NONLINEAR SCHROEDINGER EQUATIONS
WITH MAGNETIC FIELDS

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Abstract. We study the Cauchy problem for nonlinear Schrödinger equations with magnetic field. Under some growth conditions on the potentials, we show the existence of solutions in $L^2(\mathbb{R}^n)$ and in a weighted Sobolev space $\Sigma$. We also establish the continuous dependence on the initial value, and the conservation of energy when the solution is in $\Sigma$.

1. Introduction. We consider the nonlinear Schrödinger equation in $\mathbb{R}^n$:

$$i \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{j=1}^{n} (-i \partial_j - A_j(x))^2 u + V(x)u + \epsilon |u|^{p-1}u,$$

where $u = u(t, x)$ is a complex-valued function defined on $[-T, T] \times \mathbb{R}^n$ for some $T > 0$. We show the local existence of solutions for the Cauchy problem in the function space

$$\Sigma = \left\{ u \in S'(\mathbb{R}^n), (1 + |x|^2)^{1/2}u \in L^2(\mathbb{R}^n), (I - \Delta)^{1/2}u \in L^2(\mathbb{R}^n) \right\}$$

if $1 \leq p < 1 + \frac{4}{n-2}$, and in $L^2(\mathbb{R}^n)$ if $1 \leq p \leq 1 + \frac{4}{n}$, when the vector potential $A = (A_1(x), \ldots, A_n(x))$ and the scalar potential $V$ are smooth and satisfy the same growth conditions as in [15] (see below for a precise statement). We choose this power nonlinearity for simplicity, but we can allow more general terms, as in [7] or [10].

We denote by $H$ the operator associated with the steady linear equation. Such operators have been studied for example in [4], [12] or [15].

In all the paper, our assumptions will be the following:

H1: We assume that for $j \in \{1, \ldots, n\}$, $A_j(x)$ is real valued, $C^\infty$ on $\mathbb{R}^n$. If $B = (B_{jk})$ with $B_{jk} = \partial_j A_k - \partial_k A_j$, then there exists $\epsilon > 0$ such that

$$|\partial^\alpha B(x)| \leq C_\alpha (1 + |x|)^{-1-\epsilon}, \quad \forall |\alpha| \geq 1, \quad \forall x \in \mathbb{R}^n,$$

$$|\partial^\alpha A(x)| \leq C_\alpha, \forall |\alpha| \geq 1, \quad \forall x \in \mathbb{R}^n.$$  

H2: $V$ is real valued, $C^\infty$ on $\mathbb{R}^n$, $|\partial^\alpha V(x)| \leq C_\alpha$, $\forall |\alpha| \geq 2$; in addition we assume that $V$ is bounded from below; i.e., we can assume that there exists $m > 0$ such that $V(x) \geq m$, $\forall x \in \mathbb{R}^n$.

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The initial value problem for (1.1) has been studied in the space of energy in the case \( A = V = 0 \) in particular by Ginibre and Velo [7, 8] or Kato [10], and in the case \( A = 0 \) by Oh [11]. All their methods are based on some \( L^p \) estimates for the propagator of the linear equation. When \( A \) and \( V \) satisfy H1 and H2, Yajima has proved (see [15]) that near \( t = 0 \), the propagator is an integral operator and its kernel has the asymptotic form

\[
(2\pi i t)^{-n/2}a(t,x,y)e^{i\tilde{S}(t,x,y)}
\]

where \( \tilde{S} \) and \( a \) are smooth functions, \( \tilde{S} \) is real and \( a \) is bounded. This will allow us to obtain the same \( L^p \) estimates as in the usual case, and to solve the Cauchy problem in \( \Sigma \) by using an adaptation of Kato's method [9], since we have \( D(H^{1/2}) = \{ u \in L^2(\mathbb{R}^n), \langle u, Hu \rangle \in L^2(\mathbb{R}^n) \} \subset \Sigma \).

For that purpose, we will need some estimates on \( \nabla u \) and \( xu \). The fact that \( \nabla \) does not commute with \( H \) will be overcome by using an expression of \( \nabla S(t) \) which was derived by Yajima (see [15], Proposition 2.7). However, this method does not seem to be applicable to obtain regular solutions of (1.1). Hence, in order to prove the conservation of the energy

\[
E(u) = \frac{1}{2} \langle u, Hu \rangle + \frac{\epsilon}{p+1} \int |u|^{p+1} dx,
\]

we will have to consider regularized equations, as in [7] or [11]. We introduce the following notation: we set for any integer \( k \)

\[
\Sigma(k) = \{ u \in S'(\mathbb{R}^n), (I - \Delta)^{k/2} u \in L^2(\mathbb{R}^n), (1 + |x|^2)^{k/2} u \in L^2(\mathbb{R}^n) \},
\]

and for \( q \geq 1 \),

\[
\Sigma^{1,q} = \{ u \in L^q(\mathbb{R}^n), \nabla u \in L^q(\mathbb{R}^n), xu \in L^q(\mathbb{R}^n) \}.
\]

\( \Sigma(1) \) will be denoted as \( \Sigma \) and we will use \( L^q \) for \( L^q(\mathbb{R}^n) \). \( D(\mathbb{R}^n) \) is the set of \( \mathbb{R} \)-valued \( C^\infty \) functions with compact support, and \( B(\mathbb{R}^n) \) is the set of \( C^\infty \) functions having bounded derivatives of any order. For an interval \( I \), the norm in \( L^r(I, L^q(\mathbb{R}^n)) \) will be denoted as \( \| \cdot \|_{r,q} \).

Throughout the paper we will assume that \( 1 \leq p < 1 + \frac{4}{n-2} \) except in Sections 6 and 7. We set \( \frac{2}{r} = n(\frac{1}{2} - \frac{1}{p+1}) \) and \( \frac{1}{r} + \frac{1}{r'} = 1 \); hence we have \( r > 2 \).

In Section 2, we collect some useful results and estimates. In Sections 3 and 4, we study the local existence of solutions in \( \Sigma \) and the continuity with respect to the initial data. In Section 5, we establish some conservation laws for this solution. In Section 6, we study the local existence in \( L^2(\mathbb{R}^n) \) and give some global existence results. In Section 7 we give a blow-up result for the nonlinear Schrödinger equation with potential, when there is no magnetic field.

After having completed this paper, we were told of the existence of the work [3] of Cazenave and Esteban where they proved some results which partially overlap with ours. However, they consider only the case of a constant magnetic field \( B \), and they do not use the results of Yajima [15] which are basic for our paper.

