Differential and Integral Equations, Volume 3, Number 3, May 1990, pp. 401–433.

OSCILLATIONS OF SOLUTIONS OF PERTURBED AUTONOMOUS EQUATIONS WITH AN APPLICATION TO NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENTS

F.V. ATKINSON† AND L.A. PELETIER‡
Argonne National Laboratory, Argonne, IL 60439, USA

Abstract. We discuss radially symmetric, not necessarily positive, solutions of the Dirichlet problem for $\Delta u + \{f_1(u,r) + \lambda f_2(u,r)\} = 0$ in the unit ball $B_N, N > 2$, where f_1 and f_2 are suitably homogeneous functions with f_1 critical and f_2 subcritical. Estimates are obtained for asymptotic relations between $u(0)$ and λ. The method transforms the problem to a study of the zeros of certain solutions of a perturbed autonomous ordinary differential equation.

1. Introduction. As the title indicates, the problem of this paper admits both an ODE and a PDE presentation. While these are very nearly co-extensive, the ODE version has certain advantages, such as freedom from dimensional restrictions. We are concerned in this version with the asymptotics, as $s \to \infty$, of exponentially small solutions of equations of the form

$$w''(s) - w(s) + g(w(s)) + h(w(s), s) = 0, \tag{1.1}$$

where g is a nonlinear function and $h(w,s)$, qua function of s, is exponentially small as $s \to \infty$. Typical examples of recent interest for the PDE application will be such cases as

$$g(w) = w|w|^{p-1}, \quad h(w,s) = w|w|^{q-1}e^{-ms}, \quad 1 \le q < p, \ m > 0. \tag{1.2}$$

However we do not confine our attention to power-type behaviour in w.

We view (1.1) as a perturbation of the autonomous equation

$$w'' - w + g(w) = 0, \tag{1.3}$$

and this in turn as a pertubation of

$$w'' - w = 0 \tag{1.4}$$

Received May 30, 1989.
†Department of Mathematics, University of Toronto, Toronto, M5S 1A1, Canada.
‡Mathematical Institute, Leiden University, PB 9512, 2300 RA Leiden, The Netherlands.
AMS Subject Classifications:34E10, 35J60.

near $w = 0$. Of importance here are the oscillations of solutions of (1.1) of the type

$$w(s) = w(s, \gamma) \sim \gamma e^{-s}, \quad \gamma > 0; \tag{1.5}$$

such solutions exist also for (1.3) and (1.4).

The question at issue for (1.1) relates to the behaviour of the zeros of the solution $w(s, \gamma)$ as $\gamma \to \infty$. We will denote the successive zeros of $w'(s)$ in descending order by $s_1(\gamma), s_3(\gamma), \ldots$, and those of $w(s)$ by $s_2(\gamma), s_4(\gamma), \ldots$, so that, subject to justification of their existence and separation properties,

$$\cdots < s_3(\gamma) < s_2(\gamma) < s_1(\gamma) < \infty. \tag{1.6}$$

We ask whether all the $s_n(\gamma) \to \infty$ as $\gamma \to \infty$, or only the first few. The answer appears to be that they all tend to infinity if $0 < m < 1$, but not for $m \geq 1$. Indeed, for the functions g and h given by (1.2), with $q = 1$, it was shown that $s_2(\gamma) \to \infty$ if $m < 2$ [3, 4], but that $s_4(\gamma), s_6(\gamma), \ldots$, remain bounded for large values of γ if $m \geq 1$ [2]. In this paper we shall show that $s_{2n}(\gamma) \to \infty$ as $\gamma \to \infty$ for every $n \geq 1$ when $0 < m < 1$.

The solutions we are discussing, those of the type (1.5), form of course only a very special class within the totality of solutions. Some discussion of the various types of solutions was given in a similar case in the paper [1].

In the simplest version of the PDE problem to which the title refers, we ask for what values of λ the Dirichlet problem for the unit ball B_N in $\mathbf{R}^N$, $N > 2$,

(I)

$$-\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u \quad \text{in} \quad B_N, \tag{1.7}$$

$$u = 0 \quad \text{on} \quad \partial B_N, \tag{1.8}$$

has a non-trivial radially symmetric solution $u = u(r)$; here p has the critical value for Sobolev imbedding, that is

$$p = (N + 2)/(N - 2), \tag{1.9}$$

and $1 \leq q < p$. Our results will be mainly relevant to the situation in which we have in addition

$$p - q < 1; \tag{1.10}$$

however the case $1 \leq p - q < 2$ is also important, even though the results are more partial.

Since we consider only the radially symmetric case, we are concerned with the ODE problem

(II)

$$u'' + \frac{N-1}{r}u' + \lambda |u|^{q-1}u + |u|^{p-1}u = 0, \quad 0 < r < 1, \tag{1.11}$$

$$u(0) \text{ finite}, \quad u'(0) = 0, \quad u(1) = 0. \tag{1.12}$$

In this specialisation of the problem, there is no need for N to be integral.

By scaling the variables u and r one may eliminate the parameter λ; the question of the existence of a non-trivial solution for given λ is then replaced by one concerning the location of the zeros of solutions of a parameter–free equation. The scaling tranformation can be combined with an analytic transformation which removes the singularity at $r = 0$ to ∞, this making available standard results from the theory of the Emden–Fowler equation; this approach was followed in a series of papers [2-4], using the change of variables

$$t = \left(\frac{N-2}{r}\right)^{N-2}\lambda^{-2/(p-q)}, \quad y(t) = \lambda^{-1/(p-q)}u(r).$$

This led to the equation

$$y'' + t^{-k}y(|y|^{q-1} + |y|^{2k-4}) = 0, \quad k = (2N-2)/(N-2). \tag{1.13}$$

The requirement "$u(0)$ finite" in (1.12) translates into the condition

$$y(t) \to \gamma \quad \text{as} \quad t \to \infty. \tag{1.14}$$

Such solutions exist if $k > 2$, as is the case in (1.13); of course they form only a very special class within the totality of solutions, just as do the solutions of type (1.5) in the case of (1.1). However it is these solutions, and their zeros, if any, which are crucial for the eigenvalue problems (I, II).

Here we use a slightly different type of transformation which still moves the singularity from 0 to ∞. Assuming λ to be positive, we define new dependent and independent variables w and s by

$$u(r) = w(s)\lambda^{1/(p-q)}e^{s}, \tag{1.15}$$

$$r^{(N-2)/2} = \left(\frac{N-2}{2}\right)^{(N-2)/2}e^{-s}\lambda^{-1/(p-q)}. \tag{1.16}$$

Applied to (1.11), this leads to

$$w'' - w + |w|^{p-1}w + |w|^{q-1}we^{-(p-q)s} = 0, \tag{1.17}$$

which is of the form (1.1). Here the conditions $p - q \in (0,1)$, or $(0,2)$, turn out to be critical in discussing the behaviour of the zeros of solutions.

In terms the problem (1.7), the condition $p - q < 1$ means that

$$N > 2 + \frac{4}{q},$$

or $N > 6$ if $q = 1$. Likewise, the condition $p - q < 2$ means that

$$N > 2 + \frac{4}{1+q},$$

or $N > 4$ if $q = 1$. For recent references concerning special features of these various ranges of N we refer to [2, 12].

The method we will use is not restricted to nonlinearities of the special type appearing in (1.11). The signed powers of u in (1.11) can be replaced by suitable products of powers of u and r as in [9], or indeed by linear combinations of such products, or by functions of u and r satisfying suitable homogeneity and degree properties, formulated in Section 4.

We formulate our main result in the ODE setting in Section 3, in terms of the location of zeros of a parameter-free equation. The consequences for the eigenvalue problem (1.11-12) will be discussed in Section 4. The remainder of the paper is devoted to the study of solutions of equation (1.1) which have the asymptotic behaviour prescribed by (1.5).

2. The ODE framework. We now revert to the formulation (1.1), where g and h may have the form indicated in (1.2), but can be much more general. Specifically, we assume

Hg1. $g \in C^1(-\infty, \infty)$.

Hg2. $g(w) = -g(-w),\ wg(w) > 0$ if $w \neq 0$.

Hg3. For some $k > 1$ and all $w > 0$ we have

$$wg'(w) \geq kg(w). \tag{2.1}$$

We list some implications of these hypotheses. It follows from (2.1) that $w^{-k}g(w)$ is increasing on $(0, \infty)$, so that for small $w > 0$ we have

$$g(w) = O(w^k) \quad \text{as} \quad w \to 0, \tag{2.2}$$

and likewise for small $w < 0$ since $g(w)$ is odd. For large $w > 0$ we shall have

$$g(w) > \text{const.} w^k > 0, \tag{2.3}$$

with a similar result for large $w < 0$. We write

$$G(w) = \int_0^w g(v)\, dv \tag{2.4}$$

and note that $G(w) > 0$ if $w \neq 0$, by Hg2. From (2.2-3) we deduce that

$$G(w) = O(|w|^{k+1}) \tag{2.5}$$

for small w, and

$$G(w) \geq \text{const.} |w|^{k+1} > 0, \tag{2.6}$$

for large w.

We note that

$$wg(w) - (k+1)G(w) \geq 0 \tag{2.7}$$

for all w. This follows, for $w \geq 0$, by noting that the left is zero when $w = 0$ and non-decreasing for $w > 0$, by (2.1). The result is true also for $w < 0$ since the left hand side of (2.7) is an even function.

We denote by a the unique positive root of

$$\frac{w^2}{2} - G(w) = 0. \tag{2.8}$$

In justification we note that the left hand side is positive for small $w > 0$ by (2.2) and negative for large $w > 0$ by (2.3) so that a positive root a certainly exists. To see that it is unique we note that when $w = a$ the derivative of the left hand side is

$$a - g(a) = a^{-1}[a^2 - ag(a)] = a^{-1}[2G(a) - ag(a)] < 0, \tag{2.9}$$

by (2.7). Of course (2.8) will have $-a$ as a unique negative root.

The hypotheses on $h(w,s)$ are rather similar, except that we allow as a possibility linear dependence on w, and require limited exponential decay in s. We write

$$H(w,s) = \int_0^w h(v,s)dv, \tag{2.10}$$

and indicate partial derivatives of h and H with respect to their first and second arguments by the suffixes 1 and 2. We assume

Hh1. $h(w,s) \in C^1((-\infty,\infty) \times (-\infty,\infty))$

Hh2. $h(-w,s) = -h(w,s)$, $wh(w,s) > 0$ if $|w| > 0$.

Hh3. $wh_2(w,s) < 0$ if $|w| > 0$.

