ON THE INVERSION OF LAGRANGE-DIRICHLET THEOREM*

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Abstract. The inversion of the Lagrange-Dirichlet theorem is proved under the hypothesis that the potential function \( U \) of the acting force is \( h \)-differentiable, \( h > 3 \), and the lack of a local maximum of \( U \) at the equilibrium position is recognizable by means of the non-vanishing terms with lowest degree in the expansion of \( U \). This result extends a previous one relative to infinitely differentiable potential functions and is obtained by using known results concerning the existence of invariant stable manifolds.

Introduction. The Lagrange-Dirichlet theorem, as is well known, provides a sufficient condition for the stability of an equilibrium position of a conservative mechanical system. Precisely, let \( S \) be a holonomic mechanical system with a finite number \( n \) of degrees of freedom and let \( q = (q_1, \ldots, q_n) \) be a system of Lagrangian coordinates for \( S \). Let us suppose that a conservative force with potential function \( U : \Omega \rightarrow \mathbb{R}, \Omega \) neighborhood of the origin of \( \mathbb{R}^n, U \in C^h, h \geq 2 \), acts on \( S \). Finally, let \( q = 0 \) be an equilibrium position of \( S \). The L.-D. theorem assures that \( q = 0 \) is stable if \( U \) has a strict local maximum at \( q = 0 \). As also is well known, the L.-D. criterium is not invertible. Therefore, the question arises: under what additional conditions the lack of a strict local maximum of \( U \) at \( q = 0 \) implies the instability of this equilibrium position. Starting from Liapunov, many answers have been given. We will quote some of the most relevant ones. Denoting by \( U^{[i]}, i = 2, \ldots, h, \) the term of degree \( i \) in the development of \( U \) in the neighborhood of the origin, the following criteria of instability hold. The equilibrium position \( q = 0 \) is unstable if one of the following conditions holds:

1) \( U_{[2]} \) does not have a maximum at \( q = 0 \) (Liapunov [7]);
2) \( h > 2, \exists \) a positive integer \( k, 2 < k \leq h, \) such that \( U_{[2]} = \cdots = U_{[k-1]} = 0 \) and \( U_{[k]} \) has a proper minimum at \( q = 0 \) (Liapunov [7]);
3) \( U \) is an homogeneous polynomial and does not have a maximum at \( q = 0 \) (Cetaev [1]);
4) \( U \) has a proper local minimum at \( q = 0 \) (Hagedorn [3]);
5) \( h > 2, \exists \) a positive integer \( k, 2 < k \leq h, \) such that \( U_{[2]} = \cdots = U_{[k-1]} = 0, q = 0 \) is an isolated critical point for \( U_{[k]}, \) and \( U_{[k]} \) does not have a maximum at \( q = 0 \) (Palamadov [8]);

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\begin{itemize}
\item[\text{i}6)\ ] $U$ has a local minimum at $q = 0$ (Taliaferro [10]; the result was proved under the weaker assumption $U \in C^1$);
\item[\text{i}7)\ ] $U$ is analytic and \exists\ a positive integer $k$, $k > 2$, such that $U[2] = \cdots = U[k-1] = 0$ and $U[k]$ does not have a maximum at $q = 0$ (Kozlov and Palamadov [4,5]).
\end{itemize}

The last result was obtained by proving the existence of a motion of $S$ which tends asymptotically to the origin. Precisely, Kozlov and Palamadov construct a series, satisfying formally the equations of motion, whose general term tends to 0 as $t \to \infty$. By using the analyticity of $U$, they are able to prove, by means of a fixed point theorem, that the series is convergent and an asymptotic motion to the origin exists. In [6] Kozlov proved that the result holds also if $U \in C^\infty$ without being analytic.

In this paper we prove the existence of an asymptotic motion of $S$ under the following condition:

\begin{itemize}
\item[\text{i}8)\ ] $h > 3$ and \exists\ a positive integer $k$, $2 < k < h$, such that $U[2] = \cdots = U[k-1] = 0$ and $U[k]$ does not have a maximum at $q = 0$.
\end{itemize}

The result has been achieved by constructing, by means of the equations of motion, a suitable autonomous differential system of the first order and by applying to this system known results about the existence of a stable manifold. We are able, also, to provide some estimates for the asymptotic motion existing under the condition \text{i}8). Finally, an extension of the result to the case in which gyroscopic forces act on $S$ is given.