2. Some preliminary results. In this section, we recall some properties of the propagator \( \hat{S}(t) \) for the linear equation. These results have been proved in a recent paper by K. Yajima (see [15]). We first have the
Proposition 2.1. Under assumptions H1 and H2, there exists \( \gamma > 0 \) such that for every \( t \) with \( 0 < |t| \leq \gamma \), \( S(t) \) has the form

\[
S(t)v(x) = (2\pi it)^{-n/2} \int_{\mathbb{R}^n} e^{i\tilde{S}(t,x,y)}b(t, x, y)v(y) \, dy,
\]

where \( \tilde{S}(t, x, y) \) is real valued, \( C^1 \) in \((t, x, y)\), \( C^\infty \) in \((x, y)\) for every fixed \( t \), and \( b(t, x, y) \) is \( C^1 \) in \((t, x, y)\), \( C^\infty \) in \((x, y)\) with \( |\partial_\alpha \partial_\beta b(t, x, y)| \leq C_{\alpha, \beta} \) for any \( \alpha \) and \( \beta \).

For the proof of this proposition, see Theorem 1 in [15].

We then introduce the following integral operator: for \( a(t, x, y) \), a \( C(\mathbb{R}^n \times \mathbb{R}^n) \)-valued continuous function of \( t \in I \), we set

\[
I(t, a)v(x) = (2\pi it)^{-n/2} \int_{\mathbb{R}^n} e^{i\tilde{S}(t,x,y)}a(t, x, y)v(y) \, dy.
\]

Lemma 2.2. Let \( I_T = [-T, T] \subset I \); then for any \( q \) with \( 2 \leq q \leq +\infty \) there exists a constant \( C \) depending on \( a \) but not on \( t \in I \) such that for \( v \in L^q(\mathbb{R}^n) \), with \( \frac{1}{q} + \frac{1}{q'} = 1 \),

\[
\|I(t, a)v\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{|t|^{n(1/2-1/q)}} \|v\|_{L^{q'}(\mathbb{R}^n)}.
\]

Moreover, if we set \( \Lambda_a \phi(t) = \int_0^t I(t - \tau, a)\phi(\tau) \, d\tau \), then for every pair \((q, s)\) with \( q \in [2, \frac{2n}{n-2}] \) (\( q \in [2, +\infty]\) if \( n = 2 \) and \( q \in [2, +\infty]\) if \( n = 1 \)) and \( \frac{2}{s} = n(\frac{1}{2} - \frac{1}{q}) \), the following estimates hold:

\[
\begin{align*}
\|\Lambda_a \phi\|_{L^s(I_T, L^q)} &\leq C_3 \|\phi\|_{L^{s'}(I_T, L^{q'})} \quad (2.3) \\
\|\Lambda_a \phi\|_{L^s(I_T, L^2)} &\leq C_4 \|\phi\|_{L^1(I_T, L^2)} \quad (2.4) \\
\|\Lambda_a \phi\|_{L^\infty(I_T, L^2)} &\leq C_5 \|\phi\|_{L^{s'}(I_T, L^{q'})} \quad (2.5) \\
\|I(\cdot, a)v\|_{L^s(I_T, L^2)} &\leq C_6 \|v\|_{L^2}, \quad (2.6)
\end{align*}
\]

where the constants \( C_3 - C_6 \) are independent of \( T \) with \( I_T \subset I \).

Proof: The first part of the lemma is proved in [15], Proposition 3.1, and comes from the \( L^2 \)-boundedness theorem for \( I(t, a) \) (see [1]) and the Riesz-Thorin interpolation theorem. The estimates (2.3)-(2.6) then follow in the same way as for the usual Schrödinger operator (see for example [14]).

Corollary. If we set

\[
\Lambda \phi(t) = \int_0^t S(t - \tau)\phi(\tau) \, d\tau,
\]

then for every \( T \leq \gamma \), (2.3)-(2.6) hold, replacing \( I(\cdot, a) \) by \( S(\cdot) \) and \( \Lambda_a \) by \( \Lambda \).

We also have (see Proposition 2.7 in [15]):
Lemma 2.3. Let \( a(t, \cdot, \cdot) \) be a \( B(\mathbb{R}^n \times \mathbb{R}^n) \)-valued continuous function of \( t \in I \). Then there exist \( B(\mathbb{R}^n \times \mathbb{R}^n) \)-valued continuous functions \( a_{j,k}(t, x, y) \) and \( a_{ij,km}(t, x, y) \) with \( i, j = 1, \ldots, n \) and \( k, m = 1, 2 \) such that

\[
x_i I(t, a) = \sum_{j=1}^n [I(t, a_{ij,11})x_j + I(t, a_{ij,12})\partial_j] + I(t, a_{11}) \quad (2.8)
\]

\[
\partial_i I(t, a) = \sum_{j=1}^n [I(t, a_{ij,21})x_j + I(t, a_{ij,22})\partial_j] + I(t, a_{12}) \quad (2.9)
\]

3. Local existence of solutions in \( \Sigma \). Our result in this section is the following:

Theorem 3.1. Let \( u_0 \in \Sigma \). Then (1.1) has a unique maximal solution \( u \in \mathcal{C}([T^-, T^+] \cap \mathcal{L}^1 \cap \mathcal{T} \cap \Omega \cap \Sigma) \) such that \( u(0) = u_0 \).

We consider the integral form of (1.1):

\[
u(t) = S(t)u_0 - i\varepsilon \int_0^t S(t-\tau)F(u(\tau)) \, d\tau \quad (3.1)
\]

with

\[
F(u) = |u|^{p-1}u, \quad (3.2)
\]

and we set

\[
Tv(t) = S(t)u_0 - i\varepsilon \int_0^t S(t-\tau)F(v(\tau)) \, d\tau. \quad (3.3)
\]

We will use the method of Kato [10], and we introduce the following function spaces:

For \( I = [-T, T] \), \( 0 < T \leq \gamma \), we set

\[
\bar{X} = \mathcal{C}(I, L^2) \cap L^r(I, L^{p+1})
\]

\[
X = L^\infty(I, L^2) \cap L^r(I, L^{p+1})
\]

\[
X' = L^1(I, L^2) + L^r(I, L^{1+1/p})
\]

\[
X_0 = L^\infty(I, L^2 \cap L^{p+1})
\]

The following lemma can be proved exactly in the same way as Lemma 1.3 and its corollary in [10], using estimates (2.3)–(2.5) for \( \Lambda \):

Lemma 3.2. \( \Lambda F \) is continuous and bounded from \( X_0 \) into \( X \). For \( v, w \in X_0 \), we have

\[
\|
\Lambda F(v) - \Lambda F(w)\|_X \leq CT^{1-\alpha} (\|v\|_{X_0}^{p-1} + \|w\|_{X_0}^{p-1}) \|v - w\|_X
\]

with \( \alpha = n(\frac{1}{p} - \frac{1}{p+1}) \), \( 0 \leq \alpha < 1 \), and for each \( R > 0 \), \( \Lambda F \) is a contraction map on \( B_R(X_0) \) in the metric of \( X \), provided \( T \) is sufficiently small.