Hh4. $wh_1(w,s) \geq h(w,s)$ for $w > 0$.

Hh5. For some $m > 0$ we have, uniformly for bounded w,

$$e^{ms}h(w,s) \to h_0(w) \quad \text{as} \quad s \to \infty, \tag{2.11}$$

where $h_0(w) \in C^1(-\infty,\infty)$ is odd and satisfies

$$wh_0(w) > 0 \quad \text{if} \quad w > 0. \tag{2.12}$$

Hh6. $H(w,s)$ satisfies, uniformly for bounded w,

$$e^{ms}H(w,s) \to H_0(w), \quad e^{ms}H_2(w,s) \to -mH_0(w) \ \text{ as } \ s \to \infty \tag{2.13}$$

where

$$H_0(w) = \int_0^w h_0(v)dv. \tag{2.14}$$

Hh7. For any $\delta > 0$, there exists a $\sigma \in \mathbf{R}$ such that

$$h(w,s) + g(w) - w > 0 \quad \text{if} \quad \delta < w < a, \ \ s < \sigma. \tag{2.15}$$

In Hh5 we are mainly concerned with the range $0 < m < 1$, but also consider the ranges $0 < m < 2$ and $0 < m < \infty$.

Some of the above hypotheses have been included for simplicity of argument. We shall actually be concerned with the behaviour of $w(s)$ when $-a < w < a$ and s is large and positive; in particular, the behaviour of g and h when $|w| > a$ will not be material.

It follows from these hypotheses that in any strip

$$s \geq s^*, \quad |w| \leq a \tag{2.16}$$

there will hold bounds of the form

$$|h(w,s)| \leq A|w|e^{-ms} \tag{2.17}$$

$$H(w,s) \leq \tfrac{1}{2}Aw^2e^{-ms}, \tag{2.18}$$

where A is a positive constant.

3. The main results. Subject to an elementary existence proof to be given later, we denote by $w(s) = w(s,\gamma)$ the solution of (1.1) satisfying (1.7), where $\gamma > 0$. We denote by $s_1(\gamma), s_2(\gamma), \ldots$, the successive zeros of $w(s)$ and $w'(s)$, starting with the latter, and written in descending order. Their existence will be established in Section 5. Thus, as will appear,

$$w'(s_1) = 0, \quad w(s) > 0, \quad w'(s) < 0 \quad \text{for} \quad s \in (s_1, \infty), \tag{3.1}$$

$$w(s_2) = 0, \quad w(s) > 0, \quad w'(s) > 0 \quad \text{for} \quad s \in (s_2, s_1), \tag{3.2}$$

$$w'(s_3) = 0, \quad w(s) < 0, \quad w'(s) > 0 \quad \text{for} \quad s \in (s_3, s_2), \tag{3.3}$$

and so on.

We formulate our results separately for the zeros of $w(s)$ and the zeros of $w'(s)$, the former being those of principal interest. The asymptotic formulae involve certain constants P and Q which depend on the equation, and will be specified below. We shall refer to hypotheses Hg1-3 collectively as hypothesis Hg, and similarly to hypotheses Hh1-7 as hypothesis Hh.

For the zeros of $w'(s)$ we have then

Theorem 1. *Suppose* Hg *and* Hh *are satisfied.*
(a) *If* $0 < m < \infty$, *then*

$$s_1(\gamma) = \log\gamma + P + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{3.4}$$

(b) *If* $0 < m < 1$, *then for* $n = 1, 2 \ldots$

$$s_{2n+1}(\gamma) = (1-m)s_{2n-1}(\gamma) + Q + o(1), \quad \text{as} \quad \gamma \to \infty. \tag{3.5}$$

For the zeros of $w(s)$ we have the following result.

Theorem 2. *Suppose* Hg *and* Hh *are satisfied.*
(a) *If* $0 < m < 2$, *then*

$$s_2(\gamma) = \left(1 - \frac{m}{2}\right)(\log\gamma + P) + \frac{Q}{2} + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{3.6}$$

(b) *If* $0 < m < 1$, *then for* $n = 1, 2, \ldots$

$$s_{2n+2}(\gamma) = (1-m)s_{2n}(\gamma) + Q + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{3.7}$$

In view of the basic nature of these estimates we re-formulate the last results in explicit rather than in recursive form.

Theorem 3. *Suppose* Hg *and* Hh *are satisfied and* $0 < m < 1$. *Then for* $n = 1, 2, \ldots$ *we have*

$$s_{2n}(\gamma) = A_n \log\gamma + B_n + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{3.8}$$

where

$$A_n = (1-m)^{n-1}(1 - \frac{1}{2}m), \tag{3.9}$$

$$B_n = A_n P + \frac{1}{m}(1 - A_n)Q. \tag{3.10}$$

and

$$s_{2n+1}(\gamma) = (1-m)^n \log\gamma + C_n + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{3.11}$$

where

$$C_n = (1-m)^n P + \frac{1}{m}\{1 - (1-m)^n\}Q. \tag{3.12}$$

We now specify the constants P and Q. We do this in terms of other constants a, D and M. The number a has already be defined in (2.8) as the positive root of $w^2 = 2G(w)$. We define D by

$$D = \int_0^a \Big(\frac{1}{\sqrt{w^2 - 2G(w)}} - \frac{1}{w}\Big) dw, \tag{3.13}$$

and M by

$$M = \int_{-\infty}^{\infty} m e^{-ms} H_0(W(s)) ds, \tag{3.14}$$

where W is the solution of the autonomous problem

$$W'' - W + g(W) = 0, \quad W(0) = a, \quad W'(0) = 0. \tag{3.15}$$

The constants P and Q can now be written as

$$P = -(\log a + D), \tag{3.16}$$

$$Q = -2(\log a + D) + \log\frac{M}{2}. \tag{3.17}$$

For the important example, when

$$g(w) = |w|^{p-1}w \quad \text{and} \quad h(w,s) = e^{-(p-q)s}|w|^{q-1}w,$$

where

$$1 \le q < p, \quad m = p - q > 0$$

the constants a, D and M, and thus P and Q, can be computed explicitly. We find that

$$a = \Big(\frac{p+1}{2}\Big)^{1/(p-1)}, \qquad D = \frac{2}{p-1}\log 2.$$

The solution W of the associated autonomous problem (3.15) turns out to be

$$W(s) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \{\cosh(\tfrac{1}{2}(p-1)s)\}^{-2/(p-1)}.$$

If we use this expression in (3.14), and note that in this case

$$H_0(w) = \frac{|w|^{q+1}}{q+1},$$

we find for M the value

$$M = m\frac{\{2(p+1)\}^{(q+1)/(p-1)}}{(q+1)(p-1)} \cdot \frac{\Gamma\left(\frac{2q-p+1}{p-1}\right)\Gamma\left(\frac{p+1}{p-1}\right)}{\Gamma\left(2\frac{q+1}{p-1}\right)},$$

when $2q-p+1>0$ or $q+1>m$. Since by assumption $q \geq 1$ and $m<2$ this condition is always satisfied.

4. The PDE problem. In this section, we translate some of the results for the ODE problem (1.1), (1.5) to radial solutions of the eigenvalue problem

$$\text{(I)} \quad \begin{aligned} -\Delta u &= \lambda u|u|^{q-1} + u|u|^{p-1} \quad \text{in} \quad B_N, && (4.1)\\ u &= 0 \quad \text{on} \quad \partial B_N, && (4.2)\end{aligned}$$

in which

$$1 \leq q < p, \quad p = \frac{N+2}{N-2}.$$

Let $w(s,\gamma)$ be the solution of (1.1), (1.5). Then the function

$$u(r,\gamma) = \lambda^{1/(p-q)} e^s w(s,\gamma),$$

where r and s are related through

$$r^{(N-2)/2} = \left(\frac{N-2}{2}\right)^{(N-2)/2} \lambda^{-1/(p-q)} e^{-s},$$

is a solution of (4.1), (4.2) provided that λ has been chosen so that

$$1 = \left(\frac{N-2}{2}\right)^{(N-2)/2} \lambda^{-1(p-q)} \exp(-s_{2n}$$

for some $n = 1, 2, \ldots$. In this manner we ensure that the boundary condition (4.2) is satisfied. Thus we can define functions

$$\lambda_n(\gamma) = \left[\left(\frac{N-2}{2}\right)^{(N-2)/2} \exp\{-s_{2n}(\gamma)\}\right]^{p-q},$$

and intervals

$$\Lambda_n = \{\lambda_n(\gamma) : 0 < \gamma < \infty\}.$$

Plainly, if $\lambda \in \Lambda_n$ for some $n \geq 1$, then Problem I has a radial solution u_n with $n-1$ zeros, and if $\lambda \notin \Lambda_n$, no such solution exists.

It is well known (see for instance [4]) that if $q > 1$, then $\lambda_n(\gamma) \to \infty$ as $\gamma \to 0$, and that if $q = 1$, then $\lambda_n(\gamma) \to \mu_n$ as $\gamma \to 0$, where μ_n is the eigenvalue of $-\Delta$ in B_N with Dirichlet boundary conditions, corresponding to the radial eigenfunction with $n-1$ zeros. On the other hand we have shown in Theorem 3 that when

$$0 < p - q < 1 \quad \Leftrightarrow \quad \frac{4}{N-2} < q < \frac{N+2}{N-2}, \tag{4.3}$$

then $s_{2n}(\gamma) \to \infty$, and so $\lambda_n(\gamma) \to 0$ as $\gamma \to \infty$. Thus we can draw the following conclusion:

Theorem 4. (a) *Suppose $q > 1$ and*

$$\frac{4}{N-2} < q < \frac{N+2}{N-2}.$$

Then for any $\lambda > 0$ and any $n \geq 1$ there exists a solution of Problem I *with $n-1$ zeros.*

(b) *Suppose $q = 1$ and $N > 6$. Then for any $\lambda \in (0, \mu_n)$, $n \geq 1$, there exists a solution of Problem* I *with $n-1$ zeros.*

The first part of this theorem was recently proved by Jones [11] by a different method. The second part was proved for integral values of N; i.e., for $N \geq 7$, in papers by Cerami, Solimini and Strüwe [9] and Solimini [12]. They used variational methods in $\mathbf{R}^N$. Finally, it was shown by Atkinson, Brézis and Peletier [2] that Theorem 4(b) is optimal in that when $N \leq 6$, there exists a neighbourhood of $\lambda = 0$ for which there exist no solutions of Problem I which change sign.

