Differential and Integral Equations, Volume 4, Number 1, January 1991, pp. 205–213.

INTEGRAL AVERAGES AND OSCILLATION OF SECOND ORDER SUBLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The sublinear differential equation $x''(t) + a(t)f[x(t)] = 0$, $t \geq t_0 > 0$ is considered, in which $a \in C([t_0, \infty))$, $f \in C(R)$ with $yf(y) > 0$ for $y \neq 0$ and $\int_{\pm 0}^{\pm 1} [1/f(y)]\, dy < \infty$, and f has a continuous derivative on $R - \{0\}$ with $f'(y) \geq 0$ for all $y \neq 0$. No sign condition is assumed on a. Two new oscillation criteria are obtained. These criteria involve the average behavior of the integral of the coefficient a.

1. Introduction. Many physical systems are modelled by second order nonlinear ordinary differential equations. For example, the so-called Emden-Fowler equation arises in the study of gas dynamics and fluid mechanics. Also, this equation appears in the study of relativistic mechanics, nuclear physics and in the study of chemically reacting systems. The study of the Emden-Fowler equation originates from earlier theories concerning gaseous dynamics in astrophysics around the turn of the century. For a discussion on the Emden-Fowler equation, we refer to Wong [20].

Consider the second order nonlinear ordinary differential equation

$$x''(t) + a(t)f[x(t)] = 0, \tag{E}$$

where a is a continuous function on the interval $[t_0, \infty)$, $t_0 > 0$, and f is a continuous function on the real line $\mathbb{R}$. It will be supposed that f has a continuous derivative on $\mathbb{R} - \{0\}$ and satisfies

$$yf(y) > 0 \quad \text{and} \quad f'(y) \geq 0 \quad \text{for all} \quad y \neq 0.$$

Moreover, we are interested in the case where (E) is strongly sublinear in the sense that

$$\int_{+0} \frac{dy}{f(y)} < \infty \quad \text{and} \quad \int_{-0} \frac{dy}{f(y)} < \infty.$$

Note that no assumption on the sign of the coefficient a is made.

We restrict our attention to solutions of (E) which exist on some ray $[T_0, \infty)$, where $T_0 \geq t_0$ may depend on the particular solution. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*. Equation (E) is called oscillatory if all its solutions are oscillatory.

Received July 10, 1988, in revised March 1, 1989.
AMS Subject Classifications: 34C10.

For our purposes, we define

$$F(y) = \int_{+0}^{y} \frac{dz}{f(z)} \quad \text{for } y > 0, \quad F(y) = \int_{-0}^{y} \frac{dz}{f(z)} \quad \text{for } y < 0.$$

Also, we introduce the constant λ defined by

$$\lambda = \min\left\{ \frac{\inf_{y>0} F(y)f'(y)}{1 + \inf_{y>0} F(y)f'(y)}, \frac{\inf_{y<0} F(y)f'(y)}{1 + \inf_{y<0} F(y)f'(y)} \right\}, \quad (0 \le \lambda < 1).$$

The prototype of (E) is the following

$$x''(t) + a(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \quad (0 < \gamma < 1), \tag{E_0}$$

which is of particular interest. In this special case we can easily see that the constant λ is the exponent γ.

Of particular interest is the problem of finding criteria for the oscillation of the differential equation (E) when a is allowed to take on negative values for arbitrary large t. Many criteria have been found which involve the average behavior of the integral of the alternating coefficient and which are motivated by the classical averaging criterion of Wintner [17] for the linear case. For such averaging results on second order sublinear oscillation we choose to refer to the papers by Butler [1], Butler and Erbe [2], Chan and Chen [3, 4], Kamenev [5, 6], Kura [8], Kwong and Wong [9, 10], Onose [11], the author [12-16], Wong [18-22] and Yan [23]. The purpose of this paper is to proceed further in this direction to present two new oscillation criteria which include as particular cases some previous theorems. These criteria are based on the idea of the use of the n-th primitive of the function $\varphi^{\lambda}a$, where n is an integer with $n \ge 2$ and φ is a positive increasing concave function on $[t_0, \infty)$. By this idea, Kamenev [7] proved the well-known oscillation criterion for the linear case (which extends Wintner's result).

2. Main results.

In this section, we will establish two oscillation criteria for the differential equation (E).