1. Preliminaries. Let

$$L = \frac{1}{2} (\dot{q}, A\dot{q}) + U(q)$$

be the Lagrangian function of $S$, $A \in C^h[\Omega, L(\mathbb{R}^n, \mathbb{R}^n)]$, $U \in C^h[\Omega, \mathbb{R}]$, $h > 3$, $A(q) = A^T(q)$, $A(q)$ positive definite $\forall q \in \Omega$. Without loss of generality, we may assume

$$A(q) = I + \tilde{A}(q), \quad \tilde{A}(0) = 0.$$ 

Let $k$ be a positive integer, $2 < k < h$. It is easy to prove the following two lemmas.

**Lemma 1.1.** If $U[k]$ has in $e = (1,0,\cdots,0)$ a positive maximum on $S^{n-1} = \{q \in \mathbb{R}^n, \|q\| = 1\}$, then $U[k]$ has the following expression:

$$U[k](q) = (\chi/k)q_1^k + (1/2) \sum_{\alpha,\beta=2}^n v_{\alpha\beta}(q)q_\alpha q_\beta$$  \hspace{1cm} (1.1)

with $\chi > 0$ and $v_{\alpha\beta}(q) = v_{\beta\alpha}(q) \forall q \in \mathbb{R}^n$. Furthermore, the eigenvalues $\mu_\alpha$, $\alpha = 2, \cdots, n$, of the matrix $((v_{\alpha\beta}(e)))_{\alpha,\beta=2,\cdots,n}$ satisfy the condition

$$\mu_\alpha \leq \chi, \quad \alpha = 2, \cdots, n.$$  \hspace{1cm} (1.2)

**Proof:** We have

$$[\text{grad } U[k]]_{q=e} = \chi e,$$

and therefore, $(\partial U[k]/\partial q_\alpha)(e) = 0$, $\alpha = 2, \cdots, n$. Furthermore, since $U[k]$ has a positive maximum on $S^{n-1}$ at $e$, we have $\chi > 0$ and

$$\sum_{\alpha,\beta=2}^n v_{\alpha\beta}(e) u_\alpha u_\beta \leq \chi$$

for any vector $u$ tangent to $S^{n-1}$ at $e$. 

Lemma 1.2. Under the hypothesis of Lemma 1.1, the differential equation

$$\ddot{q} = \text{grad} U[k]$$  \hspace{1cm} (1.3)

in $\mathbb{R}^n$, has the solution $q = z(t)e$, with

$$z(t) = \sigma t^{-2/(k-2)}, \quad t > 0, \quad \sigma = [(2/\chi)k/(k-2)]^{1/(k-2)}. \hspace{1cm} (1.4)$$

Proof: The function $q = z(t)e$ satisfies (1.3) if $z(t)$ satisfies

$$\ddot{z} = \chi z^{k-1}. \hspace{1cm} (1.5)$$

The function (1.4) is a solution of (1.5) satisfying

$$\dot{z} = -(2\chi/k)^{1/2}z^{k/2}. \hspace{1cm} (1.6)$$

2. Existence of asymptotic motions. We will prove now our result.

Theorem 2.1. Suppose there exists a positive integer $k$, $2 < k < h$, such that

$$U = U[k] + W,$$

where $W \in C^h(\Omega, \mathbb{R})$ is of order higher than $k$ at $q = 0$. If $U[k]$ does not have a maximum at $q = 0$, then there exists a motion of $S$ defined for $t \in (0, +\infty)$ and tending to the equilibrium position $q = 0$ as $t \to \infty$.

Proof: Let $B(q) = A^{-1}(q) = I + \tilde{B}(q)$, $\tilde{B}(0) = 0$, and $\Gamma_{h,i,j}$, $h,i,j = 1, \ldots, n$, be the Christoffel symbols associated to $A(q)$. The Euler-Lagrange equations of $S$ can be written as

$$\ddot{q}_i = B_{im}[(\partial U[k]/\partial q_m) - \Gamma_{h,j,m}\dot{q}_h\dot{q}_j + (\partial W/\partial q_m)], \quad i = 1, \ldots, n. \hspace{1cm} (2.1)$$

In (2.1) and in the following formulae, the summation convention on repeated indices is assumed. First, we will show that the solutions of (2.1) defined for $t > 0$ can be determined by means of a class of solutions of an autonomous differential system of the first order in $\mathbb{R}^{2n+1}$. Second, we will examine the linear approximation of this system and prove the existence of a bidimensional stable manifold for it. This manifold contains a solution corresponding to an asymptotic motion of $S$.