In addition to shedding light on the existence of radial solutions of Problem I with a prescribed number of zeros, Theorems 1–3 contain information about the behaviour of λ_n and $\|u\|_\infty = u(0)$ as $\gamma \to \infty$. Recall that $w(s,\gamma)e^s \to \gamma$ as $s \to \infty$. Therefore, if we denote by u_n the solution corresponding to λ_n, we have

$$\begin{aligned} u_n(0,\gamma) = \gamma\lambda^{1/(p-q)}(\gamma) &= \left(\frac{N-2}{2}\right)^{(N-2)/2} \gamma \exp\{-s_{2n}(\gamma)\} \\ &= O(\gamma^{1-A_n}) \quad \text{as} \quad \gamma \to \infty, \end{aligned} \tag{4.4}$$

by Theorem 3. Because $A_n < 1$ for every $n \geq 1$, (4.4) implies that

$$\|u_n\| \to \infty \quad \text{as} \quad \gamma \to \infty$$

for every $n \geq 1$.

More specifically, we find the following relations:

Theorem 5. *Suppose*

$$\frac{4}{N-2} < q < \frac{N+2}{N-2}.$$

Then,

$$u_n(0,\gamma) \sim K_n \gamma^{1-A_n} \quad \text{as} \quad \gamma \to \infty,$$

$$\lambda_n(\gamma) \sim K_n^{p-q} \gamma^{-(p-q)A_n} \quad \text{as} \quad \gamma \to \infty,$$

where

$$K_n = \left(\frac{N-2}{2}\right)^{(N-2)/2} e^{-B_n}$$

and A_n and B_n are given in Theorem 3.

Remark. It may be shown by means of routine, though lengthy calculations that the results in Theorem 5 agree with those for $n = 1$ obtained in [4].

As announced in Section 1, we can deal, without additional difficulty, with rather more general nonlinearities than those appearing in (1.7). A special example is given by

$$-\Delta u = \lambda r^{\beta} u|u|^{q'-1} + r^{\alpha} u|u|^{p'-1}; \tag{4.5}$$

we refer to [9, 12] for the more general m–Laplacian case. Again we ask for radial solutions in B_N, finite at the origin and vanishing on the boundary. Conditions for "criticality" in this setting are discussed below, with reference to the applicability of our present ODE approach, and not with reference to the compactness or otherwise of the imbedding of associated function spaces.

For a more general formulation than (4.1), we consider the form

$$-\Delta u = f(u, r) \tag{4.6}$$

where

$$f(u, r) = f_1(u, r) + \lambda f_2(u, r) \tag{4.7}$$

and f_1, f_2 satisfy certain homogeneity conditions. We ask that

$$f_1(Ax, y) = A^p f_1(x, yA^{2/(N-2)}), \tag{4.8}$$

$$f_2(Ax, y) = A^q f_2(x, yA^{2/(N-2)}), \tag{4.9}$$

for any $A > 0$. Here f_1 is critical in the sense that, as before

$$p = (N+2)/(N-2), \tag{4.10}$$

while f_2 is "subcritical," in that

$$0 < q < p. \tag{4.11}$$

The standard case (1.11) is given by

$$f_1(u, r) = u|u|^{p-1}, \quad f_2(u, r) = u|u|^{q-1}. \tag{4.12}$$

In the case (4.1) we must have

$$p' = p + \frac{1}{2}\alpha(p-1), \quad q' = q + \frac{1}{2}\beta(p-1), \tag{4.13-14}$$

where p and q satisfy (4.10-11). The change of variables

$$u(r) = \lambda^{1/(p-q)} w(s) e^{s}, \quad r = (\lambda^{-1/(p-q)} e^{-s})^{(p-1)/2}, \tag{4.15-16}$$

a slight modification of (1.15-16), then yields

$$w'' - w + \frac{1}{4}(p-1)^2 \{ f_1(w,1) + f_2(w,1) e^{-(p-q)s} \} = 0. \tag{4.17}$$

This may be identified with (1.1) by taking

$$g(w) = \frac{1}{4}(p-1)^2 f_1(w,1), \tag{4.18}$$

$$h(w,s) = \frac{1}{4}(p-1)^2 f_2(w,1) e^{-(p-q)s}. \tag{4.19}$$

We require, of course, that these functions g and h satisfy the requirements of Section 2. For example $f_1(w,1)$ may be a signed power of w of degree > 1, or a linear combination of such signed powers with positive coefficients. The function $f_2(w,1)$ might have the same form, with degrees ≥ 1.

5. Preliminary arguments. Going back to the basic ODE problem (1.1), (1.5), we first establish the existence of the solution $w(s,\gamma)$. As in the case of the related problem (1.13-14), the existence of this solution may be seen as a consequence of much more general results. Nevertheless, an outline argument will be useful. We asume the hypotheses Hg and Hh, and that $0 < m < \infty$.

Lemma 1. *The solution $w(s,\gamma)$ exists and satisfies*

$$0 < w(s) < \gamma e^{-s}, \tag{5.1}$$

for large s.

Proof: Proceeding formally, we put $w(s) = v(s)e^{-s}$, and need that $v(s) \to \gamma$ as $s \to \infty$. This leads to the integral equation

$$v(s) = \gamma - \int_s^\infty e^{2\sigma} d\sigma \int_\sigma^\infty e^{-\tau}[g(ve^{-\tau}) + h(ve^{-\tau}, \tau)] d\tau. \tag{5.2}$$

Standard contraction–mapping arguments show that this integral equation has unique a solution for large s, satisfying $0 < v(s) < \gamma$, so that (5.1) holds for large s.

To extend the domain of $w(s)$ throughout the real axis we use an energy–type argument. We write

$$E(s) = \frac{w'^2}{2} - \frac{w^2}{2} + G(w) + H(w,s) \tag{5.3}$$

and make the following observation.

Lemma 2. *So long as the solution can be continued for decreasing s, we have*

$$E(s) > 0. \tag{5.4}$$

Proof: We have on differentiating (5.3) and using (1.1) and Hh3,

$$E'(s) = H_2(w,s) \leq 0, \tag{5.5}$$

so that $E(s)$ tends to a limit as $s \to \infty$. Since $w(s) \to 0$, so that also $H(w,s) \to 0$ by (2.18), we must have

$$E(s) \to 0 \quad \text{as} \quad s \to \infty. \tag{5.6}$$

Together with (5.5), this proves (5.4).

More precisely, the argument shows that

$$\frac{w'^2}{2} - \frac{w^2}{2} + G(w) = -H(w(s),s) - \int_s^\infty H_2(w(t),t)\,dt$$

$$= \int_s^\infty \Big\{H_2(w(s),t) - H_2(w(t),t)\Big\}\,dt, \tag{5.7}$$

from which we deduce the following uniform bound.

Lemma 3. *The solution $w(s)$ can be continued backwards throughout $(-\infty,\infty)$ and satisfies*

$$|w(s)| < a \quad \textit{for} \quad -\infty < s < \infty. \tag{5.8}$$

Proof: The inequality (5.8) certainly holds for large $s > 0$ since $w(s) \to 0$ as $s \to \infty$. We suppose by way of contradiction, that as $w(s)$ is continued backwards, $|w|$ reaches the value a, and denote by s^* the largest such value. We shall have then

$$|w(s^*)| = a, \quad |w(s)| < a \text{ for } s^* < s < \infty, \tag{5.9}$$

and so, from (5.7),

$$w'^2(s^*) = 2\int_{s^*}^\infty \{H_2(w(s^*),t) - H_2(w(t),t)\}dt. \tag{5.10}$$

Here the left hand side is non-negative, while the right hand side must be negative, since $H_2(y,t)$, which is an even function function of y and so a function of $|y|$ is decreasing in $|y|$. Thus (5.9) is impossible, and so backward continuation can go on indefinitely, with (5.8) remaining in force.

With the aid of the energy function E we can derive important oscillatory properties of w.

Lemma 4. *The derivative $w'(s)$ has a largest zero $s_1 > -\infty$.*

Proof: The set of zeros of w' is certainly bounded above. Suppose, w' does not have a largest zero. Then $w > 0$, $w' < 0$ in $\mathbf{R}$. In view of (5.8) we can conclude that for some $\ell \in (0,a]$ we have

$$w(s) \uparrow \ell \quad \text{as} \quad s \to -\infty. \tag{5.11}$$

It then follows from Hh7 and (1.1) that there exist numbers $\sigma \in \mathbf{R}$ and $\varepsilon > 0$ such that

$$w''(s) < -\varepsilon \quad \text{if} \quad s < \sigma,$$

implying that w' has a zero on $(-\infty,\sigma)$, a contradiction.

Lemma 5. *The solution $w(s)$ is oscillatory as $s \to -\infty$.*

Proof: Again we suppose to the contrary that $w(s) > 0$ in $(-\infty, s_1)$. Suppose that

$$\liminf_{s \to -\infty} w(s) > 0.$$

Then, by Hh7, $w''(s) < -\varepsilon$ for some $\varepsilon > 0$ and s sufficiently large and negative, and we obtain a contradiction. Thus we must have

$$\liminf_{s \to -\infty} w(s) = 0. \tag{5.12}$$

Suppose first that $w'(s)$ is ultimately monotone, so that $w(s)$ and $w'(s)$ both tend to zero as $s \to -\infty$. there will then be a sequence of points (σ_k) tending to $-\infty$, such that $w''(\sigma_k) \to 0$, and it will follow from (1.1) that

$$h(w(\sigma_k), \sigma_k) \to 0, \tag{5.13}$$

and so, by Hh, that

$$H(w(\sigma_k), \sigma_k) \to 0. \tag{5.14}$$

By (5.3) we will also have $E(\sigma_k) \to 0$, which contradicts (5.5-6). Suppose next that $w'(s)$ is not monotone for large $s < 0$. We then take the σ_k to be the successive points at which $w(s)$ has a minimum, so that

$$w'(\sigma_k) = 0, \quad w(\sigma_k) = 0, \quad w''(\sigma_k) \geq 0.$$

From the latter we have that $h(w(\sigma_k), \sigma_k) < w(\sigma_k)$, so that we have (5.13) again. This leads to a contradiction as before. Hence $w(s)$ must have a zero s_2 in $(-\infty, s_1)$. The argument can then be repeated to prove the existence of a sequence of zeros tending to $-\infty$.

Lemma 6. *The zeros of $w(s)$ and $w'(s)$ interlace.*

Proof: This will follow if we show that

$$ww'' < 0 \quad \text{when} \quad w' = 0. \tag{5.15}$$

We know now that (5.4) holds for all s, and so when $w' = 0$ we have

$$w^2 < 2G(w) + 2H(w, s). \tag{5.16}$$

From (2.1) we have

$$ww'' = w^2 - wg(w) - wh(w, s).$$

However we have $wg(w) \geq 2G(w)$, by (2.7), and likewise

$$wh(w, s) \geq 2H(w, s),$$

from which the result follows.

