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ON THE OPTIMAL CONTROL PROBLEMS OF PARABOLIC EQUATIONS WITH AN INFINITE NUMBER OF VARIABLES AND WITH EQUALITY CONSTRAINTS

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Abstract. Optimal control problems of systems governed by parabolic equations with an infinite number of variables and with additional equality constraints are considered. The extremum principles as well as sufficient conditions of optimality are formulated by using certain extensions of the Dubovitskii-Milyutin method.

Introduction. The problem of optimal control of systems governed by parabolic or hyperbolic equations has been discussed in several papers starting with [10].

In [7] and [8] the Dubovitskii-Milyutin method has been applied to prove the extremum principles for such systems. These results have been extended to cases of optimal control of hyperbolic or parabolic systems with additional (operator or nonoperator) equality constraints in [13] and [14] by using extensions of the Dubovitskii-Milyutin method from [12] and [15]. All these results concerned optimal control problems of systems with a finite number of variables. In [1], [2] and [3] optimal control problems of a system governed by elliptic or hyperbolic operators with an infinite number of variables have been considered. Necessary conditions of optimality have been obtained by using the method of Lions (cf. [10]). In [9] results similar to [8] in the form of the extremum principle have been obtained for the case of parabolic equations with an infinite number of variables by using the Dubovitskii-Milyutin method from [5].

In the present paper, the above results for systems governed by parabolic equations are extended to the case of optimal control with additional equality constraints and with an infinite number of variables. By using the extension of the Dubovitskii-Milyutin method from [12] and [13] the extremum principle, as well as the sufficient condition of optimality, are proved for the optimal control problem of a parabolic equation with infinitely many variables and with the additional terminal constraint $y(x,T) = 0$. Then another optimal control problem of a parabolic equation with infinitely many variables but with nonoperator equality constraint $u(\cdot) \in U$ is considered and the extremum principle is proved for it by using the generalization of the Dubovitskii-Milyutin method from [15].

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I. Preliminary definitions. Let us introduce some preliminary facts and definitions (cf. [1], [9], [11]).

Let $\big(P_k(t)\big)_{k=1}^{\infty}$ be a sequence of weights such that

$$0 < P_k(t) \in C^{\infty}(\mathbb{R}^1) \quad \text{and} \quad \int_{\mathbb{R}^1} P_k(t)\,dt = 1.$$

Let $\mathbb{R}^{\infty} = \mathbb{R}^1 \times \ldots \times \mathbb{R}^1 \times \ldots$ with boundary Γ. We introduce the measure $d\rho(x)$ in $\mathbb{R}^{\infty}$ in the following way

$$d\rho(x) = \big(P_1(x)_1 dx_1\big) \otimes \big(P_2(x_2)dx_2\big) \otimes \ldots$$

where

$$x = \big(x_k\big)_{k=1}^{\infty}.$$

Then we denote by $L_2(\mathbb{R}^{\infty}, d\rho(x)) = L_2(\mathbb{R}^{\infty})$ the space of measurable functions such that

$$\|u\|_{L_2(\mathbb{R}^{\infty})} = \left\{ \int_{\mathbb{R}^{\infty}} |u|^2 d\rho(x) \right\}^{1/2} < \infty.$$

$L_2(\mathbb{R}^{\infty})$ is a Hilbert space with a scalar product (cf. [11]):

$$(u, v)_{L_2(\mathbb{R}^{\infty})} = \int_{\mathbb{R}^{\infty}} u(x)v(x)d\rho(x).$$

Then we consider the space of ℓ-times continuously differentiable functions up to the boundary Γ of $\mathbb{R}^{\infty}$ that vanish in a neighborhood of ∞. In this space we introduce a scalar product

$$(u, v)_{W^{\ell}(\mathbb{R}^{\infty})} = \sum_{|\alpha| \le \ell} (D^{\alpha}u, D^{\alpha}v)_{L_2(\mathbb{R}^{\infty})}$$

where D^{α} is defined by

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_i)^{\alpha_1}(\partial x_2)^{\alpha_2}\ldots}, \qquad |\alpha| = \sum_{i=1}^{\infty} \alpha_i$$

where the differentiation is in the sense of a generalized function. The completion of this space gives us the Sobolev space $W^{\ell}(\mathbb{R}^{\infty})$, $\ell = 1, 2, \ldots$. Then the space $W^0(\mathbb{R}^{\infty}) = L_2(\mathbb{R}^{\infty})$ and introducing negative spaces $W^{-\ell}(\mathbb{R}^{\infty})$ we get

$$W^{\ell}(\mathbb{R}^{\infty}) \subseteq L_2(\mathbb{R}^{\infty}) \subseteq W^{-\ell}(\mathbb{R}^{\infty}) \quad \text{for} \ \ \ell = 1, 2, \ldots .$$