First step. We can assume that $U[k]$ has a positive maximum on $S^{n-1}$ at $e$. Let $z(t)$ be the function (1.4) and let us consider the time dependent transformation of Lagrangian coordinates defined by

$$q = z(t)[e + Q], \quad t > 0. \hspace{1cm} (2.2)$$

By (2.2), taking into account (1.5) and (1.6), equations (2.1) are transformed into the system

$$z\ddot{q}_i - 2(2\chi/k)^{1/2}z^{k/2}\dot{Q}_i + \chi z^{k-1}(\delta i_1 + Q_i) = B_{im}\{(\partial U[k]/\partial q_m)$$

$$- (2\chi/k)\Gamma_{h,j,m}z^k(\delta h_1 + Q_h)(\delta j_1 + Q_j) - \Gamma_{h,j,m}z^2\dot{Q}_h\dot{Q}_j$$

$$+ (2\chi/k)^{1/2}z^{(k+2)/2}(\Gamma_{h,j,m} + \Gamma_{j,h,m})(\delta h_1 + Q_h)\dot{Q}_j + (\partial W/\partial q_m)\}, \hspace{1cm} (2.3)$$
\( i = 1, \ldots, n \), where \( \delta_{ij}, i, j = 1, \ldots, n \), denote the Kronecker symbols. In (2.3), all the functions of \( q \) are evaluated for \( q = z(t)(e + Q) \). Since the function \( z(t) \) is invertible, we can assume \( z \) as independent variable. By setting \( y(z) = Q(t(z)), z > 0 \), and denoting by \('\) the derivative with respect to \( z \), the system (2.3) is transformed into the system

\[
\begin{align*}
   z^2 y_i'' + \left[ (k + 4)/2 \right] z y_i' + (k/2)(\delta_{i1} + y_i) = & \, B_{im} \left\{ (k/2)z^{1-k}(\partial U_{|k|}/\partial q_m) 
   \right. \\
   & - \, \Gamma_{h,j,m} z(\delta_{h1} + y_h)(\delta_{j1} + y_j) - (k/2\chi)\Gamma_{h,j,m} z^3 y_j y_h' \\
   & + \, (k/2\chi)^{1/2}(\Gamma_{h,j,m} + \Gamma_{j,h,m}) z^2 y_j' (\delta_{h1} + y_h) + (k/2\chi)z^{1-k}(\partial W/\partial q_m) \right\},
\end{align*}
\]

\( i = 1, \ldots, n \), where the functions of \( q \) are evaluated at \( q = z(e + y) \).

Let us reduce now (2.4) to a first order system by introducing the new system of variables

\[
v_i = z y_i' \quad i = 1, \ldots, n.
\]

We have:

\[
\begin{align*}
   \begin{cases}
      z y_i' = v_i \\
      z v_i' = \Xi_i(z, y, v)
   \end{cases}
\end{align*}
\]

where the functions \( \Xi_i \in C^{h-k} \) are defined by

\[
\Xi_i(z, y, v) = -[(k + 2)/2]v_i - (k/2)(\delta_{i1} + y_i) + B_{im} \left\{ (k/2)z^{1-k}(\partial U_{|k|}/\partial q_m) 
   \right. \\
   \left. - \, \Gamma_{h,j,m} z(\delta_{h1} + y_h)(\delta_{j1} + y_j) - (k/2\chi)\Gamma_{h,j,m} z^3 y_j y_h' \\
   + \, (k/2\chi)^{1/2}(\Gamma_{h,j,m} + \Gamma_{j,h,m}) z^2 y_j' (\delta_{h1} + y_h) + (k/2\chi)z^{1-k}(\partial W/\partial q_m) \right\}, 
\]

\( i = 1, \ldots, n \).

It is easy to see that, for a fixed \( z_0 \in (0, +\infty) \), the solutions of (2.5) defined for \( z \in (0, z_0] \) are in a one to one correspondence with the solutions of the autonomous system in \( \mathbb{R}^{2n+1} \)

\[
\begin{align*}
   \begin{cases}
      dz/d\phi = -z \\
      dy_i/d\phi = -v_i \\
      dv_i/d\phi = -\Xi_i(z, y, v)
   \end{cases}
\end{align*}
\]

defined for \( \phi \in [0, +\infty) \) for which \( z(0) = z_0 \).