We assemble certain bounds on $w'^2(s)$ arising from the above arguments. From (5.3-4)and (5.7) we have

$$w'^2(s) > w^2(s) - 2G(w(s)) - 2H(w(s), s) \tag{5.17}$$

$$\geq w^2(s) - 2G(w(s)) - 2H(a, s), \tag{5.18}$$

and on the other hand

$$w'^2(s) \leq w^2(s) - 2G(w(s)) + 2H(a, s). \tag{5.19}$$

In particular, we have from (5.17-19)

Lemma 7. *For large s*

$$w'(s) = 0 \quad \Rightarrow \quad |w(s)| = a + O(e^{-ms}). \tag{5.20}$$

In addition to the zeros $s_n(\gamma), n = 1, 2, \ldots$, of $w'(s)$ and $w(s)$ characterised in (1.6) and further described in (3.1-3), which form the principal object of our study, it is convenient to use certain intermediate points $t_n = t_n(\gamma)$ which are such that

$$w(t_n) = (-1)^{[(n-1)/2]} \frac{a}{2} \tag{5.21}$$

$$\ldots < s_3 < t_3 < s_2 < t_2 < s_1 < t_1 < \infty. \tag{5.22}$$

Here $[x]$ denotes as usual $\sup\{k \in \mathbf{Z} : k \leq x\}$ (See Fig. 1.).

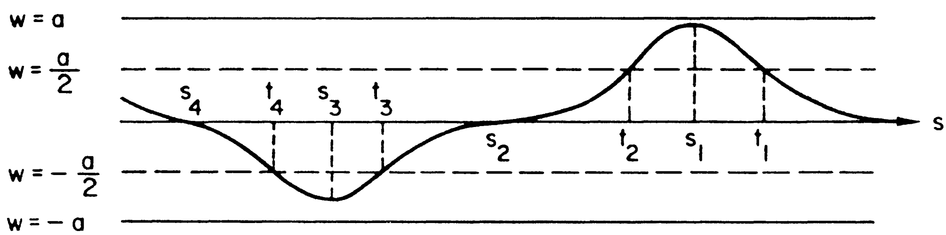

Fig. 1. The point s_n and t_n.

It is clear from Lemma 7 that if $s_1(\gamma)$, for example, is suitably large, then $t_1(\gamma)$ must exist in $(s_1(\gamma), \infty)$. In fact we shall find that if $0 < m < 1$, then $t_n(\gamma)$ exists up to any assigned number k, provided that γ is chosen sufficiently large.

In intervals such as (t_2, t_1) or (t_4, t_3) which contain a zero of w', we approximate the differential equation by means of (1.3) or (3.15) and compare $w(s)$ to the function $W(s)$ which has been suitably translated. In other intervals $(t_1, \infty), (t_3, t_2)$ we use methods of approximate integration to deduce information about the location of the zeros s_{2n} of w.

6. Existence of and approximation to $t_1(\gamma)$. We establish successively the existence, for large γ, of the numbers $t_n(\gamma)$, $n = 1, 2, \ldots$; that of the $s_n(\gamma)$ has been dealt with in Section 5. We start with $t_1 = t_1(\gamma)$, defined formally as the value, if any, such that

$$w(t_1) = \tfrac{1}{2}a, \quad w' < 0 \quad \text{in} \quad [t_1, \infty). \tag{6.1}$$

We choose a number S, dependent on γ, and aim to show that a contradiction results if either t_1 does not exist, or it does exists and $t_1 < S$. We restrict S in the first place so that so long as the solution remains in the strip

$$\Sigma = \{(s, w) : s \geq S, \ \ 0 < w \leq \tfrac{1}{2}a\}, \tag{6.2}$$

we must have $w'^2(s) > 0$, and so $w'(s) < 0$. From Lemma 2 it follows that

$$w'^2(s) \geq w^2(s) - 2G(w(s)) - 2H(w(s), s),$$

and so, by (2.18), that

$$w'^2(s) \geq w^2(s)(1 - Ae^{-ms}) - 2G(w(s)). \tag{6.3}$$

By the hypotheses of Section 2, there will be a constant $B \in (0,1)$ such that

$$w^2 - 2G(w) \geq Bw^2 \quad \text{if} \quad |w| \leq \tfrac{1}{2}a \tag{6.4}$$

so that

$$w'^2 \geq w^2(B - Ae^{-ms}) \geq \tfrac{1}{2}Bw^2 \quad \text{if} \quad s \geq S, \tag{6.5}$$

where

$$S > \frac{1}{m}\log\left(\frac{2A}{B}\right). \tag{6.6}$$

Note that (6.6) implies that $Ae^{-mS} < 1/2$.

Thus, if $(s, w) \in \Sigma$ and S satisfies (6.6), then, by (6.3),

$$\frac{w'}{\sqrt{(1 - Ae^{-ms})w^2 - 2G(w)}} \leq -1. \tag{6.7}$$

We rewrite this inequality in the form

$$\left(\frac{1}{\sqrt{w^2 - \frac{2G(w)}{1-Ae^{-ms}}}} - \frac{1}{w}\right)w' \leq -\frac{w'}{w} - \sqrt{1 - Ae^{-ms}} \tag{6.8}$$

or again, since $s \geq S$ and $w'(s) < 0$,

$$\left(\frac{1}{\sqrt{w^2 - \frac{2G(w)}{1-Ae^{-mS}}}} - \frac{1}{w}\right)w' \leq \left(-1 - \frac{w'}{w}\right) + (1 - \sqrt{1 - Ae^{-ms}}). \tag{6.9}$$

We then integrate (6.9) over (S, σ) and let $\sigma \to \infty$. The last term on the right hand side gives

$$\int_S^\infty (1 - \sqrt{1 - Ae^{-ms}})ds < \frac{A}{m}e^{-mS}. \tag{6.10}$$

The first term on the right hand side of (6.9) gives, in view of the condition (1.5) at $s = \infty$,

$$[S + \log w(S)] - \lim_{\sigma\to\infty}[\sigma + \log w(\sigma)] = S + \log w(S) - \log\gamma \leq S + \log(a/2) - \log\gamma.$$

As to the left hand side of (6.9), we observe that the integrand is positive, and that the range of integration is from $w = w(S) < \frac{1}{2}a$ to $w = 0$. Hence, rearranging, we have

$$S > \log\gamma - \log\frac{a}{2} - \frac{A}{m}e^{-mS} - J, \tag{6.11}$$

where

$$J = \int_0^{a/2}\left(\frac{1}{\sqrt{w^2 - \frac{2G(w)}{1-Ae^{mS}}}} - \frac{1}{w}\right)dw. \tag{6.12}$$

For convenience we set

$$X = w^2 - \frac{2G(w)}{1 - Ae^{-mS}}. \tag{6.13}$$

Then

$$X < w^2 - 2G(w) \tag{6.14}$$

and

$$1 - \frac{X}{w^2 - 2G(w)} = \frac{2G(w)}{w^2 - 2G(w)} \cdot \frac{Ae^{-mS}}{1 - Ae^{-mS}}. \tag{6.15}$$

When $(s, w) \in \Sigma$ it follows from (6.4) that the first factor in the right hand side of (6.15) is bounded above by $(1 - B)/B$. Hence, in view of (6.6) we can conclude that

$$X > (1 - B_1 e^{-mS})[w^2 - 2G(w)], \tag{6.16}$$

where $B_1 = 2A(1 - B)/\{B(2 - B)\}$. We accordingly impose the further restriction on S that

$$B_1 e^{-mS} < 1. \tag{6.17}$$

Write

$$I_1 = \int_0^{a/2} \Big(\frac{1}{\sqrt{w^2 - 2G(w)}} - \frac{1}{w} \Big) dw. \tag{6.18}$$

Plainly, by (6.14),

$$J > I_1. \tag{6.19}$$

On the other hand, using the identity

$$\frac{1}{p} - \frac{1}{q} = \frac{q^2 - p^2}{pq(p + q)} \qquad p,\ q > 0, \tag{6.20}$$

we can rewrite J, with

$$p = \sqrt{X} \quad \text{and} \quad q = w,$$

as

$$J = \frac{1}{1 - Ae^{-mS}} \int_0^{a/2} \frac{2G(w)\, dw}{w\sqrt{X}(w + \sqrt{X})}. \tag{6.21}$$

Hence, using the lower bound (6.16) for X, we find that

$$J < K(S) \int_0^{a/2} \frac{2G(w)\, dw}{w\sqrt{w^2 - 2G(w)}(w + \sqrt{w^2 - 2G(w)})},$$

where

$$K(S) = (1 - Ae^{-mS})^{-1}(1 - B_1 e^{-mS})^{-1}. \tag{6.22}$$

Using (6.20) once again we find that

$$J < K(S) I_1. \tag{6.23}$$

It thus follows from (6.11) that

$$S > \log\gamma - \log\frac{a}{2} - \frac{A}{m} e^{-mS} - K(S) I_1. \tag{6.24}$$

We now take

$$S = \log \gamma - \log \frac{a}{2} - I_1 - \eta, \tag{6.25}$$

where $\eta > 0$ is fixed. Then

$$K(S) = 1 + O(\gamma^{-m}) \quad \text{as} \quad \gamma \to \infty,$$

whence for sufficiently large γ (6.6) and (6.17) will hold, while (6.24) will be false, so that we have a contradiction. We thus have

Lemma 8. *Suppose $0 < m < \infty$. For large γ, $t_1(\gamma)$ exists and satisfies*

$$t_1(\gamma) > \log \gamma - \log \frac{a}{2} - I_1 - o(1) \quad \text{as} \quad \gamma \to \infty, \tag{6.26}$$

where I_1 is defined in (6.18).

The lower bound for $t_1(\gamma)$ is easily completed to an asymptotic formula.

Lemma 9. *Suppose $0 < m < \infty$. For large γ, $t_1(\gamma)$ exists and satisfies*

$$t_1(\gamma) = \log \gamma - \log \frac{a}{2} - I_1 + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{6.27}$$

where I_1 is given in (6.18).