Theorem 1. *Let n be an integer with $n \ge 2$ and φ be a positive and twice continuously differentiable function on the interval $[t_0, \infty)$ with*

$$\varphi' \ge 0 \text{ and } \varphi'' \le 0 \text{ on } [t_0, \infty).$$

Equation (E) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} [\varphi(s)]^{\lambda} a(s)\, ds = \infty. \tag{I}$$

Proof: Assume, for the sake of contradiction, that there exists a nonoscillatory solution x on an interval $[T_0, \infty)$, $T_0 \ge t_0$, of the differential equation (E). Without loss of generality, we can suppose that $x(t) \ne 0$ for all $t \ge T_0$. Furthermore, we define

$$w(t) = [\varphi(t)]^{\lambda} F[x(t)], \qquad t \ge T_0$$

and we obtain for $t \geq T_0$

$$w'(t) = \lambda[\varphi(t)]^{\lambda-1}\varphi'(t)F[x(t)] + [\varphi(t)]^{\lambda}\frac{x'(t)}{f[x(t)]} = \lambda\frac{\varphi'(t)}{\varphi(t)}w(t) + [\varphi(t)]^{\lambda}\frac{x'(t)}{f[x(t)]}\,.$$

Therefore, for every $t \geq T_0$, we get

$$\begin{aligned} w''(t) &= \lambda\Big\{\frac{\varphi''(t)}{\varphi(t)} - \Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^2\Big\}w(t) + \lambda\frac{\varphi'(t)}{\varphi(t)}w'(t) \\ &\quad + \lambda[\varphi(t)]^{\lambda-1}\varphi'(t)\frac{x'(t)}{f[x(t)]} + [\varphi(t)]^{\lambda}\Big\{\frac{x''(t)}{f[x(t)]} - \Big[\frac{x'(t)}{f[x(t)]}\Big]^2 f'[x(t)]\Big\} \\ &= [\varphi(t)]^{\lambda}\frac{x''(t)}{f[x(t)]} + \lambda\Big\{\frac{\varphi''(t)}{\varphi(t)} - \Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^2\Big\}w(t) + \lambda\frac{\varphi'(t)}{\varphi(t)}w'(t) \\ &\quad + \lambda\frac{\varphi'(t)}{\varphi(t)}\Big[w'(t) - \lambda\frac{\varphi'(t)}{\varphi(t)}w(t)\Big] - \frac{1}{w(t)}\Big[w'(t) - \lambda\frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^2 F[x(t)]f'[x(t)] \\ &\leq [\varphi(t)]^{\lambda}\frac{x''(t)}{f[x(t)]} + \lambda\Big\{\frac{\varphi''(t)}{\varphi(t)} - (1+\lambda)\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^2\Big\}w(t) + 2\lambda\frac{\varphi'(t)}{\varphi(t)}w'(t) \\ &\quad - \frac{\lambda}{1-\lambda}\frac{1}{w(t)}\Big[w'(t) - \lambda\frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^2 \\ &= [\varphi(t)]^{\lambda}\frac{x''(t)}{f[x(t)]} + \lambda\frac{\varphi''(t)}{\varphi(t)}w(t) - \frac{\lambda}{1-\lambda}\frac{1}{w(t)}\Big[w'(t) - \frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^2. \end{aligned}$$

Hence, from (E) it follows that

$$w''(t) \leq -[\varphi(t)]^{\lambda}a(t) + \lambda\frac{\varphi''(t)}{\varphi(t)}w(t) - \frac{\lambda}{1-\lambda}\frac{1}{w(t)}\Big[w'(t) - \frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^2 \tag{1}$$

for all $t \geq T_0$. This gives

$$[\varphi(t)]^{\lambda}a(t) \leq -w''(t), \quad t \geq T_0$$

and consequently for $t \geq T_0$

$$\begin{aligned} \int_{T_0}^{t}(t-s)^{n-1}[\varphi(s)]^{\lambda}a(s)\,ds &\leq -\int_{T_0}^{t}(t-s)^{n-1}w''(s)\,ds \\ &= (t-T_0)^{n-1}w'(T_0) - (n-1)\int_{T_0}^{t}(t-s)^{n-2}w'(s)\,ds \\ &= \begin{cases} (t-T_0)w'(T_0) - w(t) + w(T_0), & \text{if } n=2 \\ (t-T_0)^{n-1}w'(T_0) - (n-1)(n-2)\int_{T_0}^{t}(t-s)^{n-3}w(s)\,ds & \\ \quad +(n-1)(t-T_0)^{n-2}w(T_0), & \text{if } n>2 \end{cases} \\ &\leq (t-T_0)^{n-1}w'(T_0) + (n-1)(t-T_0)^{n-2}w(T_0). \end{aligned}$$