We then introduce the spaces:

\[
\bar{Y} = \{ v \in \bar{X}, \nabla v \in \bar{X}, xv \in \bar{X} \} \subset \mathcal{C}(I, \Sigma)
\]

\[
Y = \{ v \in X, \nabla v \in X, xv \in X \}
\]

\[
Y' = \{ v \in X', \nabla v \in X', xv \in X' \}
\]
with the norms
\[ \|v\|_Y = \|v\|_X + \|\nabla v\|_X + \|xv\|_X \]
\[ \|f\|_{Y'} = \|f\|_{X'} + \|\nabla f\|_{X'} + \|xf\|_{X'}. \]
Then we have by the Sobolev imbedding theorem: \( Y \subset L^\infty(I, H^1) \subset X_0 \) and the constant \( C \) appearing in \( \|v\|_{X_0} \leq C\|v\|_Y \) is independent of \( T \).

**Lemma 3.3.** \( S(\cdot) \) is a bounded linear operator from \( \Sigma \) into \( \overline{Y} \), \( \Lambda \) is bounded from \( Y' \) into \( \overline{Y} \). In both cases the associated norms are independent of \( T \); their supremum is denoted by \( M \).

**Proof:** For \( v \in S(\mathbb{R}^n) \) (the Schwartz space), \( S(\cdot)v \in C(I, S(\mathbb{R}^n)) \subset \overline{Y} \) (see [15]). Hence we only have to show
\[ \|S(\cdot)v\|_Y \leq M\|v\|_\Sigma \quad \text{and} \quad \|\Lambda f\|_{Y'} \leq M\|f\|_{Y'} \]
with a constant \( M \) independent of \( T \).

Using (2.6) we have \( \|S(\cdot)v\|_X \leq C\|v\|_{L^2} \leq C\|v\|_\Sigma \) and using (2.3), (2.5) with \( s = r \) and \( q = p + 1 \),
\[ \|\Lambda \phi\|_X \leq C\|\phi\|_{L^{r'}(I, L^{1+1/p})} \leq C\|\phi\|_{X'} \leq C\|\phi\|_{Y'}, \]
where \( C \) does not depend on \( T \). As stated in Lemma 2.3, there exists \( a_{ik} \) and \( a_{ij,km} \) in \( C(I, B(\mathbb{R}^n \times \mathbb{R}^n)) \) such that
\[ x_iS(\cdot)v = \sum_{j=1}^{n}[I(\cdot, a_{ij,11})x_jv + I(\cdot, a_{ij,12})\partial_jv] + I(\cdot, a_{i1}) \]
and
\[ x_i\Lambda \phi = \sum_{j=1}^{n}[\Lambda a_{ij,11}(x_j\phi) + \Lambda a_{ij,12}(\partial_j\phi)] + \Lambda a_{i1}\phi. \]
Then using estimates (2.6), (2.3) and (2.5) of Lemma 2.2 with \( s = r \) and \( q = p + 1 \), we obtain the following inequalities:
\[ \|x_iS(\cdot)v\|_X \leq C\|v\|_\Sigma \]
\[ \|x_i\Lambda \phi\|_X \leq C\|\phi\|_{Y'}. \]
The terms \( \|\partial_jS(\cdot)v\|_X \) and \( \|\partial_j\Lambda \phi\|_X \) are treated in the same way, using (2.9).

**Lemma 3.4.** \( F \) is bounded from \( Y \) into \( Y' \) and
\[ \|F(v)\|_{Y'} \leq CT^{1-\alpha}\|v\|_Y^p, \quad \forall v \in Y, \text{ with } \alpha = n\left(\frac{1}{2} - \frac{1}{p+1}\right). \]

**Proof:** For the terms \( \|F(v)\|_{X'} \) and \( \|\nabla F(v)\|_{X'} \), the proof is as in [10], Lemma 2.2. Hence we consider the term \( \|xF(v)\|_{X'} \): for \( v \in Y \), we have \( \|xF(v)\|_{r',1+(1/p)} \leq \|v\|_{\infty,p+1}^{r-1}\|xv\|_{r',p+1} \). Hence, with \( 1 - \alpha = \frac{1}{r'} - \frac{1}{r} \):
\[ \|xF(v)\|_{X'} \leq \|xF(v)\|_{r',1+(1/p)} \leq CT^{1-\alpha}\|v\|_{\infty,p+1}^{p-1}\|xv\|_{r,p+1} \leq CT^{1-\alpha}\|v\|_{X_0}\|xv\|_X \leq CT^{1-\alpha}\|v\|_Y^p. \]
Lemma 3.5. Let $u_0 \in \Sigma$ and $T$ be defined as in (3.3). Then for every $R > M\|u_0\|_{\Sigma}$, if $T$ is sufficiently small, $T$ maps $B_R(Y)$ into itself, and is a contraction in the metric of $X$.

Proof: It follows that of Lemma 2.3 in [10]: For $u_0 \in \Sigma$, $S(\cdot)u_0 \in Y$ and for $v \in B_R(Y)$,

$$
\|Tv\|_Y \leq \|S(\cdot)u_0\|_Y + \|\Lambda F(v)\|_Y \leq M\|u_0\|_{\Sigma} + M\|F(v)\|_Y
$$

$$
\leq M\|u_0\|_{\Sigma} + MCT^{1-\alpha}R^p.
$$

Then, if we choose $T$ such that $MCT^{1-\alpha}R^p \leq R - M\|u_0\|_{\Sigma}$, $T$ maps $B_R(Y)$ into itself. Using Lemma 3.2, we have for $v, w \in B_R(Y)$:

$$
\|Tv - Tw\|_X \leq CT^{1-\alpha}\left(\|v\|^{-1}_Y + \|w\|^{-1}_Y\right)\|v - w\|_X
$$

$$
\leq 2CT^{1-\alpha}R^{p-1}\|v - w\|_X.
$$

Hence, if $T$ is sufficiently small, $T$ is a contraction in the $X$-norm.

Proof of Theorem 3.1: We take $R > M\|u_0\|_{\Sigma}$, and $T$ small enough for $T$ to be a contraction in $B_R(Y)$ in the $X$-norm. Since $B_R(Y)$ endowed with the $X$-norm is complete, $T$ has a unique fixed point $u \in Y$. Using Lemma 3.3, we have $u = Tu \in \bar{Y} \subset C(I, \Sigma)$. By iteration, we can construct the maximal solution $u \in C([T^-, T^+], \Sigma) \cap L_{loc}^1(T^-, T^+, \Sigma^{1,p+1})$ of the integral equation (3.1).