Proof: The existence of t_1 being assured by the last section, we use the inequality

$$w'^2 < w^2 - 2G(w), \quad t_1 \le s < \infty, \tag{6.28}$$

which follows from (5.7), since $w(s) > w(t)$ for $t_1 \le s < t < \infty$. We deduce that

$$\frac{w'}{\sqrt{w^2 - 2G(w)}} > -1 \quad \text{on} \quad [t_1, \infty),$$

or

$$\left(\frac{1}{\sqrt{w^2 - 2G(w)}} - \frac{1}{w}\right) w' > -1 - \frac{w'}{w} \quad \text{if} \quad t_1 \le s < \infty. \tag{6.29}$$

Proceeding as with (6.9), we integrate (6.29) over (t_1, σ), and let $\sigma \to \infty$. Remembering the definition (6.18) of I_1, we obtain

$$-I_1 > \log \gamma - \log \frac{a}{2} + t_1(\gamma). \tag{6.30}$$

This together with (6.26) proves (6.27).

We remark that the $o(1)$ term in (6.27) must be negative. To extend the approximation to $s_2(\gamma)$ and $t_2(\gamma)$ we use a different argument. We consider $w(s)$ for $s \le t_1$ as the solution of a backward initial value problem, and compare this problem with another such problem. We have of course

$$w(t_1) = \frac{a}{2}. \tag{6.31}$$

We also have, according to (5.17), (5.19), Lemma 9 and Hh6, that

$$|w'^2(t_1) - w^2(t_1) + 2G(w(t_1))| < 2H(a/2, t_1) = o(1) \quad \text{as} \quad \gamma \to \infty$$

and so

$$w'(t_1) = -\sqrt{(a^2/4) - 2G(a/2)} + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{6.32}$$

so that $w(s)$ is continued for $s \leq t_1$ as the solution of (1.1) with the data (6.31-32).

We resume this argument in Section 8.

7. A comparison function. To estimate the function $w(s)$ on (t_2, t_1) we use as a comparison function the solution of the autonomous equation

$$W'' - W + g(W) = 0 \quad \text{on} \quad (-\infty, \infty) \tag{7.1}$$

with the initial data

$$W(0) = a, \quad W'(0) = 0. \tag{7.2}$$

In this section we note some basic properties of this function. We remark to begin with that it is an even function. We have the first integral

$$W'^2(s) - W^2(s) = 2G(W(s)) \equiv \text{constant} = 0. \tag{7.3}$$

Thus for $s > 0$ the integration can be completed in the form

$$s = \int_W^a \frac{du}{\sqrt{u^2 - 2G(u)}}. \tag{7.4}$$

Here the integral diverges as $W \to 0$, and so we have

$$0 < W(s) < a, \quad W'(s) < 0 \quad \text{for} \quad s > 0. \tag{7.5}$$

For $s < 0$ we have $0 < W(s) < a$, $W'(s) > 0$.

The integral (7.4) has the alternative form

$$s = \int_W^a \Big(\frac{1}{\sqrt{u^2 - 2G(u)}} - \frac{1}{u} \Big)\, du + \log \frac{a}{W} \quad \text{for} \quad s > 0. \tag{7.6}$$

In this version, the integral converges as $W \to 0$: we write

$$D = \int_0^a \Big(\frac{1}{\sqrt{u^2 - 2G(u)}} - \frac{1}{u} \Big)\, du. \tag{7.7}$$

Then (7.6) yields for large $s > 0$,

$$s = D + \log \frac{a}{W(s)} + o(1) \quad \text{as} \quad s \to \infty. \tag{7.8}$$

Rearranging this, and also taking into account the result for large negative s, we have

$$W(s) \sim a e^{D - |s|} \quad \text{as} \quad |s| \to \infty. \tag{7.9}$$

There will be a unique $T > 0$ such that $W(T) = a/2$. By (7.4) and (7.6), it will have the representations

$$T = \int_{a/2}^{a} \frac{du}{\sqrt{u^2 - 2G(u)}} \tag{7.10}$$

and

$$T = \int_{a/2}^{a} \Big(\frac{1}{\sqrt{u^2 - 2G(u)}} - \frac{1}{u} \Big)\, du + \log 2. \tag{7.11}$$

8. Approximation to s_1, t_2. As a first observation we note that the existence of s_1 and t_2 is ensured for large γ by Lemmas 5 and 8. To obtain approximations to s_1 and t_2 we go back to the comparison argument initiated at the end of Section 6. We define

$$\hat{w}(s) = w(s + t_1 - T). \tag{8.1}$$

Then, by (6.31-32),

$$\hat{w}(T) = \frac{a}{2}, \quad \hat{w}'(T) = -\sqrt{(a^2/4) - 2G(a/2)} + o(1), \tag{8.2-3}$$

for large γ. Also, by transformation from (1.1),

$$\hat{w}''(s) = \hat{w}(s) - g(\hat{w}(s)) - h(\hat{w}(s), s + t_1 - T), \tag{8.4}$$

and here by Hh5,

$$h(\hat{w}(s), s + t_1 - T) = O(e^{-m(s+t_1-T)}) = O(e^{-mt_1}),$$

if s is bounded below. Hence by Lemma 9

$$h(\hat{w}(s), s + t_1 - T) = O(\gamma^{-m}). \tag{8.5}$$

We compare this with the differential equation for $W(s)$ and the data when $s = T$, namely

$$W''(s) = W(s) - g(W(s)), \tag{8.6}$$

$$W(T) = \frac{a}{2}, \quad W'(T) = -\sqrt{(a^2/4) - 2G(a/2)} \tag{8.7}$$

and deduce that over any bounded s–interval,

$$\hat{w}(s) = W(s) + o(1), \quad \hat{w}'(s) = W'(s) + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{8.8}$$

In particular, we deduce that for some $\hat{s} = o(1)$ we have

$$\hat{w}'(\hat{s}) = 0, \tag{8.9}$$

and that

$$\hat{w}(-T) = \frac{a}{2} + o(1)), \quad \hat{w}'(-T) = \sqrt{(a^2/4) - 2G(a/2)} + o(1). \tag{8.10}$$

Moreover, there will be a $\hat{t} = -T + o(1)$ such that

$$\hat{w}(\hat{t}) = \frac{a}{2}, \quad \hat{w}'(\hat{t}) = \sqrt{(a^2/4) - 2G(a/2)} + o(1). \tag{8.11}$$

Reverting to the original solution $w(s)$, we obtain from (8.9) the required zero s_1, of $w'(s)$, with

$$s_1 = \hat{s} + t_1 - T = t_1 - T + o(1), \tag{8.12}$$

and likewise t_2 with

$$t_2 = \hat{t} + t_1 - T = t_1 - 2T + o(1). \tag{8.13}$$

By means of the approximation to t_1 given in Lemma 9, we can put (8.12-13) in more explicit terms. This yields the following Lemma.

Lemma 10. *Suppose $0 < m < \infty$. For large $\gamma, s_1(\gamma)$ and $t_2(\gamma)$ exist and satisfy*

$$s_1(\gamma) = \log\gamma - (\log a + D) + o(1) \quad \text{as} \quad \gamma \to \infty \tag{8.14}$$

$$t_2(\gamma) = \log\gamma - (\log a + D) - T + o(1) \quad \text{as} \quad \gamma \to \infty \tag{8.15}$$

where D and T are defined in, respectively, (7.7) and (7.11).

Note that (8.14) is the first assertion of Theorem 1. More generally, we have

$$w(s_1 + s) = W(s) + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{8.16}$$

uniformly for bounded s–intervals. This is essentially (8.8), since

$$\begin{aligned} w(s_1 + s) &= w(\hat{s} + t_1 - T + s) = w(t_1 - T + s) + o(1) \\ &= \hat{w}(s) + o(1) = W(s) + o(1). \end{aligned} \tag{8.17}$$

Because $W(s) > 0$ for all s, it thus follows that $s_2(\gamma)$, if it exists, must satisfy

$$s_1(\gamma) - s_2(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty. \tag{8.18}$$

9. Approximation to $E(s_1)$. In passing from one extremum to the next, the solution spends, roughly speaking, a finite time near each extremum and, for large γ, a much longer time in the neighbourhood of zero (See Fig. 1). This longer time is closely related to the value of $|w'|$ at the zero, which by (5.3) is equal to $\sqrt{2E}$. In the case of the interval (s_3, s_1) we need to estimate $E(s_2)$. As a first step in this direction we estimate $E(s_1)$ which by (5.5) will be a lower bound for $E(s)$ when $s \leq s_1$.

We define the integrals

$$M_1 = m\int_{-\infty}^{0} e^{-ms} H_0(W(s))\,ds, \quad M_2 = m\int_{0}^{\infty} e^{-ms} H_0(W(s))\,ds, \tag{9.1}$$

and write

$$M = M_1 + M_2.$$

Convergence is ensured by the bounds in Section 7. We prove next

Lemma 11. *Suppose $0 < m < \infty$. Then*

$$E(s_1) \sim M_2 e^{-ms_1} \sim M_2 L\gamma^{-m} \quad \text{as} \quad \gamma \to \infty, \tag{9.2}$$

where

$$L = \exp\{m(\log a + D)\}. \tag{9.3}$$

Proof: We have by (5.5-6) that

$$E(s_1) = -\int_{s_1}^{\infty} H_2(w(s), s)\,ds = -\int_{0}^{\infty} H_2(w(s_1 + t), s_1 + t)\,dt.$$

Because $s_1 \to \infty$ as $\gamma \to \infty$, it follows from Hh5 and the dominated convergence theorem that

$$E(s_1) \sim me^{-ms_1} \int_0^\infty e^{-mt} H_0(w(s_1+t))dt \quad \text{as} \quad \gamma \to \infty. \tag{9.4}$$

By Lemma 10 we have

$$e^{-ms_1} \sim L\gamma^{-m}, \tag{9.5}$$

and so to complete the proof we need to show that as $\gamma \to \infty$,

$$\int_0^\infty e^{-mt} H_0(w(s_1+t))dt \to \int_0^\infty e^{-mt} H_0(W(t))dt, \tag{9.6}$$

It follows from (8.16) that (9.6) is true when both integrals are taken over $(0, R)$ for any fixed $R > 0$. The full result then follows by uniform convergence, since both w and W are bounded.

By a more detailed argument we show later that

$$E(s_2) - E(s_1) \sim M_1 e^{-ms_1} \sim M_1 L\gamma^{-m}, \tag{9.7}$$

so that

$$E(s_2) \sim Me^{-ms_1} \sim ML\gamma^{-m} \quad \text{as} \quad \gamma \to \infty. \tag{9.8}$$

10. The interval (s_3, s_1): existence of t_3. Having dealt with the atypical interval (s_1, ∞), we have reached the stage of dealing in a routine manner with successive intervals (s_3, s_1), $(s_5, s_3), \ldots$. The process is fully routine in the case $0 < m < 1$; we deal later with the special case $1 \leq m < 2$, for which we only consider the existence and estimation of $s_2(\gamma)$.