Thus

$$\frac{1}{t^{n-1}}\int_{T_0}^{t}(t-s)^{n-1}[\varphi(s)]^{\lambda}a(s)\,ds \leq \Big(1-\frac{T_0}{t}\Big)^{n-1}w'(T_0) + (n-1)\Big(1-\frac{T_0}{t}\Big)^{n-2}\frac{w(T_0)}{t}$$

for $t \geq T_0$, which gives

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{T_0}^{t} (t-s)^{n-1}[\varphi(s)]^{\lambda} a(s)\, ds \leq w'(T_0).$$

But, for $t \geq T_0$

$$\begin{aligned}
&\frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1}[\varphi(s)]^{\lambda} a(s)\, ds \\
&\quad \leq \frac{1}{t^{n-1}} \int_{t_0}^{T_0} (t-s)^{n-1}[\varphi(s)]^{\lambda} |a(s)|\, ds + \frac{1}{t^{n-1}} \int_{T_0}^{t} (t-s)^{n-1}[\varphi(s)]^{\lambda} a(s)\, ds \\
&\quad \leq \int_{t_0}^{T_0} [\varphi(s)]^{\lambda} |a(s)|\, ds + \frac{1}{t^{n-1}} \int_{T_0}^{t} (t-s)^{n-1}[\varphi(s)]^{\lambda} a(s)\, ds
\end{aligned}$$

and so

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1}[\varphi(s)]^{\lambda} a(s)\, ds \leq \int_{t_0}^{T_0} [\varphi(s)]^{\lambda} |a(s)|\, ds + w'(T_0) < \infty,$$

which contradicts condition (I).

Corollary 1. *Let n be an integer with $n \geq 2$ and let $\beta \in [0, \lambda]$. Equation (E) is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} s^{\beta} a(s)\, ds = \infty.$$

Proof: It suffices to apply Theorem 1 with

$$\varphi(t) = 1, \quad t \geq t_0 \text{ if } \lambda = 0, \text{ and } \varphi(t) = t^{\beta/\lambda}, \quad t \geq t_0 \text{ if } \lambda > 0.$$

Remark 1. By applying Theorem 1 with $n = 2$, we obtain a previous result due to the author [14, Theorem 1]. Note that Theorem 1 in [14] extends previous oscillation criteria given by Kamenev [5], Kura [8] and the author [13]. Moreover, Theorem 1 has been recently proved by Wong [22] for the special case of the differential equation (E_0). But this Wong's result can be obtained from the main result given by the author in [12] (by choosing $\rho = \varphi^{\gamma}$). Also, note that, for the particular case of the equation (E_0), Corollary 1 has previously given by Yan [23] and the author [12].

Theorem 2. *Suppose that $\lambda > 0$. Let n be an integer with $n \geq 2$ and φ be a positive and twice continuously differentiable function on the interval $[t_0, \infty)$ such that*

$$(\varphi')^2 \leq c\varphi(-\varphi'') \text{ on } [t_0, \infty),$$

where c is a positive constant.

Equation (E) is oscillatory if there exists a continuous function A on $[t_0, \infty)$ with

$$\int_{t_0}^{\infty} \frac{[A_+(T)]^2}{T}\, dT = \infty, \tag{II}$$

where $A_+(T) = \max\{A(T), 0\}$, $T \geq t_0$, *and such that*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1}[\varphi(s)]^\lambda a(s)\,ds \geq A(T) \text{ for every } T \geq t_0. \tag{III}$$