**Second step.** We will consider now the linear approximation of (2.7) around the origin of \( \mathbb{R}^{2n+1} \). We have

\[
\Xi_i(z, y, v) = p_i z - (k/2\chi)(\chi \delta_{ij} - V_{ij})y_j - [(k + 2)/2]v_i + N_i(z, y, v),
\]

where

\[
p_i = (k/2\chi)\left\{ \chi(\partial B_{11}/\partial q_1)(0) + [z^{-k}(\partial W/\partial q_1)(z_e)](0) + \Gamma_{i,1,1}(0) \right\} \quad i = 1, \ldots, n,
\]

\[
V_{ij} = (\partial^2 U_{|k|}/\partial q_i \partial q_j)(e), \quad i, j = 1, \ldots, n,
\]

and the functions \( N_i(z, y, v), i = 1, \ldots, n \), are infinitesimal of order \( > 1 \) at \((0,0,0)\). By (1.1), we have

\[
V_{11} = (k - 1)\chi, \quad V_{1\alpha} = 0, \quad V_{\alpha\beta} = v_{\alpha\beta}(e), \quad \alpha, \beta = 2, \ldots, n,
\]
and the linear approximation of (2.7) is written as

\[
\begin{align*}
\frac{dz}{d\phi} &= -z \\
\frac{dy_i}{d\phi} &= -v_i, \quad i = 1, \cdots, n, \\
\frac{dv_1}{d\phi} &= -p_1 z - (k/2)(k-2)y_1 + [(k+2)/2]v_1 \\
\frac{dv_\alpha}{d\phi} &= -p_\alpha z + (k/2\chi)(\chi \delta_{\alpha\beta} - v_{\alpha\beta}(e))y_\beta + [(k+2)/2]v_\alpha, \quad \alpha = 2, \cdots, n.
\end{align*}
\]

Let us denote by \( A \) the matrix of the coefficients of (2.8). We are going to show that \( A \) has two real negative eigenvalues and all the others are real non negative. Let us consider first the case

A) \( k \neq 4 \). In this case \( \lambda = -1 \) is eigenvalue of \( A \) with corresponding eigenvectors given by

\[
\begin{pmatrix}
1, a_1, \cdots, a_n, a_1, \cdots, a_n
\end{pmatrix},
\]

with \((a_1, \cdots, a_n)\) solution of the system

\[
\begin{align*}
(k^2 - 3k - 4)a_1 &= -2p_1 \\
(k+2)\delta_{\alpha\beta} - (k/2\chi)v_{\alpha\beta}(e) &a_\beta = p_\alpha.
\end{align*}
\]

System (2.10) has a unique solution as

\[
(k^2 - 3k - 4) \neq 0 \quad \text{for } k \neq 4,
\]

and, by (1.2),

\[
\det \left( \begin{pmatrix}
(k+2)\delta_{\alpha\beta} - (k/2\chi)v_{\alpha\beta}(e) &
\end{pmatrix}_{\alpha,\beta=2,\cdots,n} \right) \neq 0.
\]

Therefore, \( \lambda = -1 \) is a simple eigenvalue of \( A \). Furthermore, the solutions of the equation

\[
\lambda^2 - [(k+2)/2] \lambda - (k/2)(k-2) = 0
\]

are eigenvalues of \( A \) with corresponding eigenvectors given by

\[
\begin{pmatrix}
0, 1, 0, \cdots, 0, -\lambda, 0, \cdots, 0
\end{pmatrix}.
\]

Thus, the negative solution of (2.11), which is different from \(-1\) as \( k \neq 4 \), is another real negative simple eigenvalue of \( A \). The other eigenvalues of \( A \) are given by the positive solution of (2.11) and by the values of \( \lambda \) for which the following condition is satisfied:

\[
\det \left( \begin{pmatrix}
\lambda^2 - [(k+2)/2] \lambda + (k/2) &
\end{pmatrix}_{\alpha,\beta=2,\cdots,n} \right) = 0.
\]

Indeed, for any solution \( \lambda \) of (2.13) we have the eigenvector of \( A \)

\[
\begin{pmatrix}
0, 0, a_2, \cdots, a_n, 0, -\lambda a_2, \cdots, -\lambda a_n
\end{pmatrix},
\]
with \((a_2, \cdots, a_n)\) non null solution of the system

\[
[[\lambda^2 - [(k + 2)/2]\lambda + (k/2)]\delta_{\alpha\beta} - (k/2)\chi v_{\alpha\beta}(e)]a_\beta = 0, \quad \alpha = 2, \cdots, n.
\]

For any eigenvalue \(\mu_\alpha, \alpha = 2, \cdots, n\), of the matrix \((v_{\alpha\beta})_{\alpha,\beta=2,\cdots,n}\) we have two solutions of (2.13) given by

\[
\lambda_\alpha^\pm = (k + 2)/4 \pm \{[(k + 2)/4]^2 - (k/2)(1 - \mu_\alpha/\chi)\}^{1/2},
\]

which are distinct and non negative because of (1.2).

