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NON-EXISTENCE OF RADIALLY SYMMETRIC NON-NEGATIVE SOLUTIONS FOR A CLASS OF SEMI-POSITONE PROBLEMS

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Abstract. We study the radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)); \quad \| x \| < 1, \;\; x \in R^N \quad (N \geq 2)$$
$$u(x) = 0; \quad \| x \| = 1$$

where $\lambda > 0$, $f(0) < 0$ (semi-positone) and f superlinear. We establish that there exists a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ there are no non-negative solutions.

1. Introduction. Here we consider the radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)); \quad \| x \| < 1, \quad x \in R^N, \;\; N \geq 2 \tag{1.1}$$

$$u(x) = 0; \quad \| x \| = 1 \tag{1.2}$$

where $\lambda > 0$ and $f : [0, \infty) \to R$ is such that $f' \geq 0$. It is well known that the study of (1.1)-(1.2) is equivalent to

$$-u'' - (n/r)u' = \lambda f(u); \quad 0 < r < 1 \tag{1.3}$$

$$u'(0) = 0 \tag{1.4}$$

$$u(1) = 0 \tag{1.5}$$

where $n = N - 1$. We will assume that there exists $\alpha > 1$ such that

$$\liminf_{u \to \infty} f(u)/u^{\alpha} > 0 \tag{1.6}$$

and

$$f(0) < 0. \tag{1.7}$$

Our main result is given in Theorem 1.1.

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Theorem 1.1. *Under the above assumptions, there exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, then (1.1)-(1.2) has no non-negative radially symmetric solution.*

Remark 1.1. From [1] it is known that if u_λ is a positive solution of (1.1)-(1.2), that is, if $u_\lambda > 0$ for $\| x \| < 1$, then u must be radially symmetric and decreasing on $[0,1]$. Further, in [2] it was established that if u is non-negative, radially symmetric and $N \geq 2$, then u_λ must be positive and hence, radially decreasing. However, this is not the case when $N = 1$ (see [3]).

This work extends some of the work in [3] where various cases of non-positone problems were studied when $N = 1$. It also is a continuation of the work in [2], [4] where existence results were established when λ was small. Our methods are based on qualitative behavior of possible non-negative solutions near the boundary as well as energy estimates. In particular, we establish the following two main Lemmas under the above assumption on f.

Lemma 1.1. *There exists $\lambda_1 > 0$ such that, if $\lambda > \lambda_1$ and u_λ is a non-negative solution of (1.3)-(1.5), then there exists $t_1(\lambda) \in (0, 1/2]$ such that $u_\lambda(t_1) = (\beta + \theta)/2$. Here, β, θ are the positive zeroes of $f(s)$ and $F(s) = \int_0^s f(t)\,dt$, respectively.*

Lemma 1.2. *There exists $\lambda_2 > 0$ such that, if $\lambda > \lambda_2$ and u_λ is a non-negative solution of (1.3)-(1.5) and c is any number such that $c > 1$, then there exists $t_2(\lambda) \in [3/4, 1)$ such that $u_\lambda(t_2) = \beta/c$.*

We will prove these Lemmas and Theorem 1.1 in section 2. We close this section by introducing the following which we will use in section 2.

Let c (> 1) be any positive number and let

$$M := \max\{F(x)/(\beta/c) \leq x \leq [(\beta + \theta)/2]\}. \tag{1.8}$$

Clearly, $M < 0$. Further, from (1.6) there exists $k > 0$ (k independent of λ) such that

$$f(u) \geq k u^\alpha \quad \forall u \geq (\beta + \theta)/2. \tag{1.9}$$

Now, let $u := u(t, d, \lambda)$ be the solution of

$$-u_{tt} - [(N-1)/t]u_t = \lambda f(u), \quad u(0) = d, \quad u'(0) = 0 \tag{1.10}$$

and let

$$E(t) = \lambda F\big(u(t)\big) + [u'(t)]^2/2. \tag{1.11}$$

Then,

$$E'(t) = \{\lambda f(u(t)) + u''(t)\}u'(t) = -[(N-1)/t][u'(t)]^2 \leq 0. \tag{1.12}$$