In the case at hand, taking it that $0 < m < 1$, we commence the process with an $o(1)$ approximation to $s_1(\gamma)$, knowing that $s_1(\gamma)$ has the properties,

$$s_1(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty, \tag{10.1}$$

$$w(s_1) = a + o(1) \quad \text{as} \quad \gamma \to \infty, \quad w'(s_1) = 0, \tag{10.2}$$

$$\log E(s_1) = -ms_1 + \log M_2 + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{10.3}$$

In this and the next three sections we use (10.1-3) as a basis for showing that $s_3(\gamma)$ has properties similar to (10.1-3), apart from a sign change in (10.2); it will then follow that $t_3(\gamma)$ exists for large γ. The argument will thus provide a recursive procedure for successively estimating s_3, s_5, and so on, and en route doing the same for the zeros $s_2, s_4, \ldots$ and the intermediate points $t_n(\gamma)$.

We know already on the basis of (10.1-2) that $t_2(\gamma)$ exists for large γ with the estimate

$$t_2(\gamma) = s_1(\gamma) - T + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{10.4}$$

and satisfies

$$w(t_2) = \frac{a}{2}, \quad w'(t_2) = \sqrt{(a^2/4) - 2G(a/2)} + o(1) \ \text{ as } \ \gamma \to \infty. \tag{10.5}$$

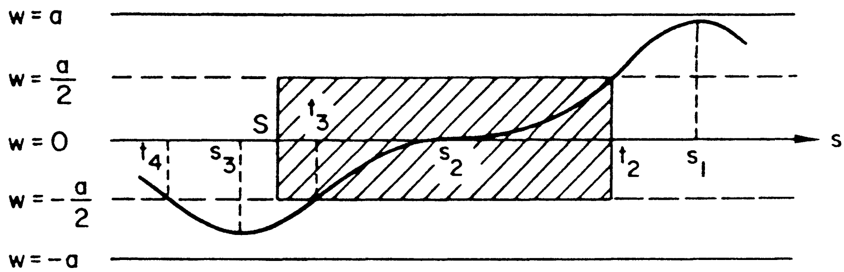

Fig. 2. The rectangle P.

Arguing as in Section 6, we choose a number $S < t_2$ (not the same as in Section 6 but with a similar role) and consider the continuation of the trajectory $(s, w(s))$ for decreasing s in a rectangle (see Fig. 2)

$$P = \{(s,w) : S \leq s \leq t_2,\ |w| \leq \frac{a}{2}\}. \tag{10.6}$$

We claim that for suitable S and large γ the trajectory will leave this rectangle by its lower side so that t_3 exists with

$$S \leq t_3 < s_2. \tag{10.7}$$

Our choice for S will be of the form

$$S = S(\gamma) = (1-m)s_1(\gamma) - A_1, \tag{10.8}$$

for some constant A_1. For $(s,w) \in P$, it follows from (5.5) that $E(s) \geq E(t_2) \geq E(s_1)$ and so

$$w'^2(s) > w^2(s) - 2G(w(s)) - 2H(w(s), s) + 2E(s_1). \tag{10.9}$$

By Lemma 10, $s_1(\gamma) \to \infty$ as $\gamma \to \infty$ and hence, because $0 < m < 1$, it follows from (10.8) that $S(\gamma) \to \infty$ as $\gamma \to \infty$. Therefore, still proceeding as in Section 6, we find that (6.6) is satisfied for γ large enough and that we obtain, in place of (6.5),

$$w'^2(s) \geq \frac{1}{2}Bw^2(s) + 2E(s_1).$$

Because $E(s_1) > 0$ by Lemma 2, we conclude in particular, that $w'(s) > 0$ so long as the trajectory lies in P. This means that the trajectory cannot leave the rectangle by its upper side; we need to reject the possibility that it does so by the left vertical side. Suppose to the contrary that it does, and that

$$-\frac{a}{2} < w(s) < \frac{a}{2} \quad \text{for} \quad s \in (S, t_2). \tag{10.10}$$

Since $w' > 0$ in the rectangle we may then also assume that

$$w(S) < w(s) < \frac{a}{2} \quad \text{for} \quad s \in (S, t_2). \tag{10.11}$$

We have from (10.9) and (10.11), as in (6.3), the bound

$$w'^2(s) > w^2(s)(1 - Ae^{-ms}) - 2G(w(s)) + 2E(s_1). \tag{10.12}$$

We now write

$$p^2 = w^2(s)(1 - Ae^{-mS}) - 2G(w(s)) + 2E(s_1), \tag{10.13}$$

$$q^2 = w^2(s)(1 - Ae^{-mS}) + 2E(s_1), \tag{10.14}$$

$(p, q > 0)$. Then we conclude from (10.12) upon integration over (S, t_2) that

$$t_2 - S < \int_{-a/2}^{a/2} \frac{dw}{p} = \int_{-a/2}^{a/2} \left(\frac{1}{p} - \frac{1}{q}\right) dw + \int_{-a/2}^{a/2} \frac{dw}{q} = J_1 + J_2. \tag{10.15}$$

We use (6.20) to write J_1 as

$$J_1 = 2\int_{-a/2}^{a/2} \frac{G(w)\,dw}{pq(p+q)}. \tag{10.16}$$

For a rough bound we can disregard the last term in (10.13-14) and say, as in (6.3-5), that p^2, $q^2 \geq Cw^2$, where C is some positive constant. Remembering (2.5), this yields

$$J_1 = O(1) \quad \text{as} \quad \gamma \to \infty. \tag{10.17}$$

The integral J_2 can be evaluated; we find that

$$J_2 = \frac{2}{\sqrt{1 - Ae^{-mS}}} \sinh^{-1}\left(\frac{a}{2}\sqrt{\frac{1 - Ae^{-mS}}{2E(s_1)}}\right). \tag{10.18}$$

By (10.8) we have

$$e^{-mS} = O(e^{-m(1-m)s_1}) = o(1) \quad \text{as} \quad \gamma \to \infty \tag{10.19}$$

and by (5.6) and (10.1), $E(s_1) \to 0$ as $\gamma \to \infty$. Thus, since

$$\sinh^{-1} x = \log(2x) + O(x^{-2}) \quad \text{as} \quad x \to \infty, \tag{10.20}$$

we find that

$$J_2 = 2\log a - \log 2 - \log E(s_1) + o(1) \tag{10.21}$$

$$= 2\log a - \log 2 - \log M_2 + ms_1 + o(1), \tag{10.22}$$

where we have used (9.2). Hence (10.15) gives

$$t_2 - S < ms_1 + O(1). \tag{10.23}$$

Together with (8.15) and (10.8), this gives a contradiction for large γ if A_1 is chosen suitably large. This proves

Lemma 12. *Suppose $0 < m < 1$. Then, for large γ, $t_2(\gamma)$ exists, and $s_2(\gamma)$ and $t_2(\gamma)$ satisfy*

$$(1-m)s_1(\gamma) - A_1 < t_3(\gamma) < s_2(\gamma), \tag{10.24}$$

where A_1 is some constant.

Additional, somewhat rough conclusions can be drawn at this point. If $m < 1$ we have by (10.1) that

$$t_3(\gamma),\ s_2(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty. \tag{10.25}$$

From this it follows that $E(t_3) \to 0$ and $H(w(t_3), t_3) \to 0$ as $\gamma \to \infty$ so that

$$w'^2(t_3) - w^2(t_3) + 2G(w(t_3)) = o(1), \tag{10.26}$$

and so

$$w'(t_3) = \sqrt{(a^2/4) - 2G(a/2)} + o(1). \tag{10.27}$$

The approximation argument of Section 8 can now be repeated to show that s_3 and t_4 satisfy

$$s_3(\gamma) = t_3(\gamma) - T + o(1), \quad t_4(\gamma) = t_3(\gamma) - 2T + o(1) \quad \text{as} \quad \gamma \to \infty \tag{10.28}$$

and

$$w(s_3) = -a + o(1), \quad w'(s_3) = 0, \tag{10.29}$$

$$w(t_4) = -\frac{a}{2}, \quad w'(t_4) = -\sqrt{(a^2/4) - 2G(a/2)} + o(1). \tag{10.30}$$

We remark that the existence of t_3 implies the existence of t_4.

It follows from (10.24) and (10.28) that

$$s_3 > t_4 > (1-m)s_1 + O(1), \tag{10.31}$$

and in particular, since $s_1(\gamma) \to \infty$ as $\gamma \to \infty$, that s_3 satisfies at least the first of (10.1-3).

The same approximation argument shows furthermore that

$$w(s_3(\gamma) + t) = -W(t) + o(1) \quad \text{as} \quad \gamma \to \infty, \tag{10.32}$$

uniformly in any bounded t–interval, from which we deduce that

$$s_2(\gamma) - s_3(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty, \tag{10.33}$$

and also that

$$s_3(\gamma) - s_4(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty. \tag{10.34}$$

11. The interval (s_3, s_1): the interior. We now approximate to $E(s)$ in the interior of (s_3, s_1). We know that $E(s_1)$ is a lower bound for $E(s)$ in (s_3, s_1) and we have determined its asymptotic behaviour. However our estimate of $s_1 - s_3$, and likewise that of $s_1 - s_2$, is more closely determined by the magnitude of $E(s_2)$. Somewhat more generally, we need

Lemma 13. *Suppose $0 < m < 1$. For any $\varepsilon > 0$ there exists a number $R > 0$ such that for*

$$s_3 + R < s < s_1 - R \tag{11.1}$$

and γ sufficiently large we have

$$|e^{ms_1}E(s) - M| < \varepsilon. \tag{11.2}$$

Proof: We show first that

$$E(s_2) \sim Me^{-ms_1} \quad \text{as} \quad \gamma \to \infty. \tag{11.3}$$

By a modification of (9.4), we have

$$E(s_2) \sim me^{-ms_1} \int_{s_2 - s_1}^{\infty} e^{-mt} H_0(w(s_1 + t))dt \quad \text{as} \quad \gamma \to \infty, \tag{11.4}$$

where we have used the fact that $s_2 \to \infty$, by Lemma 12. Thus to prove (11.3) we need to show that as $\gamma \to \infty$

$$\int_{s_2 - s_1}^{0} e^{-mt} H_0(w(s_1 + t))dt \to \int_{-\infty}^{0} e^{-mt} H_0(W(t))dt. \tag{11.5}$$