Let $I = [-T, T] \subset [T^-, T^+]$; then $F(u) \in X' \subset L^r(I, H^{-1}) + L^1(I, L^2) \subset L^1(I, H^{-1})$ and $\nabla F(u) \in X' \subset L^1(I, H^{-1})$. Hence, $F(u) \in L^1(I, L^2) \subset L^1(I, \Sigma(-1))$. Since $H$ is bounded from $\Sigma$ into $\Sigma(-1)$, $u$ satisfies (1.1) in $\Sigma(-1)$ with $u(0) = u_0$.

4. Continuity with respect to the initial data. In this section, we show that the solutions $u \in C(I, \Sigma)$ of (1.1) depend continuously on the initial data in the following sense:

Theorem 4.1. Let $u_0 \in C(I, \Sigma)$ be a solution of (1.1) with initial data $u_0$, and let $u_0^m \in \Sigma$ with $u_0^m \to u_0$ in $\Sigma$ as $m \to +\infty$. Then, the solution $u_m$ of (1.1) with $u_m(0) = u_0^m$ exists on $I$ provided $m$ is sufficiently large, and $u_m \to u$ in $C(I, \Sigma)$ as $m \to +\infty$.

We will use Kato’s notation, so we refer to [10]. Here $F'(z)$ is the derivative of $F$ considered as a differentiable map from $\mathbb{R}^2$ into itself; i.e.,

$$
F'(z) \cdot \zeta = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (F(z + \epsilon \zeta) - F(z)).
$$

Proof of Theorem 4.1: It is sufficient to prove the theorem in the case of a small interval $I$; then it can be extended to the whole interval by iteration. Since $(u_0^m)$ is bounded in $\Sigma$, we can take $I = [-T, T]$, with $T$ small enough to be able to construct $u^m$ (for sufficiently large $m$) and $u$ in $B_R(Y)$ satisfying

$$
u^m = S(\cdot)u_0^m - i\epsilon\Lambda F(u^m)
$$

$$
u = S(\cdot)u_0 - i\epsilon\Lambda F(u)
$$

as desired.
with, for example, $R = 2M\|u_0\|_\Sigma$. Then we have:

$$u_m - u = S(\cdot)(u^m_0 - u_0) - i\epsilon \Lambda (F(u_m) - F(u)) \tag{4.1}$$

and, for $j = 1, \ldots, n$, $\partial_j (u_m - u) = \partial_j S(\cdot)(u^m_0 - u_0) - i\epsilon \partial_j \Lambda (F(u_m) - F(u))$. So we can find $B(\mathbb{R}^n \times \mathbb{R}^n)$-valued continuous functions $a_{ij}, b_{ij}$ and $c_j$ such that

$$\partial_j (u_m - u) = \sum_{i=1}^n [I(\cdot, a_{ij})(\partial_i u_0^m - \partial_i u_0) + I(\cdot, b_{ij})(x_i u_0^m - x_i u_0)]$$

$$+ I(\cdot, c_j)(u_0^m - u_0)$$

$$- i\epsilon \left[ \sum_{i=1}^n (\Lambda a_{ij} (F'(u_m)\partial_i u_m - F'(u)\partial_i u)$$

$$+ \Lambda b_{ij} (x_i F(u_m) - x_i F(u))) + \Lambda c_j (F'(u_m) - F'(u)) \right]$$

and we have a similar equality for $x_j (u_m - u)$. Hence we obtain by using Lemma 3.3

$$\|u_m - u\|_Y \leq C_1 \|u_0^m - u_0\|_\Sigma + C_2 \sum_{i=1}^n \|F'(u_m)\partial_i u_m - F'(u)\partial_i u\|_{X'}$$

$$+ C_3 \|xF(u_m) - xF(u)\|_{X'} + C_4 \|F(u_m) - F(u)\|_{X'},$$

where $C_1, \ldots, C_4$ do not depend on $T$. But we have (see Section 3)

$$\|F(u_m) - F(u)\|_{X'} \leq CT^{1-\alpha} (\|u_m\|_{Y}^{p-1} + \|u\|_{Y}^{p-1}) \|u_m - u\|_X$$

$$\leq 2CT^{1-\alpha} R^{p-1} \|u_m - u\|_Y, \tag{4.2}$$

and similarly,

$$\|xF(u_m) - xF(u)\|_{X'} \leq 2CT^{1-\alpha} R^{p-1} \|x u_m - x u\|_X \leq 2CT^{1-\alpha} R^{p-1} \|u_m - u\|_Y,$$

while

$$\|F'(u_m)\partial_i u_m - F'(u)\partial_i u\|_{X'} \leq \|F'(u_m)\partial_i u - \partial_i u_m\|_{X'}$$

$$+ \|F'(u_m)\partial_i u - F'(u)\partial_i u\|_{X'}.$$

Using $|F'(u)| \leq C|u|^{p-1}$ and $|F'(u) \cdot v| \leq |F'(u)||v|$ (see the Appendix in [10]), we obtain in the same way as before:

$$\|F'(u_m)\partial_i u_m - \partial_i u\|_{X'} \leq CT^{1-\alpha} R^{p-1} \|u_m - u\|_Y.$$

Thus, if we choose $T$ such that $(nC_2 + 2C_3 + 2C_4) T^{1-\alpha} R^{p-1} \leq \frac{1}{2}$, we have

$$\|u_m - u\|_Y \leq 2C_1 \|u_0^m - u_0\|_\Sigma + 2C_2 \sum_{i=1}^n \|F'(u_m)\partial_i u - F'(u)\partial_i u\|_{X'},$$
and it remains to establish that $\|F'(u_m)\partial_1 u - F'(u)\partial_1 u\|_{X'} \to 0$ as $m \to +\infty$. But we have $F'(u_m) \cdot v \to F'(u) \cdot v$ in $X'$ as $m \to +\infty$ if $u_m \to u$ in $X$, and $v \in X$ (see [10], Lemma 5.2), and one can easily show, using (4.1) and (4.2), that provided $T$ is sufficiently small, $u_m \to u$ in $X$ as $m \to +\infty$. This completes the proof of Theorem 4.1.

5. Conservation laws. In this section, we establish that if $u \in C(I, \Sigma)$ is a solution of (1.1), then the energy

$$E(u) = \frac{1}{2} \langle u, Hu \rangle + \frac{\epsilon}{p+1} \int |u|^{p+1} dx,$$

and the $L^2$-norm of $u$ are conserved during the time the solution is well defined.

For this purpose, we introduce as in [11] a regularized integral equation. Most of the results we need can be proved exactly in the same way as in [7] or [11], to which we will refer.