2. Proofs of Lemmas and Theorem 1.1.

Proof of Lemma 1.1: Now, if u is a non-negative solution of (1.3)-(1.5), then $d \geq \theta$. Now suppose $u \geq (\theta + \beta)/2$ for all $t \in [0, t_1]$ for some $t_1 > 0$. Multiplying (1.10) by t^{N-1} we obtain

$$-\big(t^{N-1}u'(t)\big)' = \lambda f(u(t))t^{N-1}$$

and hence, integrating on $[0,t]$ for all $t \in [0,t_1]$ we obtain

$$\begin{aligned}
-t^{N-1}u'(t) &= \int_0^t \lambda f(u(t))t^{N-1}\,dt \\
&\geq \int_0^t \lambda k u^{\alpha}(t)t^{N-1}\,dt \quad \text{(by (1.9))} \\
&= \int_0^t \lambda k u^{\alpha}(t)[t^N/N]'\,dt \\
&= \lambda k u^{\alpha}(t)[t^N/N] - \int_0^t \lambda k[t^N/N]\alpha u^{\alpha-1}(t)u'(t)\,dt \\
&\geq \lambda k u^{\alpha}(t)[t^N/N],
\end{aligned}$$

since $u'(t) \leq 0$ (see Remark 1.1). Hence,

$$u'(t) \leq -\lambda k u^{\alpha}(t)t/N \quad \forall t \in [0,t_1]$$

and so

$$u'(t)/u^{\alpha}(t) \leq -\lambda k t/N \quad \forall t \in [0,t_1].$$

Now, integrating on $[0,t]$ where $t \in [0,t_1]$ we obtain

$$[1/u^{\alpha-1}(t)] - [1/d^{\alpha-1}] \geq -\lambda k t^2(1-\alpha)/(2N).$$

That is,

$$\begin{aligned}
[1/u^{\alpha-1}(t)] &\geq [1/d^{\alpha-1}] + \lambda k t^2(\alpha-1)/(2N) \\
&\geq \lambda k t^2(\alpha-1)/(2N)
\end{aligned}$$

and hence,

$$u(t) \leq \{(2N)/[\lambda k t^2(\alpha-1)]\}^{1/(\alpha-1)} \tag{2.1}$$

from which Lemma 1.1 follows easily.

Proof of Lemma 1.2: Now suppose $u := u_\lambda$ is a non-negative solution of (1.3)-(1.5) and let $b_\lambda \in (0,1)$ be such that $u(b_\lambda) = \beta/c$ and $u(t) \leq \beta/c$ for $t \in [b_\lambda, 1]$. Then, Lemma 1.2 will be true if

$$b_\lambda \to 1 \quad \text{as} \quad \lambda \to \infty. \tag{2.2}$$

Now, let $t \in [b_\lambda, 1]$. Then,

$$-u'' - [(N-1)/t]u' = \lambda f(u) \leq \lambda[-k_1],$$

where $-k_1 = f(\beta/c)$. That is,

$$u'' + [(N-1)/t]u' \geq \lambda k_1,$$

and so multiplying by t^{N-1} and integrating from b_λ to 1, we have

$$\int_{b_\lambda}^1 \{t^{N-1}u'\}'\,dt \geq \int_{b_\lambda}^1 \lambda k_1 t^{N-1}\,dt.$$

Hence,

$$u'(1) - b_\lambda^{N-1} u'(b_\lambda) \geq (\lambda k_1/N)\{1 - b_\lambda^N\},$$

that is,

$$b_\lambda^{N-1}\{-u'(b_\lambda)\} - \{-u'(1)\} \geq (\lambda k_1/N)\{1 - b_\lambda^N\}.$$

So we have

$$b_\lambda^{N-1}|u'(b_\lambda)| - |u'(1)| \geq (\lambda k_1/N)\{1 - b_\lambda^N\}, \tag{2.3}$$

since $u'(t) \leq 0$ $\forall t \in [0,1]$. But multiplying by $t^{2(N-1)}u'$ and integrating from b_λ to 1, we get

$$\int_{b_\lambda}^1 \{ - u''u't^{2(N-1)} - [(N-1)/t](u')^2 t^{2(N-1)}\}\, dt = \int_{b_\lambda}^1 \lambda f(u) u' t^{2(N-1)}\, dt,$$

that is,

$$\int_{b_\lambda}^1 -\{t^{N-1}u'\}' u' t^{(N-1)}\, dt = \int_{b_\lambda}^1 \lambda\{F(u)\}' t^{2(N-1)}\, dt. \tag{2.4}$$