Proof: Let x be a solution on an interval $[T_0, \infty)$, $T_0 \geq t_0$, of the differential equation (E) with $x(t) \neq 0$ for all $t \geq T_0$. Moreover, let $w(t) = [\varphi(t)]^\lambda F[x(t)]$, $t \geq T_0$. Then, as in the proof of Theorem 1, (1) holds for every $t \geq T_0$. Thus, for any t, T with $t \geq T \geq T_0$, we obtain

$$\begin{aligned}
&\int_T^t (t-s)^{n-1}[\varphi(s)]^\lambda a(s)\,ds + \lambda \int_T^t (t-s)^{n-1} \frac{-\varphi''(s)}{\varphi(s)} w(s)\,ds \\
&\qquad + \frac{\lambda}{1+\lambda} \int_T^t (t-s)^{n-1} \frac{1}{w(s)} \left[w'(s) - \frac{\varphi'(s)}{\varphi(s)} w(s)\right]^2 ds \\
&\leq -\int_T^t (t-s)^{n-1} w''(s)\,ds = (t-T)^{n-1} w'(T) - (n-1)\int_T^t (t-s)^{n-2} w'(s)\,ds \\
&= \begin{cases} (t-T)w'(T) - w(t) + w(T), & \text{if } n = 2 \\ (t-T)^{n-1}w'(T) - (n-1)(n-2)\int_T^t (t-s)^{n-3} w(s)\,ds & \\ \qquad +(n-1)(t-T)^{n-2} w(T), & \text{if } n > 2. \end{cases}
\end{aligned}$$

Hence, we have for $t \geq T \geq T_0$

$$\begin{aligned}
&\frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1}[\varphi(s)]^\lambda a(s)\,ds + \lambda \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1} \frac{-\varphi''(s)}{\varphi(s)} w(s)\,ds \\
&\qquad + \frac{\lambda}{1-\lambda} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1} \frac{1}{w(s)} \left[w'(s) - \frac{\varphi'(s)}{\varphi(s)} w(s)\right]^2 ds \\
&\leq \begin{cases} (1-\frac{T}{t})w'(T) - \frac{w(t)}{t} + \frac{w(T)}{t}, & \text{if } n = 2 \\ (1-\frac{T}{t})^{n-1} w'(T) - (n-1)(n-2)\frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3} w(s)\,ds & \\ \qquad +(n-1)(1-\frac{T}{t})^{n-2} \frac{w(T)}{t}, & \text{if } n > 2. \end{cases}
\end{aligned}$$

So, for every $T \geq T_0$

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1}[\varphi(s)]^\lambda a(s)\,ds + \lambda \int_T^\infty \frac{-\varphi''(s)}{\varphi(s)} w(s)\,ds \\
&\qquad + \frac{\lambda}{1-\lambda} \int_T^\infty \frac{1}{w(s)} [w'(s) - \frac{\varphi'(s)}{\varphi(s)} w(s)]^2\,ds \\
&\leq \begin{cases} w'(T) - \liminf_{t\to\infty} \frac{w(t)}{t}, & \text{if } n = 2 \\ w'(T) - (n-1)(n-2) \liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3} w(s)\,ds, & \text{if } n > 2 \end{cases}
\end{aligned}$$

and consequently, by condition (III), we get

$$\begin{aligned}
w'(T) &\geq A(T) + \lambda \int_T^\infty \frac{-\varphi''(s)}{\varphi(s)} w(s)\,ds + \frac{\lambda}{1-\lambda} \int_T^\infty \frac{1}{w(s)} \left[w'(s) - \frac{\varphi'(s)}{\varphi(s)} w(s)\right]^2 ds \\
&\quad + \begin{cases} \liminf_{t\to\infty} \frac{w(t)}{t}, & \text{if } n = 2 \\ (n-1)(n-2) \liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3} w(s)\,ds, & \text{if } n > 2 \end{cases}
\end{aligned}$$

for all $T \geq T_0$. This shows that

$$w'(T) \geq A(T) \quad \text{for every } T \geq T_0, \tag{2}$$

$$\int_{T_0}^{\infty} \frac{-\varphi''(s)}{\varphi(s)} w(s)\, ds < \infty, \tag{3}$$

$$\int_{T_0}^{\infty} \frac{1}{w(s)} \left[w'(s) - \frac{\varphi'(s)}{\varphi(s)} w(s) \right]^2 ds < \infty \tag{4}$$

and

$$\begin{cases} \liminf_{t\to\infty} \frac{w(t)}{t} < \infty, & \text{if } n = 2 \\ \liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{T_0}^{t} (t-s)^{n-3} w(s)\, ds < \infty, & \text{if } n > 2. \end{cases} \tag{5}$$