Let us consider, now , the case

B) \(k = 4\). Also in this case, \(\lambda = -1\) is eigenvalue of \(A\). If \(p_1 = 0\), the vectors

\[
(0,1,0,\cdots,0,1,0,\cdots,0), \quad (1,1,a_2,\cdots,a_n,1,a_2,\cdots,a_n)
\]

with \((a_2,\cdots,a_n)\) unique solution of the system

\[
[3\chi\delta_{\alpha\beta} - v_{\alpha\beta}(e)]a_\beta = (\chi/2)p_\alpha, \quad \alpha = 2, \cdots, n,
\]

are eigenvectors corresponding to it. Its multiplicity is equal to two because one can show, as in case A), that there are other \(2n-1\) real non-negative eigenvalues. Therefore, \(\lambda = -1\) is a semisimple eigenvalue. If \(p_1 \neq 0\), we have only one eigenvector corresponding to \(\lambda = -1\). Nevertheless, its algebraic multiplicity is equal to two, because there exist only other \(2n-1\) eigenvalues of \(A\) which are real and non-negative. A basis of its generalized eigenspace is given by the vectors

\[
(1,-p_1/5,b_2,\cdots,b_n,0,b_{n+2},\cdots,b_{2n}), \quad (0,1,0,\cdots,0,1,0,\cdots,0),
\]

with suitable values of \(b_2,\cdots,b_n,b_{n+2},\cdots,b_{2n}\).

Now, by using known results about the existence of invariant manifolds [9], we can conclude that for the differential system (2.7), a stable \(C^{n-1}\)-bidimensional invariant manifold \(M\) exists. This manifold contains the origin of \(\mathbb{R}^{2n+1}\), it is tangent at the origin to the generalized eigenspace of the negative eigenvalues of \(A\). All the solutions of (2.7) which start from points of \(M\) in a sufficiently small neighborhood \(U\) of the origin, tend asymptotically to the origin as \(\phi \to +\infty\). As there are points \((z_0,y_0,v_0)\in M \cap U\), with \(z_0 > 0\), then there are solutions of (2.4) tending to the origin of \(\mathbb{R}^n\) as \(z \to 0^+\) and therefore, there are motions of \(S\) which are asymptotic to the equilibrium position \(q = 0\).

**Corollary 2.2.** Assume now \(h > 4, 2 < k < h - 1\). Let \(\lambda^-\) be the negative solution of (2.11). Then, the asymptotic motion existing under the hypothesis of Theorem 2.1 admits a parametrization such that the following estimates hold:

i) if \(\lambda^-\) is a semisimple eigenvalue of \(A\), then \(q(t) = z(t)[1 + O(z(t))];\)

ii) if \(\lambda^-\) is not a semisimple eigenvalue of \(A\), then \(q(t) = z(t)[1 + O(|z(t)| \log z(t))].\)

**Proof:** If \(\lambda^-\) is a semisimple eigenvalue of \(A\), the eigenspace of the negative eigenvalues of \(A\) admits as basis the vectors (2.9) and (2.12) with \(\lambda = \lambda^-\) if \(\lambda^- \neq -1\), and the vectors (2.14) if \(\lambda^- = -1\). Therefore, if \((\xi, \eta)\) are coordinates associated to these bases, the manifold \(M\) is represented in a neighborhood of the origin of \(\mathbb{R}^{2n+1}\) by

\[
\begin{align*}
z &= \xi \\
y_1 &= \gamma_1 \xi + \eta, \quad y_\alpha = a_\alpha \xi + \Phi_\alpha(\xi, \eta) \\
v_1 &= \gamma_1 \xi - \lambda^- \eta + \Psi_1(\xi, \eta), \quad v_\alpha = a_\alpha \xi + \Psi_\alpha(\xi, \eta), \quad \alpha = 2, \cdots, n,
\end{align*}
\]
where \( \gamma = a_1 \) if \( \lambda^- \neq -1 \), \( \gamma = 1 \) if \( \lambda^- = -1 \), and \( \Psi_1, \Phi_\alpha, \Psi_\alpha, \alpha = 2, \ldots, n \), are infinitesimal at \((0, 0)\) of order \( \geq 2 \).