This is certainly true when both integrals are taken over an arbitrary interval $(-R, 0)$, by continuous dependence on initial data as in (9.6). To complete the proof we need to show that

$$\limsup_{\gamma \to \infty} \int_{s_2 - s_1}^{-R} e^{-mt} H_0(w(s_1 + t))dt \to 0 \tag{11.6}$$

as $R \to \infty$. We use the bound (see (6.3))

$$w'^2 > w^2(1 - Ae^{-ms_2}) - 2G(w), \quad s_2 \le s \le s_1, \tag{11.7}$$

and choose $\rho > 0$ so that

$$w^2(1 - Ae^{-ms_3}) - 2G(w) \ge m^2 w^2, \quad 0 \le |w| \le \rho, \tag{11.8}$$

for large γ. This is possible since $s_3 \to \infty$, as just proved in (10.31), $G(w) = O(|w|^k)$, where $k > 1$ and $m < 1$. We also choose $R > 0$ so that $W(-R) < \rho$ and note that

$$w(s_1 - R) \to W(-R), \tag{11.9}$$

by (8.16) so that

$$0 \le |w(s)| \le \rho, \quad s_2 \le s_1 - R, \tag{11.10}$$

for γ sufficiently large. Consequently, for large γ,

$$w'(s) \ge mw(s) > 0 \qquad s_2 < s \le s_1 - R, \tag{11.11}$$

so that $w(s)e^{-ms}$ is increasing in this interval. We deduce that

$$0 < w(s) \leq \rho e^{m(s+R-s_1)} \qquad s_2 \leq s \leq s_1 - R, \tag{11.12}$$

from which (11.6) follows easily. This proves (11.3).

The same argument proves (11.2) for $s_2 < s \leq s_1 - R$ and suitably large R. For $[s_3 + R, s_2)$, we replace (11.9) by an approximation (see (10.32))

$$w(s_3 + R) \to -W(-R) \in (-\rho, 0) \tag{11.13}$$

and so we have instead of (11.11)

$$w'(s) \geq -mw(s) > 0, \qquad s_3 + R \leq s < s_2, \tag{11.14}$$

which means that $w(s)e^{ms}$ is negative and increasing, and $|w(s)|e^{ms}$ is decreasing. Thus

$$|w(s)| \leq |w(s_3 + R)|e^{m(-s+s_3+R)}, \qquad s_3 + R \leq s \leq s_2, \tag{11.15}$$

and the proof may be completed as before.

12. The interval (s_3, s_1): approximation to s_2. The rough lower bound obtained in (10.24) for s_2 can now be replaced by an estimate with an $o(1)$ error, comparable to those for s_1 and t_2 given by (8.14-15). We prove

Lemma 14.. *Suppose* $0 < m < 1$. *Then*

$$s_2(\gamma) = (1 - \frac{1}{2}m)s_1(\gamma) - (\log a + D) + \frac{1}{2}\log\frac{M}{2} + o(1) \quad \textit{as} \quad \gamma \to \infty. \tag{12.1}$$

Proof: Since $t_2 = s_1 - T + o(1)$, it will be sufficient for (12.1) to approximate to $t_2 - s_2$ with an $o(1)$ error. By (5.3) we have

$$t_2 - s_2 = \int_0^{a/2} \frac{dw}{\sqrt{w^2 - 2G(w) - 2H(w,s) + 2E(s)}}, \tag{12.2}$$

where $w = w(s)$. We manipulate the right hand side as in (6.20-21), writing

$$p^2 = w^2 - 2G(w) - 2H(w,s) + 2E(s), \tag{12.3}$$

$$q^2 = w^2 - 2G(w) + 2E(s), \tag{12.4}$$

$$u^2 = w^2 + 2E(s), \tag{12.5}$$

$$v^2 = w^2 + 2E(s_2), \tag{12.6}$$

$(p, q, u, v > 0)$ so that (12.2) is equivalent to

$$t_2 - s_2 = \int_0^{a/2} \frac{dw}{p}. \tag{12.7}$$

We proceed to estimate the effect of these successive changes of integrand. In what follows all integrals will be taken over $(0, a/2)$ except where otherwise indicated.

To begin with we observe that

$$\int \Big(\frac{1}{p} - \frac{1}{q}\Big)\, dw = o(1) \quad \text{as} \quad \gamma \to \infty. \tag{12.8}$$

Indeed, as in (6.20), the left hand side equals

$$\int \frac{2H(w,s)\, dw}{pq(p+q)}. \tag{12.9}$$

Using (2.18) and (6.3-5), we see that this integral is of order

$$e^{-ms_2} \int \frac{w^2\, dw}{(w^2 + E(s_1))^{3/2}} = O(e^{-ms_2} |\log E(s_1)|), \tag{12.10}$$

which verifies (12.8) in view of (9.2) and (10.24).

Next we claim that

$$\int \Big(\frac{1}{q} - \frac{1}{u}\Big)\, dw = I_1 + o(1), \tag{12.11}$$

where I_1 is given by (6.18). To prove this we first write the left hand side of (12.11) as

$$\int \frac{2G(w)\, dw}{qu(q+u)}$$

and then let γ tend to infinity. It follows from (10.24) that $s_2(\gamma) \to \infty$ and hence, because $s \geq s_2$, that the term $E(s_2)$ in the expressions for q and u tends to zero. Thus we end up in the limit with the integral

$$\int \frac{2G(w)\, dw}{w\sqrt{w^2 - 2(G(w)}(\sqrt{w^2 - 2G(w)} + w)},$$

which, using (6.20) once again, can be seen to be equal to I_1.

The next step is to show that

$$\int \Big(\frac{1}{u} - \frac{1}{v}\Big)\, dw = o(1), \tag{12.12}$$

or, what is effectively the same

$$\int \frac{E(s_2) - E(s)}{(w^2 + E(s_1))^{3/2}}\, dw = o(1). \tag{12.13}$$

We choose $\varepsilon > 0$ and break up this integral into integrals over the sub–intervals

$$(0,\, w(s_1 - R)) \quad \text{and} \quad (w(s_1 - R), a/2),$$

where R has been chosen as in Lemma 13. On the first subinterval we have then by Lemmas 11 and 13 that

$$0 < E(s_2) - E(s) < C\varepsilon E(s_1),$$

where C is some positive constant. Hence the contribution of this subinterval is bounded by

$$C\epsilon E(s_1)\int \frac{dw}{(w^2+E(s_1))^{3/2}} < C\varepsilon.$$

As to the second subinterval, recall that $w(s_1-R) \to W(-R)$ as $\gamma \to \infty$ and so, since $w' > 0$, w is bounded away from zero here. Hence, because $E(s_2) \to 0$ as $\gamma \to \infty$ the contribution of this subinterval tends to zero. This proves (12.12).

The final step consists of calculating the last integral. We find, using (10.20),

$$\begin{aligned}\int \frac{dw}{v} &= \sinh^{-1}\Big(\frac{a}{2}\frac{1}{\sqrt{2E(s_2)}}\Big) = \log\Big(\frac{a}{\sqrt{2E(s_2)}}\Big) + O(E(s_2)) \\ &= \log a - \frac{1}{2}\log(2M) + \frac{1}{2}ms_1 + o(1). \qquad (12.14)\end{aligned}$$

Assembling these results, we have

$$t_2 - s_2 = I_1 + \log a - \frac{1}{2}\log(2M) + \frac{1}{2}ms_1 + o(1). \qquad (12.15)$$

or, by (10.4),

$$s_2 = (1-\frac{1}{2}m)s_1 - (I_1+T) - \log a + \frac{1}{2}\log(2M) - \frac{1}{2}ms_1 + o(1). \qquad (12.16)$$

Remembering the definitions (6.18), (7.11) and (3.13) of respectively I_1, T and D, we find that

$$I_1 + T = D + \log 2$$

and so we conclude that

$$s_2 = (1-\frac{1}{2}m)s_1 - (\log a + D) + \frac{1}{2}\log\frac{M}{2} + o(1) \quad \text{as} \quad \gamma \to \infty,$$

which is what we set out to prove.

Thus, explicitly, the greatest zero of $w(s)$ has the asymptotic form

$$s_2(\gamma) = (1-\frac{1}{2}m)\log\gamma - (2-\frac{1}{2}m)(\log a + D) + \frac{1}{2}\log\frac{M}{2} + o(1), \qquad (12.17)$$

or, in the notation of (3.15-16),

$$\begin{aligned} s_2 &= (1-\frac{1}{2}m)(\log\gamma + P) + \frac{1}{2}Q + o(1) \qquad (12.18)\\ &= (1-\frac{1}{2}m)s_1 + \frac{1}{2}Q + o(1), \qquad (12.19)\end{aligned}$$

in accordance with part (a) of Theorem 2.

13. The interval (s_3, s_1): approximation to s_3. We now complete the discussion of the interval (s_3, s_2), still assuming that $0 < m < 1$, by finding an $o(1)$ approximation to s_3. The argument follows closely the lines of Section 12. We first state the main result.

Lemma 15. *Suppose that* $0 < m < 1$. *Then,*

$$s_3(\gamma) = (1-m)s_1(\gamma) - 2(\log a + D) + \log \frac{M}{2} + o(1) \quad \text{as} \quad \gamma \to \infty. \tag{13.1}$$

Proof: In view of (10.28) it is sufficient to obtain an $o(1)$ approximation to t_3. We have by (5.3),

$$s_2 - t_3 = \int_{-a/2}^{0} \frac{dw}{\sqrt{w^2 - 2G(w) - 2H(w,s) + 2E(s)}} \tag{13.2}$$

and we claim that this integral can be estimated as in the previous section, to give precisely the same estimate as was obtained there for $t_2 - s_2$, namely (12.15). Thus we assert that

$$s_2 - t_3 = I_1 + \log a - \frac{1}{2}\log(2M) + \frac{1}{2}ms_1 + o(1). \tag{13.3}$$

To verify this, we review the steps taken in Section 12 to prove (12.15).

In the first step, (12.8-10), we now use the fact that

$$H(w,s) = O(e^{-m_3 s}),$$

and deduce from (10.3) and (10.31) that

$$e^{-ms_3} = O\big((E(s_1))^{1-m}\big),$$

so that

$$e^{-ms_3}\log E(s_1) = o(1).$$

The second step, (12.11), is unaffected: we still have $E(s) \to 0$ as $\gamma \to \infty$ because $s \geq s_3$ and $s_3 \to \infty$ by (10.31).