Let $h \in D(\mathbb{R}^n)$ be an even function such that $h \geq 0$, supp $h \subset B(0,1)$ and $\|h\|_{L^1(\mathbb{R}^n)} = 1$. Here $B(0,1)$ is the ball in $\mathbb{R}^n$ centered at 0, with radius 1. Let $g \in D(\mathbb{R}^n)$ be such that $0 \leq g \leq 1$, supp $g \subset B(0,2)$ and $g \equiv 1$ on $B(0,1)$. Then we set for a positive integer $m$,

$$h_m(x) = m^n h(mx), \quad g_m(x) = g\left(\frac{x}{m}\right),$$

and we consider the following integral equation with $u_0 \in \Sigma$:

$$u(t) = S(t)(h_m \ast g_m u_0) - i\epsilon \int_0^t S(t-\tau)[h_m \ast g_m F(h_m \ast u(\tau))] d\tau. \quad (5.1)$$

We set $F_m(u) = h_m \ast g_m F(h_m \ast u)$. The following result can easily be deduced from the proof of Theorem 3.5 in [11], and Lemma 3.3, using

$$\|h_m \ast g_m u_0\|_{L^{p+1}} \leq \|h_m\|_{L^1} \|u_0\|_{L^{p+1}} \leq \|u_0\|_{L^{p+1}}.$$

Proposition 5.1. For any $\rho > 0$, there exists a $T(\rho) > 0$ with $T(\rho) \leq \gamma$ independent of $m$, such that for any $u_0 \in \Sigma$ with $\|u_0\|_{\Sigma} < \rho$, equations (5.1) and (3.1) have unique solutions in the ball with radius $\rho$ in $C([-T(\rho), T(\rho)], L^{p+1})$.

The following proposition is proved as in [7], Proposition 3.1.

Proposition 5.2. Let $\rho > 0$ and $T(\rho)$ be defined as in Proposition 5.1. Let $u_0 \in \Sigma$ be such that $\|u_0\|_{\Sigma} < \rho$; let $u_m$ be the solution of (5.1) in $C([-T(\rho), T(\rho)], L^{p+1})$ and let $u$ be the solution of (3.1) in the same space. Then $u_m$ tends to $u$ in $C([-T(\rho), T(\rho)], L^{p+1})$, as $m \to +\infty$.

Lemma 5.3. Let $u \in L^{p+1}(\mathbb{R}^n)$; then for any integer $m$, $h_m \ast g_m u \in S$ and $F_m(u) \in S$ ($S$ is the space of smooth rapidly decreasing functions). If $u(\tau)$ is an $L^{p+1}$-valued continuous function of $\tau \in I$, with $I$ an interval of $\mathbb{R}$, then for any integers $m, k$, $F_m(u) \in C(I, \Sigma(k))$.

Proof: It follows exactly those of Lemmas 4.3 and 4.4 in [11], since to prove the second part of the lemma, it suffices to show that for any $\alpha$ and $\beta$, $x^\alpha \partial^\beta_x F_m(u)$ is $L^2(\mathbb{R}^n)$-valued and continuous on $I$.

Then we can prove the following proposition in the same way as Oh did in [11]:
Proposition 5.4. Let $I$ be an interval of $\mathbb{R}$, $u_0 \in \Sigma$ and $u_m \in C(I, L^{p+1})$ be a solution of (5.1). Then, for any integer $k$, $u_m$ is in $C^1(I, \Sigma(k))$ and satisfies

$$i \frac{du_m}{dt}(t) = Hu_m(t) + \epsilon F_m(u_m(t)).$$

(5.2)

Furthermore, if we set

$$E_m(u) = \frac{1}{2} \langle u, Hu \rangle + \frac{\epsilon}{p+1} \langle g_m F(h_m \ast u), h_m \ast u \rangle,$$

then we have

$$\frac{d}{dt} \|u_m(t)\|_{L^2}^2 = 0 \quad \text{and} \quad \frac{d}{dt} E_m(u_m(t)) = 0 \quad \text{for any} \quad t \in I.$$

Proof: Since $S(\cdot)$ is $L(\Sigma(k))$-valued for any $k$ (see [15]), the preceding lemma and (5.1) yield: $u_m \in C(I, \Sigma(k))$. Now, by differentiating the right-hand side of (5.1) with respect to $t$, and by using the fact that $H$ is bounded from $\Sigma(k)$ into $\Sigma(k-2)$ for any $k$, we obtain $u_m \in C^1(I, \Sigma(k))$ and (5.2). The last equalities then follow by very classical computations, using the fact that $h$ is even (see [11] for example).

Lemma 5.5. Let $u_m \in C(I, L^{p+1})$ be a solution of (5.1). Then, for any $t \in I$, the sequence $u_m(t)$ is bounded in $\Sigma$.

Proof: We will show, by using (5.1), that

$$\|u_m(t)\|_{\Sigma} \leq C_1 + C_2 \int_0^t \|u_m(\tau)\|_{\Sigma} d\tau,$$

where $C_1$ and $C_2$ are independent of $m$ and $t$ (we will consider if necessary that $t$ is in a compact subinterval of $I$). As stated in Lemma 5.2, $u_m$ tends to $u$ in $C(I, L^{p+1})$, hence $u_m(t)$ is bounded in $L^{p+1}$ for any $t \in I$. Now we have

$$\|h_m \ast g_m u_0\|_{L^2} \leq \|h_m\|_{L^1} \|u_0\|_{L^2} \leq \|u_0\|_{\Sigma}$$

and

$$\|\partial_j (h_m \ast g_m u_0)\|_{L^2} \leq \|h_m \ast (\partial_j g_m) u_0\|_{L^2} + \|h_m \ast g_m \partial_j u_0\|_{L^2}$$

$$\leq \|\partial_j g\|_{L^\infty} \|u_0\|_{L^2} + \|\partial_j u_0\|_{L^2} \leq C \|u_0\|_{\Sigma}$$

$$\|x_j (h_m \ast g_m u_0)\|_{L^2} \leq \|x_j h_m \ast g_m u_0\|_{L^2} + \|h_m \ast g_m x_j u_0\|_{L^2}$$

$$\leq C \|u_0\|_{\Sigma},$$

since $x_j h_m$ is bounded in $L^1(\mathbb{R}^n)$.