If \( \lambda^- \) is not a semisimple eigenvalue of \( \mathcal{A} \), the basis of the generalized eigenspace of the eigenvalue \(-1\) of \( \mathcal{A} \) is given by the vectors (2.15). Therefore, if \((\xi, \eta)\) are coordinates associated to this basis, the manifold \( M \) is represented by

\[
\begin{align*}
\{ z = \xi \\
y_1 &= -(p_1/5)\xi + \eta, \quad y_\alpha = b_\alpha \xi + \Phi'_\alpha(\xi, \eta) \\
v_1 &= \xi + \Psi'_1(\xi, \eta), \quad v_\alpha = b_{n+\alpha} \eta + \Psi'_\alpha(\xi, \eta), \quad \alpha = 2, \ldots, n,
\end{align*}
\]

where \( \Psi'_1, \Phi'_\alpha, \Psi'_\alpha, \alpha = 2, \ldots, n \), are infinitesimal at \((0, 0)\) of order \( \geq 2 \). Thus, in both cases, differential system (2.7) restricted on the stable manifold \( M \) assumes the form

\[
\begin{align*}
\frac{dz}{d\phi} &= -z \\
\frac{d\eta}{d\phi} &= \lambda^- \eta + \eta z + n(z, \eta),
\end{align*}
\]

where \( \eta = 0 \) in case \( \lambda^- \) is semisimple eigenvalue, \( \eta = -p_1/5 \) in the other case, and \( n(z, \eta) \) is of order \( \geq 2 \) at \((0, 0)\). Let us consider the first case. If \( \lambda^- \leq -2 \), by an obvious application of the Gronwall inequality, we have:

\[
|\eta(\phi, \eta_0, z_0)| \leq z_0 e^{-\phi},
\]

for \( z_0 > 0 \) sufficiently small, and \( \eta_0 = O(z_0^2) \). If \( \lambda^- > -2 \), we consider the Banach space \( X \) defined by

\[
X = \{ u \in C([0, +\infty), \mathbb{R}) : u(\phi)e^{2\phi} \text{ bounded, } \|u\| = \sup_{0, +\infty} (u(\phi)e^{2\phi}) \},
\]

and, for \( c > 0 \), the closed subset \( S \) of \( X \) defined by \( S = \{ u \in X : \|u\| \leq c \} \). The map \( F : S \to X \):

\[
(Fu)(\phi) = \int_{-\infty}^{\phi} e^{-(s-\phi)\lambda^-} n(z_0 e^{-s}, u(s))\, ds,
\]

admits a fixed point \( \eta(\cdot) \), provided that the values \( c \) and \( z_0 \) are suitably small. Obviously, \((z_0 e^{-\phi}, \eta(\phi))\) satisfies system (2.17). Then, i) follows now by means of (2.16).

To complete the proof we observe that ii) is a direct consequence of the presence of the secular term \( (\eta \neq 0) \) in (2.17).

An extension of the result obtained by Furta [2], concerning the "gyroscopic" case, is given by the following.

**Corollary 2.3.** Let us consider the Lagrangian function

\[
\mathbf{L} = \frac{1}{2}(\dot{q}, A\dot{q}) + U(q) + (G(q), \dot{q})
\]

where \( A \) and \( U \) satisfy the same assumptions as in Theorem 2.1, and \( G = (G_1, \ldots, G_n) \), \( G_i \in C^1[\Omega, \mathbb{R}] \), \( i = 1, \ldots, n \), with

\[
G(q) = G_{[s]}(q) + H(q),
\]
s being an integer, \( s \in [(k + 2)/2, h) \), \( H(q) \) of order greater than \( s \) at 0. Then, again, the corresponding E.-L. system admits a solution tending to the origin as \( t \to +\infty \).

**Proof:** The proof is achieved by observing that the equations of motion are obtained by system (2.1) by adding the gyroscopic terms:

\[
B_{im}[(\partial G_i/\partial q_m) - (\partial G_m/\partial q_i)]\dot{q}_i, \quad i = 1, \cdots, n.
\]

Then we can associate to the equations of motion a linear system which differs from (2.8) only if \( s = (k + 2)/2 \); in such case in (2.8) \( p_i, i = 1, \cdots, n \), has to be replaced by

\[
p_i - (k/2\chi)[(\partial G_i/\partial q_1) - (\partial G_1/\partial q_i)]_{[k/2]}(e), \quad i = 1, \cdots, n.
\]

Therefore, the existence of the asymptotic motion is proved as above.

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**REFERENCES**