Now, let

$$I = \int_{b_\lambda}^1 -\{t^{N-1}u'\}' u' t^{(N-1)}\, dt.$$

Then, integrating by parts we obtain

$$I = b_\lambda^{2(N-1)}[u'(b_\lambda)]^2 - [u'(1)]^2 - I.$$

Hence,

$$I = (1/2)\{b_\lambda^{2(N-1)}[u'(b_\lambda)]^2 - [u'(1)]^2\}. \tag{2.5}$$

But from (2.4), integrating by parts and using (1.5), we get

$$\begin{aligned} I &= -\lambda F(u(b_\lambda)) b_\lambda^{2(N-1)} - \int_{b_\lambda}^1 \lambda F(u) 2(N-1) t^{2N-3}\, dt \\ &\leq -\lambda F(u(b_\lambda)) b_\lambda^{2(N-1)} - \int_{b_\lambda}^1 \lambda F(u(b_\lambda)) 2(N-1) t^{2N-3}\, dt \\ &= -\lambda F(u(b_\lambda))\{b_\lambda^{2(N-1)} + [1 - b_\lambda^{2(N-1)}]\} \\ &= -\lambda F(u(b_\lambda)). \end{aligned} \tag{2.6}$$

Now, combining (2.5), (2.6) we obtain

$$\begin{aligned} \{-\lambda F(u(b_\lambda))\}^{1/2} &\geq (1/2)^{1/2}\{b_\lambda^{2(N-1)}[u'(b_\lambda)]^2 - [u'(1)]^2\}^{1/2} \\ &\geq (1/2)^{1/2}\{b_\lambda^{N-1}|u'(b_\lambda)| - |u'(1)|\} \end{aligned}$$

and hence, using (2.3), we have

$$\{-\lambda F(u(b_\lambda))\}^{1/2} \geq (1/2)^{1/2}(\lambda k_1/N)\{1 - b_\lambda^N\}. \tag{2.7}$$

But, $\{-\lambda F(u(b_\lambda))\}^{1/2} \leq \{\lambda k_2\}^{1/2}$, where $-k_2 = F(\beta/c)$. Hence, from (2.7), we obtain

$$1 \geq (\lambda)^{1/2}\{k_1/[N k_2^{1/2}]\}\{1 - b_\lambda^N\}, \tag{2.8}$$

from which (2.2) follows easily and Lemma 1.2 is proven.

Remark 2.1. See [5], where a similar idea as in Lemma 1.2 was used in dealing with the case $f(u) = u^p - \epsilon$, $\lambda = 1$ and ϵ large.

We now prove Theorem 1.1.

Proof of Theorem 1.1: Now suppose Theorem 1.1 is not true. Let λ_1, λ_2, t_1, t_2 be as in Lemmas 1.1 and 1.2 and let $\lambda_3 \geq \max\{\lambda_1, \lambda_2\}$. Then, by the mean value theorem there exists $t_3 \in [t_1, t_2]$ such that

$$\begin{aligned} |u'(t_3)| &= |\{u(t_2) - u(t_1)\}/\{t_2 - t_1\}| \\ &\leq |\{(\beta/c) - [(\beta+\theta)/2]\}|/(1/4) \\ &= 4|(2\beta - \beta c - \theta c)/(2c)| \\ &= (2/c)\{(\beta+\theta)c - 2\beta\} = (2/c)\mu \text{ (say)}. \end{aligned} \tag{2.9}$$

But,

$$E(t_3) = \lambda F(u(t_3)) + \{u'(t_3)\}^2/2$$

and using (1.8), (2.9)

$$\leq \lambda M + (2\mu^2)/(c^2).$$

But $M < 0$. Hence, there exists $\lambda_4 > \lambda_3$ such that if $\lambda > \lambda_4$ then

$$E(t_3) < 0. \tag{2.10}$$

But $t_3 \leq 1$ and by (1.12) and (2.10) we will have

$$E(1) < 0.$$

This is a contradiction, since $E(1) = \{u'(1)\}^2/2 \geq 0$. Hence the result.

REFERENCES

[1] B. Gidas, W.N. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle,* Commun. Math Phys., 68 (1979), 209-243.

[2] A. Castro and R. Shivaji, *Non-negative solutions for a class of radially symmetric non-positone problems,* Proc. Amer. Math. Soc., to appear.

[3] A. Castro and R. Shivaji, *Non-negative solutions for a class of non-positone problems,* Proc. Royal Soc. of Edinburgh, 108A (1988), 291-302.

[4] J. Smoller and A. Wasserman, *Existence of positive solutions for semilinear elliptic equations in general domains,* Arch. Rational Mech. Anal., Vol. 98, No. 3 (1987), 229-249.

[5] M. Ramaswamy and P.N. Srikanth, *Symmetry breaking for a class of semilinear elliptic problems,* Trans. Amer. Math. Soc., Vol. 304, No. 2 (1987), 839-845.