Let us consider now the term $F_m(u_m)$:

$$\|F_m(u_m)\|_{L^2} = \|h_m \ast g_m F(h_m \ast u_m)\|_{L^2}$$

$$\leq C \|h_m\|_{L^q} \|F(h_m \ast u_m)\|_{L^{1+(1/p)}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{p+1}$$

$$\leq C \|h\|_{L^q} \|h_m \ast u_m\|_{L^{p+1}} \leq C \|h\|_{L^q} \|u_m\|_{L^{p+1}} \leq C' \|u_m\|_{\Sigma},$$
since $\Sigma \subset L^{p+1}$ and $u_m$ is bounded in $L^{p+1}$ (see Proposition 5.2). Similarly,

$$
\| \partial_j(h_m * g_m F(h_m * u_m)) \|_{L^2} \leq \| h_m * \partial_j g_m F(h_m * u_m) \|_{L^2} + \| h_m * g_m \partial_j F(h_m * u_m) \|_{L^2}.
$$

The first term can be bounded like the preceding ones, since $\partial_j g_m$ is bounded in $L^\infty$. For the second term, we have:

$$
\| h_m * g_m \partial_j F(h_m * u_m) \|_{L^2} \leq C \| h_m \|_{L^\infty} \| \partial_j F(h_m * u_m) \|_{L^{1+1/p}} 
$$

$$
\leq C \| h_m \|_{L^\infty} \| u_m \|_{L^{p+1}} \| \partial_j(u_m) \|_{L^{p+1}} \leq C \| u_m \|_{L^{p+1}} \| h_m * \partial_j u_m \|_{L^{p+1}} 
$$

$$
\leq C \| h_m \|_{L^\infty} \| \partial_j u_m \|_{L^2} \leq C \| u_m \|_{\Sigma}.
$$

Finally,

$$
\| x_j(h_m * g_m F(h_m * u_m)) \|_{L^2} 
$$

$$
\leq \| (x_j h_m) * g_m F(h_m * u_m) \|_{L^2} + \| h_m * g_m x_j F(h_m * u_m) \|_{L^2}.
$$

The first term can be bounded like the preceding ones, and

$$
\| h_m * g_m x_j F(h_m * u_m) \|_{L^2} \leq C \| h_m \|_{L^\infty} \| x_j F(h_m * u_m) \|_{L^{1+1/p}} 
$$

$$
\leq C \| h_m \|_{L^\infty} \| u_m \|_{L^{p+1}} \| x_j(u_m) \|_{L^{p+1}} 
$$

$$
\leq C \| (x_j h_m) * u_m \|_{L^{p+1}} + \| h_m * x_j u_m \|_{L^{p+1}} 
$$

$$
\leq C + C' \| u_m \|_{\Sigma}.
$$

Hence, we obtain

$$
\| u_m(t) \|_{\Sigma} \leq C_1 + C_2 \int_0^t \| u_m(\tau) \|_{\Sigma} d\tau,
$$

and by Gronwall’s Lemma, the sequence $(u_m(t))$ is bounded in $\Sigma$.

**Theorem 5.6.** Let $u \in C(I, \Sigma)$ be a solution of $(3.1)$ with initial data $u_0 \in \Sigma$. Then for any $t \in I$,

$$
\| u(t) \|_{L^2} = \| u_0 \|_{L^2}
$$

$$
E(u(t)) = E(u_0),
$$

with $E(u) = \frac{1}{2} \langle u, Hu \rangle + \frac{\epsilon}{p+1} \int |u|^{p+1} dx$.

**Proof:** It follows that of Ginibre and Velo [7], Proposition 3.4 and Lemma A.2.1, by setting $H = \Sigma$, $V = L^{p+1}$ and $q(u) = \frac{1}{2} \langle u, Hu \rangle$. Note however that we had to show that the sequence of regularized solutions was bounded in $\Sigma$ since it does not come from the conservation of energy as was the case in [7].
6. Local existence of solutions in $L^2(\mathbb{R}^n)$. In this section, we assume $1 < p \leq 1 + \frac{4}{n}$. Then the following result holds:

**Theorem 6.1.** Let $u_0 \in L^2(\mathbb{R}^n)$. Then (3.1) has a unique maximal solution $u \in C([T^-_2, T^+_2] \cap L^2_{\text{loc}}(T^-_2, T^+_2, L^{p+1}))$. Furthermore,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in (T^-_2, T^+_2).$$

If $T^+_2$ (respectively $-T^-_2$) $< +\infty$, then

$$\|u\|_{L^r(0, T^+_2, L^{p+1})} = +\infty \quad \text{(respectively } \|u\|_{L^r(T^-_2, 0, L^{p+1})} = +\infty).$$

We will use a fixed-point method. The following lemma establishes some useful inequalities:

**Lemma 6.2.** Let $0 < T \leq \gamma$ and $I_T = [-T, T]$; then there exists a real $q$ with $1 < q \leq +\infty$, such that for every $u, v \in L^r(I_T, L^{p+1})$,

$$\|\Lambda F(u) - \Lambda F(v)\|_{r, p+1} \leq C T^{p/q} (\|u\|_{r, p+1}^{p-1} + \|v\|_{r, p+1}^{p-1}) \|u - v\|_{r, p+1} \quad (6.1)$$

$$\|\Lambda F(u) - \Lambda F(v)\|_{\infty, 2} \leq C T^{p/q} (\|u\|_{r, p+1}^{p-1} + \|v\|_{r, p+1}^{p-1}) \|u - v\|_{r, p+1} \quad (6.2)$$

**Proof:** We have $\|F(u) - F(v)\|_{L^{1+1/(1/p)}} \leq C (\|u\|_{L^{p+1}}^{p-1} + \|v\|_{L^{p+1}}^{p-1}) \|u - v\|_{L^{p+1}}$; hence, using Hölder’s inequality,

$$\|F(u) - F(v)\|_{r', 1+1/(1/p)} \leq C (\|u\|_{L^{p+1}(I_T, L^{p+1})}^{p-1} + \|v\|_{L^{p+1}(I_T, L^{p+1})}^{p-1}) \|u - v\|_{L^{q}(I_T, L^{p+1})},$$

with $q_1 = \frac{4p(p+1)}{n+4-n+4p}$, $q_1 \leq r$. Finally, with $\frac{1}{q} + \frac{1}{r'} = \frac{1}{q_1}$, i.e., $q = \frac{4p}{n+4-n+4p}$, we obtain

$$\|F(u) - F(v)\|_{r', 1+1/(1/p)} \leq C \left( \int_{-T}^{T} dt \right)^{p/q} (\|u\|_{r, p+1}^{p-1} + \|v\|_{r, p+1}^{p-1}) \|u - v\|_{r, p+1}.$$

We then deduce (6.1) and (6.2) by using (2.3) and (2.5) with $s = r$ and $q = p + 1$.