In the third step, (12.12-13), the integral (12.13) is now broken up into integrals over the subintervals

$$(-a/2, w(s_3 + R)) \quad \text{and} \quad (w(s_3 + R), 0). \tag{13.4}$$

The contribution of the first subinterval tends to 0 as $\gamma \to \infty$ for any fixed positive R, since $s_3 \to \infty$, while for the second subinterval we use Lemma 13 as before.

The computation in the last step now yields

$$\int_{-a/2}^{0} \frac{dw}{v} = -\sinh^{-1}\Big(-\frac{a}{2}\frac{1}{\sqrt{2E(s_2)}}\Big) = \sinh^{-1}\Big(\frac{a}{2}\frac{1}{\sqrt{2E(s_2)}}\Big)$$

which results in (12.14) again. We conclude that

$$s_2(\gamma) - t_3(\gamma) = t_2(\gamma) - s_2(\gamma) + o(1) \quad \text{as} \quad \gamma \to \infty$$

or, remembering that $t_3 = s_3 + T + o(1)$ by (10.28) and $t_2 = s_1 - T + o(1)$ by Lemma 10,

$$s_2(\gamma) - s_3(\gamma) = s_1(\gamma) - s_2(\gamma) + o(1) \quad \text{as} \quad \gamma \to \infty \tag{13.5}$$

which is equivalent to (13.1).

This result may be put explicitly rather than in recurrence form. Then it becomes

$$s_3(\gamma) = (1-m)\log\gamma - (3-m)(\log a + D) + \log\frac{M}{2} + o(1) \quad \text{as} \quad \gamma \to \infty. \qquad (13.6)$$

We have thus established that s_3 has the properties (10.1-2), except for a change of sign in (10.2). From (10.2) we deduce, as in Section 8, that

$$w(s_3 + t) = -W(t) + o(1) \quad \text{as} \quad \gamma \to \infty, \qquad (13.7)$$

uniformly on bounded t-intervals.

It remains to discuss the analogue of (10.3). Since $s_3 \to \infty$, we have, as in Section 9,

$$E(s_3) = -\int_{s_3}^{\infty} H_2(w(s), s)\,ds \sim me^{-ms_3}\int_0^{\infty} e^{-mt}H_0(w(s_3+t))\,dt \text{ as } \gamma \to \infty \quad (13.8)$$

and so, using (13.7),

$$E(s_3) \sim M_2 e^{-ms_3} \quad \text{as} \quad \gamma \to \infty, \qquad (13.9)$$

which is the required property.

14. The recurrence relations. The analogues of (10.1-3) having now been established for s_3, we can repeat the process for the intervals (s_5, s_3), and so on. We can thus generalise (13.1) to the recurrence relation (3.5) for the zeros of w' of Theorem 1; for this purpose we must have $0 < m < 1$. The initial value of (3.4) for s_1 was obtained in (8.14), for general $m > 0$. This completes the proof of Theorem 1.

If $0 < m < 1$ we derive estimates for the zeros s_{2n+2} of w from the zeros s_{2n+1} of w', using as a pattern the case $n = 0$, that is to say (12.19); the proof for $n > 0$ is the same as that for $n = 0$. Thus

$$s_{2n+2}(\gamma) = (1 - \frac{1}{2}m)s_{2n+1}(\gamma) + \frac{1}{2}Q + o(1) \quad \text{as} \quad \gamma \to \infty. \qquad (14.1)$$

and so we can obtain $s_2, s_4, \ldots$ to show that the same recurrence relation applies to the $s_1, s_3, \ldots$, as stated in (3.7).

15. The first zero $s_2(\gamma)$ for $1 \le m < 2$. In this section, we prove Part (a) of Theorem 2 for $1 \le m < 2$, the region $0 < m < 1$ having been dealt with in Lemma 14.

In addition, we shall show that if $m > 1$, then $|w(s)| \to 0$ as $\gamma \to \infty$ uniformly on $(-\infty, s_2)$. Thus, if $m > 1$ and γ is large, only t_1 and t_2 exist, whilst if $m < 1$, t_n exists up to any given n, provided γ is chosen large enough.

Lemma 16. *Suppose $0 < m < 2$. Then for large values of γ,*

$$s_2(\gamma) > (1 - \frac{1}{2}m)\log\gamma - A_1, \tag{15.1}$$

where A_1 is some constant.

Proof: We know from Lemma 8 that for any $m > 0$, t_2 exists, provided that γ is large enough. Proceeding as in the proof of Lemma 12, we now consider the solution in a rectangle.

$$S \le s \le t_2, \quad 0 \le w \le \frac{1}{2}a, \tag{15.2}$$

where, in place of (10.8),

$$S = (1 - \frac{1}{2}m)\log\gamma - A_1, \tag{15.3}$$

and A_1 is chosen suitably large. Supposing that the trajectory does not reach the lower side of the rectangle one arrives at a contradiction.

On the interval $[s_2, s_1)$ we now have, as in Lemma 13,

Lemma 17. *Suppose $0 < m < 2$. For any $\epsilon > 0$ there exists an $R > 0$ such that for*

$$s_2 \le s < s_1 - R, \tag{15.4}$$

and for sufficiently large γ, we have

$$|e^{ms_1}E(s) - M| < \epsilon. \tag{15.5}$$

Because by Lemma 16, $s_2(\gamma) \to \infty$ as $\gamma \to \infty$, the proof of Lemma 17 is the same as that of Lemma 13.

Finally, because the argument in Lemma 12 is unaffected once Lemma 17 is established, we may conclude that Lemma 15 continues to hold for values of $m \in [1, 2)$.

Lemma 18. *Suppose $0 < m < 2$. Then,*

$$s_2(\gamma) = (1 - \frac{1}{2}m)s_1(\gamma) - (\log a + D) + \log\frac{M}{2} + o(1) \quad \textit{as} \quad \gamma \to \infty, \tag{15.6}$$

or

$$s_2(\gamma) = (1 - \frac{1}{2}m)\log\gamma - (2 - \frac{1}{2}m)(\log a + D) + \log\frac{M}{2} + o(1) \quad \textit{as} \quad \gamma \to \infty. \tag{15.7}$$

In contrast to the proof of Lemma 15, the proof of Lemma 18 cannot be repeated to give estimates for $s_4, s_6, \ldots$ and so on. To do that one needs to be sure that the sequence $\{t_n\}$ does not break off. In the next two lemmas we shall show, by way of example, that if $m \in (1, 2)$ and

$$h(w, s) = e^{-ms}w, \tag{15.8}$$

then even t_3 does not exist if γ is large enough.

We begin with a corrollary of Lemma 17.

Lemma 19. *Suppose* $0 < m < 2$. *Then,*

$$w'(s_2) = \sqrt{2M} a^{m/2} e^{mD/2} \gamma^{-m/2} [1 + o(1)] \quad \textit{as} \quad \gamma \to \infty. \tag{15.9}$$

Proof: Since $E(s_2) = w'^2(s_2)/2$, (15.8) follows at once from (15.5) and (8.14).

Lemma 20. *Suppose* $1 < m < 2$. *Then*

$$|w(s_3(\gamma))| = O(\gamma^{1-m}) \quad \textit{as} \quad \gamma \to \infty.$$

Proof: Observe that on (s_4, s_2)

$$\{e^{-2s}(e^s w)'\}' = e^{-s}(w'' - w) = -e^{-s}\{g(w) + h(w,s)\} > 0, \tag{15.10}$$

because $w < 0$ on s_4, s_2. Integration of (15.10) from $s \in (s_4, s_2)$ to s_2 yields

$$(e^s w(s))' < w'(s_2) e^{2s - s_2}$$

and so, after another integration over (s_1, s_2), with $s \in (s_4, s_2)$ we obtain

$$|w(s)| < \frac{1}{2} w'(s_2) e^{s_2 - s}.$$

This implies, by Lemmas 18 and 19, that

$$|w(s)| < C\gamma^{1-m} e^{-s} \quad \text{for} \quad s_4 < s < s_2, \tag{15.11}$$

where C is some positive constant. Hence, on any interval (σ, s_2), provided $\sigma \geq s_4$,

$$|w(s)| = O(\gamma^{1-m}) \quad \text{as} \quad \gamma \to \infty. \tag{15.12}$$

Thus, it only remains to exclude the possibility that $s_3(\gamma) \to -\infty$ as $\gamma \to \infty$. To do this we return to the equation and write it as

$$w'' + w\big(-1 + w^{-1} g(w) + e^{-ms}\big) = 0.$$

By (15.11), $w^{-1} g(w) \to 0$ as $\gamma \to \infty$ on (σ, s_2) and so, by the oscillation properties of the resulting equation, w cannot have one sign on arbitrary long intervals $(\sigma, 0)$. This means that $s_4(\gamma)$ and hence also $s_3(\gamma)$ is bounded below as $\gamma \to \infty$.

REFERENCES

[1] F.V. Atkinson, *On linear perturbations of nonlinear differential equations*, Canad. J. Math., 6 (1964), 561-571.

[2] F.V. Atkinson, H. Brezis, and L.A. Peletier, *Solutions qui changent de signe d'équations elliptiques avec exposant de Sobolev critique*, C.R. Acad. Sci. Paris, Série II, 306 (1988), 711-714.

[3] F.V. Atkinson, H. Brezis, and L.A. Peletier, *Nodal solutions of elliptic equations with critical Sobolev exponents*, to appear in J. Diff. Equations.

[4] F.V. Atkinson and L.A. Peletier, *Large solutions of elliptic equations involving critical exponents*, Asymptotic Anal., 1 (1988), 139-160.

[5] F.V. Atkinson and L.A. Peletier, *Emden-Fowler equations involving critical exponents*, Nonlinear Anal. TMA, 10 (1986), 755-776.

[6] H. Brezis, *Some variational problems with lack of compactness*, Proc. Symp. Pure Math., 45 (1986), 165-201.
[7] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), 434-477.
[8] G. Cerami, S. Solimini, and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. Functional Anal., 69 (1986), 289-306.
[9] H. Egnell, *Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal., 104 (1988), 57-77.
[10] D. Fortunato and E. Janelli, *Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains*, to appear.
[11] C.K.R.T. Jones, *Radial solutions of a semilinear elliptic equation at a critical exponent*, Preprint University of Maryland, MD87-49-CKRT, 1987.
[12] M.C. Knaap and L.A. Peletier, *Quasi-linear elliptic equations with nearly critical growth*, to appear in Comm. PDE.
[13] S. Solimini, *On the existence of infitely many radial solutions for some elliptic problems*, ISAS, 93/85/MC, Trieste 1985.