Now, we will establish that (5) implies

$$\liminf_{t\to\infty} \frac{w(t)}{t} < \infty \tag{6}$$

in both cases where $n = 2$ or $n > 2$. Clearly, this is true for $n = 2$. So, we consider the case where $n > 2$. Suppose that (6) fails; i.e.,

$$\lim_{t\to\infty} \frac{w(t)}{t} = \infty.$$

Let ρ be a number with $0 < \rho < 1/(n-1)(n-2)$. Moreover, let us consider an arbitrary positive constant μ. Then there exists a $T_1 \geq T_0$ such that

$$\frac{w(t)}{t} \geq \frac{\mu}{\rho} \quad \text{for all } t \geq T_1.$$

Thus, we obtain for $t \geq T_1$

$$\begin{aligned} \frac{1}{t^{n-1}} \int_{T_0}^{t} (t-s)^{n-3} w(s)\, ds &\geq \frac{1}{t^{n-1}} \int_{T_1}^{t} (t-s)^{n-3} w(s)\, ds \\ &\geq \frac{\mu}{\rho} \frac{1}{t^{n-1}} \int_{T_1}^{t} (t-s)^{n-3} s\, ds = \frac{\mu}{\rho} \frac{1}{t^{n-1}} \int_{0}^{t-T_1} r^{n-3}(t-r)\, dr \\ &= \frac{\mu}{\rho} \left[\frac{1}{n-2} \left(1 - \frac{T_1}{t}\right)^{n-2} - \frac{1}{n-1} \left(1 - \frac{T_1}{t}\right)^{n-1} \right]. \end{aligned}$$

Since

$$\lim_{t\to\infty} \left[\frac{1}{n-2} \left(1 - \frac{T_1}{t}\right)^{n-2} - \frac{1}{n-1} \left(1 - \frac{T_1}{t}\right)^{n-1} \right] = \frac{1}{(n-2)(n-1)} > \rho,$$

we can choose a $T_2 \geq T$ so that

$$\frac{1}{n-2} \left(1 - \frac{T_1}{t}\right)^{n-2} - \frac{1}{n-1} \left(1 - \frac{T_1}{t}\right)^{n-1} \geq \rho, \quad t \geq T_2.$$

So, we have

$$\frac{1}{t^{n-1}}\int_{T_0}^{t}(t-s)^{n-3}w(s)\,ds \geq \mu \quad \text{for every } t \geq T_2,$$

which gives

$$\lim_{t\to\infty}\frac{1}{t^{n-1}}\int_{T_0}^{t}(t-s)^{n-3}w(s)\,ds = \infty,$$

since $\mu > 0$ is arbitrary. This contradicts (5). Thus, (6) has been established.

The remainder of the proof proceeds as exactly in the proof of Theorem 1 in [15]. So, we can easily see that

$$\limsup_{t\to\infty}\frac{t\varphi'(t)}{\varphi(t)} < \infty. \tag{7}$$

Also, for every $t \geq T_0$

$$\begin{aligned}
&\int_{T_0}^{t}\frac{1}{w(s)}\big[w'(s)-\frac{\varphi'(s)}{\varphi(s)}w(s)\big]^2\,ds \\
&\quad = \int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,ds - 2\int_{T_0}^{t}\frac{\varphi'(s)}{\varphi(s)}w'(s)\,ds + \int_{T_0}^{t}\big[\frac{\varphi'(s)}{\varphi(s)}\big]^2 w(s)\,ds \\
&\quad = \int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,ds - 2\frac{\varphi'(t)}{\varphi(t)}w(t) + 2\frac{\varphi'(T_0)}{\varphi(T_0)}w(T_0) \\
&\qquad - 2\int_{T_0}^{t}\frac{-\varphi''(s)}{\varphi(s)}w(s)\,ds - \int_{T_0}^{t}\big[\frac{\varphi'(s)}{\varphi(s)}\big]^2 w(s)\,ds \\
&\quad \geq \int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,ds - 2\frac{\varphi'(t)}{\varphi(t)}w(t) + 2\frac{\varphi'(T_0)}{\varphi(T_0)}w(T_0) - (c+2)\int_{T_0}^{t}\frac{-\varphi''(s)}{\varphi(s)}w(s)\,ds
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_{T_0}^{\infty}\frac{1}{w(s)}\big[w'(s)-\frac{\varphi'(s)}{\varphi(s)}w(s)\big]^2\,ds \\
&\quad \geq \int_{T_0}^{\infty}\frac{[w'(s)]^2}{w(s)}\,ds - 2\big[\limsup_{t\to\infty}\frac{t\varphi'(t)}{\varphi(t)}\big]\big[\liminf_{t\to\infty}\frac{w(t)}{t}\big] \\
&\qquad + 2\frac{\varphi'(T_0)}{\varphi(T_0)}w(T_0) - (c+2)\int_{T_0}^{\infty}\frac{-\varphi''(s)}{\varphi(s)}w(s)\,ds.
\end{aligned}$$