**Remark 6.3.** In the same was as in this lemma, we can easily show, by using the proof of Lemmas 3.4 and 2.3, that

$$\|\partial_j (\Lambda F(u))\|_{q, s} \leq C T^{p/q} \|u\|_{r, p+1}^{p-1} \|u\|_{L^r(I_T, \Sigma^{1, p+1})} \quad (6.3)$$

$$\|x_j (\Lambda F(u))\|_{q, s} \leq C T^{p/q} \|u\|_{r, p+1}^{p-1} \|u\|_{L^r(I_T, \Sigma^{1, p+1})} \quad (6.4)$$

with $(q, s) = (r, p + 1)$ and $(\infty, 2)$. Hence we have
Proposition 6.4. There exists a $\delta > 0$ such that if $u_0 \in L^2(\mathbb{R}^n)$ and $0 < T \leq \gamma$ are such that $\|S(\cdot)u_0\|_{L_r^r(I_T, L^{p+1})} \leq \delta$, with $I_T = [-T, T]$, then (3.1) has a unique solution $u \in C(I_T, L^2) \cap L^r(I_T, L^{p+1})$. In addition, for all $t \in I$, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $u$ depends continuously on $u_0 \in C(I_T, L^2) \cap L^r(I_T, L^{p+1})$.

Proof: Let $\delta > 0$, and let $u_0 \in \Sigma$ and $T \leq \gamma$ satisfy $\|S(\cdot)u_0\|_{L_r^r(I_T, L^{p+1})} \leq \delta$. We set

$$E = \{u \in L^r(I_T, L^{p+1}), \|u\|_{r,p+1} \leq 2\delta\}.$$ 

For $u \in E$, we still consider

$$Tu(t) = S(t)u_0 - i\epsilon \int_0^t S(t - \tau)F(u(\tau))\,d\tau.$$ 

Then, using (6.1), we have for any $u$ and $v$ in $E$:

$$\|Tu - Tv\|_{L_r^r(I_T, L^{p+1})} \leq 2C_1(2\delta)^{p-1}T^{p/q}\|u - v\|_{r,p+1}$$

$$\leq 2C_1(2\delta)^{p-1}\gamma^{p/q}\|u - v\|_{r,p+1}.$$ 

Hence, if we choose $\delta$ such that $2C_1(2\delta)^{p-1}\gamma^{p/q} \leq \frac{1}{2}$, $T$ is a contraction from $E$ into itself and by the fixed-point theorem, we obtain a unique solution $u \in E$ of (3.1). Applying (6.2) we have $u \in L^\infty(I_T, L^2)$.

If $u_0^m$ tends to $u_0$ in $L^2$, with $u_0^m \in L^2$, then from (2.6), we have for sufficiently large $m$ $\|S(\cdot)u_0^m\|_{r,p+1} \leq \delta$; we can then construct the solution $u_m \in E$ with $u_m(0) = u_0^m$. Then

$$\|u_m - u\|_{r,p+1} \leq \|S(\cdot)(u_0 - u_0^m)\|_{r,p+1} + \|\Lambda F(u_m) - \Lambda F(u)\|_{r,p+1},$$

and applying (2.6) and (6.1) with $2C_1(2\delta)^{p-1}\gamma^{p/q} \leq \frac{1}{2}$, we obtain

$$\|u_m - u\|_{r,p+1} \leq 2C_5\|u_0 - u_0^m\|_{L^2}.$$ 

Hence $u_m \to u$ in $L^r(I_T, L^{p+1})$ as $m \to \infty$. Applying (6.2), we see that $u_m \to u$ in $L^\infty(I_T, L^2)$.

If $u_0 \in \Sigma$, then from Theorem 3.1, there exists $\tau \geq 0$, with $\tau \leq T$, such that $u \in C([-\tau, \tau], \Sigma) \cap L^r(-\tau, \tau, \Sigma^{1,p+1})$. Let us show that we can take $\tau = T$; otherwise we would have $\|u(t)\|_{\Sigma} \to +\infty$ when $t \not\to T$ or $t \not\to -\tau$; but from (6.3) we have

$$\|u\|_{L^r(I_T, \Sigma^{1,p+1})} \leq C\|u_0\|_{\Sigma} + C_3T^{p/q}\|u\|_{r,p+1}^{p-1}\|u\|_{L^r(I_T, \Sigma^{1,p+1})}$$

$$\leq C\|u_0\|_{\Sigma} + C_3T^{p/q}(2\delta)^{p-1}\|u\|_{L^r(I_T, \Sigma^{1,p+1})};$$

hence, for $\delta$ sufficiently small, we have

$$\|u\|_{L^r(I_T, \Sigma^{1,p+1})} \leq C\|u_0\|_{\Sigma};$$

then, applying (6.4), we see that $\|u\|_{L^\infty(I_T, \Sigma)} \leq C\|u_0\|_{\Sigma}$ which is a contradiction. Hence $\tau = T$. Then we have $\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \forall t \in I_T$. 


For \( u_0 \in L^2 \), let \( u_0^m \in \Sigma \) with \( u_0^m \to u_0 \) in \( L^2 \) as \( m \to +\infty \); then the corresponding solution \( u^m \) tends to \( u \) in \( L^\infty (I_T, L^2) \), hence \( u \in \mathcal{C}([-T, T], L^2) \) and \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \forall t \in I_T \).

**Proof of Theorem 6.1:** Let \( u_0 \in \Sigma \). Since \( \|S(\cdot)u_0\|_{L^r(I_T, L^{p+1})} \to 0 \) as \( T \to 0 \), we can apply Proposition 6.4 for sufficiently small \( T \). Then, by iteration, we obtain a unique maximal solution \( u \in \mathcal{C}([T^-_2, T^+_2], L^2) \cap L^\infty_{\text{loc}}(T^-_2, T^+_2, L^{p+1}) \). It remains to show the last point of Theorem 6.1, but the proof follows exactly that of [2], Theorem 1, by writing \( L^r(0, T^+_2, L^{p+1}) \) instead of \( L^0(0, T^+, L^p) \).

**Remark 6.5.** In the case \( A(x) = 0 \); i.e., if we consider the following equation:

\[
\frac{i}{\partial t} \partial u = -\frac{1}{2} \Delta u + Vu + \varepsilon |u|^{p-1}u, \quad 1 < p < 1 + \frac{4}{n}
\]

where \( V \) satisfies assumption H2, then in [11], Y.G. Oh has proved the global existence of solutions in \( \mathcal{C}([\mathbb{R}, D(H^{1/2})) \subset \mathcal{C}(\mathbb{R}, H^1) \subset \mathcal{C}(\mathbb{R}, L^{p+1}) \). Then using Theorem 6.1 we can deduce from this result that if \( u_0 \in L^2(\mathbb{R}^n) \), there exists a unique global solution \( u \in \mathcal{C}(\mathbb{R}, L^2) \) for this equation.

**Remark 6.6.** Since \( A \) is real valued, we have \( |\nabla u| \leq |(-i\nabla - A)u| \) and \( D(H^{1/2}) \subset L^{p+1}(\mathbb{R}^n) \) if \( 1 \leq p \leq 1 + \frac{4}{n-2} \) (see [5]). In the case \( 1 \leq p < 1 + \frac{4}{n} \), it then follows from the conservation laws and the Gagliardo-Nirenberg inequalities for \( |u| \) that the solutions of (1.1) with \( u(0) = u_0 \in \Sigma \) are global in \( D(H^{1/2}) \) (see for example [11]).

