{m}(p_i + k_i) \neq 1$.*

(ii) *$f_i \in C[\mathbb{R}, \mathbb{R}]$, $uf_i(u) > 0$ for $u \neq 0$ and $\lim_{u \to 0}(f_i(u)/u) = 1$.*

(iii) *There exists a positive number $\delta$ such that either $f_i(u) \leq u$ for $0 \leq u \leq \delta$ or $f_i(u) \geq u$ for $-\delta \leq u \leq 0$.*

*Then every solution of Eq. (33) is oscillatory if and only if every solution of the linearized equation*

$$\dot{x}(t) + \sum_{i=1}^{m} p_i x([t - k_i]) = 0$$

*is oscillatory.*

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