This, because of (3), (4), (6) and (7), ensures that

$$\int_{T_0}^{\infty}\frac{[w'(s)]^2}{w(s)}\,ds < \infty.$$

Next, by the Schwarz inequality, we obtain for $t \geq T_0$

$$\begin{aligned} w(t) &= \left[[w(T_0)]^{1/2} + \left\{[w(t)]^{1/2} - [w(T_0)]^{1/2}\right\}\right]^2 \\ &\leq 2w(T_0) + 2\left\{[w(t)]^{1/2} - [w(T_0)]^{1/2}\right\}^2 \\ &= 2w(T_0) + \tfrac{1}{2}\left\{\int_{T_0}^{t} \frac{w'(s)}{[w(s)]^{1/2}}\, ds\right\}^2 \\ &\leq 2w(T_0) + \tfrac{1}{2}\left(\int_{T_0}^{t} ds\right)\int_{T_0}^{t} \frac{[w'(s)]^2}{w(s)}\, ds \\ &\leq 2w(T_0) + \tfrac{1}{2}t\int_{T_0}^{t} \frac{[w'(s)]^2}{w(s)}\, ds \end{aligned}$$

and hence

$$w(t) \leq Nt \quad \text{for all } t \geq T_0, \tag{9}$$

where

$$N = \frac{2w(T_0)}{T_0} + \frac{1}{2}\int_{T_0}^{\infty} \frac{[w'(s)]^2}{w(s)}\, ds$$

(and, by (8), N is finite). Finally, in view of (2), (8) and (9), we have

$$\int_{T_0}^{\infty} \frac{[A_+(T)]^2}{T}\, dT \leq \int_{T_0}^{\infty} \frac{[w'(T)]^2}{T}\, dT \leq N\int_{T_0}^{\infty} \frac{[w'(T)]^2}{w(T)}\, dT < \infty,$$

which contradicts condition (III).

Corollary 2. *Suppose that $\lambda > 0$. Let n be an integer with $n \geq 2$ and let $\beta \in [0, \lambda)$. Equation (E) is oscillatory if there exists a continuous function A on $[t_0, \infty)$ such that (II) holds and*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{T}^{t} (t-s)^{n-1} s^{\beta} a(s)\, ds \geq A(T) \quad \textit{for every } T \geq t_0.$$

Proof: The corollary follows immediately from Theorem 2 by taking

$$\varphi(t) = t^{\beta/\lambda}, \qquad t \geq t_0.$$

Remark 2. Letting $n = 2$ in Theorem 2, we obtain Theorem 1 of [15]. Moreover, for the special case of the differential equation (E_0), we have $\lambda = \gamma$ and hence the oscillation criterion of Yan [23, Theorem 2] (see also Kura [8, Theorem 2] for the particular case $n = 2$) is a consequence of Corollary 2.

Remark 3. As in [14, 15], we can extend our results to the more general case of the damped differential equation

$$x''(t) + g(t)x'(t) + a(t)f[x(t)] = 0,$$

where g is a continuous function on the interval $[t_0, \infty)$. Also, we can obtain sufficient conditions for all solutions x of the forced equation

$$x''(t) + a(t)f[x(t)] = b(t),$$

where b is continuous on $[t_0, \infty)$, to satisfy $\liminf_{t\to\infty} |x(t)| = 0$.

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