In fact, one can easily see from the proof of Theorem 6.1 that since the solutions are global in \( L^2(\mathbb{R}^n) \), they are global in \( \Sigma \).

### 7. A blow-up result for Schrödinger equation with potential.

In this section we consider the following equation:

\[
\begin{cases}
  i \frac{\partial u}{\partial t} = -\frac{1}{2} \Delta u + Vu - |u|^{p-1}u \\
u(0) = u_0,
\end{cases}
\]  

(7.1)

where \( V \) satisfies H2 with, in addition, \( \frac{\partial V}{\partial r} \geq 0 \) in \( \mathbb{R}^n \), \( r = |x| \). Y.G. Oh has proved the global existence of solution in \( D(H^{1/2}) \subset H^1 \), with \( H = -\frac{1}{2} \Delta + V \), for \( 1 \leq p < 1 + \frac{4}{n} \) (see [11]). Here, we will show that if \( p > 1 + \frac{4}{n} \) and under some assumptions on the initial data \( u_0 \), the \( H^1 \) norm of the solution blows up in finite time.

We will follow Glassey (see [9]). In all of this section, we assume that \( 1 + \frac{4}{n} < p < 1 + \frac{4}{n-2} \). We first establish some conservation laws:

**Lemma 7.1.** Let \( u_0 \in \Sigma \) and \( u(t) \) be a solution of (7.1) in \( \mathcal{C}(I, \Sigma) \) with \( I = [0, T] \). Then for any \( t \in [0, T] \), we have

i) \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \)

ii) \( \frac{1}{2} \int |\nabla u|^2 \, dx + \int V(x)|u|^2 \, dx - \frac{1}{p+1} \int |u|^{p+1} \, dx = E_0 \)

iii) \( \frac{d}{dt} \int |u|^2 |u|^2 \, dx = 2 \text{Im} \int ru_r \bar{u} \, dx \)

iv) \( \frac{d}{dt} \int \text{Im}(ru_r \bar{u}) \, dx = \int |\nabla u|^2 \, dx - \int rV_r |u|^2 \, dx - n(1 - \frac{2}{p+1}) \int |u|^{p+1} \, dx \).
Proof: i) and ii) were proved in Section 5.

Multiplying (7.1) by $2\bar{u}$ and taking the real part, we obtain \( \frac{\partial}{\partial t}|u|^2 = -\nabla (\text{Im} \, \bar{u} \nabla u) \).

Then if we multiply this inequality by $|x|^2$ and integrate by parts over $\mathbb{R}^n$, we obtain iii).

In order to obtain iv), we multiply (7.1) by $2r \bar{u}$, and take the real part, then integrate each term by parts; we obtain (see [9])

\[
-\text{Re} \int r \bar{u}_t u \, dx = \left(1 - \frac{n}{2}\right) \int |\nabla u|^2 \, dx
\]

\[
2\text{Re} \int r \bar{u}_t V(x) u \, dx = -n \int V(x)|u|^2 \, dx - \int r \bar{u}_t |u|^2 \, dx
\]

\[
-2\text{Re} \int r \bar{u}_t |u|^{p-1} u \, dx = \frac{2n}{p+1} \int |u|^{p+1} \, dx
\]

and

\[
\text{Re} \left[ 2i \int r \bar{u}_t u \, dx \right] = \frac{d}{dt} \text{Re} \left[ i \int r \bar{u}_t \, dx \right] + n \text{Re} \left[ i \int \bar{u}_t \, dx \right].
\]

Then, using (7.1),

\[
\text{Re} \left[ 2i \int r \bar{u}_t u \, dx \right] = \frac{d}{dt} \text{Im} \int r \bar{u}_t \, dx - \frac{n}{2} \int |\nabla u|^2 \, dx
\]

\[
- n \int V(x)|u|^2 \, dx + n \int |u|^{p+1} \, dx
\]

and we obtain iv).

Theorem 7.2. Let $u(t)$ be a solution of (7.1) in $\Sigma$. We assume that

1) $E_0 \leq 0$

2) $\text{Im} \int r \bar{u}_t u \, dx < 0$;

then there exists $T < +\infty$ such that $\lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty$.

Proof: We set when $u$ is well defined:

\[
y(t) = -\text{Im} \int r \bar{u}_t \, dx;
\]

then $y(0) > 0$ and from iv) we have

\[
y'(t) \geq -\int |\nabla u|^2 \, dx + \int r \bar{u}_t |u|^2 \, dx + n \left[1 - \frac{2}{p+1}\right] \int |u|^{p+1} \, dx
\]

\[
\geq -\int |\nabla u|^2 \, dx + \frac{n(p-1)}{p+1} \int |u|^{p+1} \, dx.
\]

Using ii) we have

\[
y'(t) \geq -\int |\nabla u|^2 \, dx + (n(p-1)) \left[\frac{1}{4} \int |\nabla u|^2 \, dx + \int V|u|^2 \, dx - E_0\right]
\]

\[
\geq \left[1 - \frac{n(p-1)}{4}\right] \int |\nabla u|^2 \, dx, \quad \text{since } E_0 \leq 0.
\]
Let $C_n = 1 - \frac{n(p - 1)}{4} > 0$; then

$$y'(t) \geq C_n \int |\nabla u|^2 \, dx.$$  

Hence, since $y(0) > 0$ we have $y(t) > 0$ as long as $u$ is well defined. Using 3) we have

$$\frac{d}{dt} \int r^2 |u|^2 \, dx = -2y(t) < 0,$$

hence

$$\int r^2 |u|^2 \, dx \leq \int r^2 |u_0|^2 \, dx = d_0^2.$$  

Then by Schwarz' inequality:

$$y'(t) \leq \left( \int r^2 |u|^2 \, dx \right)^{1/2} \left( \int |u_r|^2 \, dx \right)^{1/2} \leq d_0 \|\nabla u\|_{L^2}^2.$$  

Using (7.2) we have $y'(t) \geq (C_n/d_0^2)y^2(t)$, $y(0) > 0$. Then if $0 \leq t < d_0^2/(C_n y(0))$ we have

$$y(t) \geq \frac{y(0)d_0^2}{d_0^2 - C_n y(0)t}$$

and

$$\|\nabla u(t)\|_{L^2} \geq \frac{y(t)}{d_0} \geq \frac{y(0)d_0}{d_0^2 - C_n y(0)t}.$$  

Hence there exists $T \leq d_0^2/(C_n y(0))$ such that $\lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty$.

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