Then in the similar way as for the case of the finite number of variables we introduce the subspace

$$W_0^{\ell}(\mathbb{R}^{\infty}) = \big\{u : u \in W^{\ell}(\mathbb{R}^{\infty}),\ D^{\alpha}u = 0 \text{ on } \Gamma,\ |\alpha| \le \ell - 1,\ \ell > 1\big\}$$

and $W_0^{-\ell}(\mathbb{R}^{\infty})$ is its dual.

For $W^{\ell}(\mathbb{R}^{\infty})$ we get a similar chain

$$W_0^{\ell}(\mathbb{R}^{\infty}) \subseteq L_2(\mathbb{R}^{\infty}) \subseteq W_0^{-\ell}(\mathbb{R}^{\infty}) \quad \text{for} \ \ \ell = 1, 2, \ldots$$

where

$$\|u\|_{W_0^\ell(\mathbb{R}^\infty)} \geq \|u\|_{L_2(\mathbb{R}^\infty)} \geq \|u\|_{W_0^{-\ell}(\mathbb{R}^\infty)}.$$

Let us denote by $L_2(0,T;W_0^\ell(\mathbb{R}^\infty))$ the space of measurable functions $f(t)$ on $[0,T]$ with a norm

$$\|f\|_{L_2(0,T;W_0^\ell(\mathbb{R}^\infty))} = \left(\int_0^T \|f(t)\|^2_{W_0^\ell(\mathbb{R}^\infty)}dt\right)^{\frac{1}{2}} < +\infty$$

and with a scalar product (cf. [11])

$$(f,g)_{L_2(0,T;W_0^\ell(\mathbb{R}^\infty))} = \int_0^T (f(t),g(t))_{W_0^\ell(\mathbb{R}^\infty)}dt.$$

Then we denote

$$Q = R^\infty \times [0,T], \quad \Sigma = \Gamma \times [0,T], \quad L_2(Q) = L_2(0,T;L_2(\mathbb{R}^\infty)).$$

As a conjugate space to $L_2(0,T;W^\ell(\mathbb{R}^\infty))$ we have $L_2(0,T;W_0^{-\ell}(\mathbb{R}^\infty))$ and we get a chain of inclusions (cf. [11])

$$L_2(0,T;W_0^\ell(\mathbb{R}^\infty)) \subseteq L_2(0,T;L_2(\mathbb{R}^\infty)) \subseteq L_2(0,T;W_0^{-\ell}(\mathbb{R}^\infty)).$$

II. Formulation of the problem. Consider the following optimization problem:

$$I(y,u) = \int_Q f(x,t,y,u)d\rho(x)dt \longrightarrow \min \tag{1}$$

under the constraints

$$\frac{\partial y}{\partial t} + A(t)y = u, \qquad x \in \mathbb{R}^\infty, \quad t \in (0,T), \tag{2}$$

$$y(x,t) = 0, \qquad x \in \Gamma, \quad t \in (0,T), \tag{3}$$

$$y(x,0) = y_0(x), \quad x \in \mathbb{R}^\infty, \tag{4}$$

$$y(x,T) = y_1(x), \quad x \in \mathbb{R}^\infty, \tag{5}$$

$$u \in U, \tag{6}$$

where $f : \mathbb{R}^n \times \mathbb{R}^1_+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, U is a closed, convex subset with nonempty interior of the space $L_2(Q)$, $y_0(x)$, $y_1(x)$ are fixed functions of the space $L_2(\mathbb{R}^\infty)$.

The operator $A(t)$ has the form

$$(A(t))u(x) = -\sum_{k=1}^\infty \frac{1}{P_k(x_k,t)}\frac{\partial^2}{\partial x_k^2}\sqrt{P_k(x_k,t)}u(x) + q(x,t)u(x),$$

where $q(x,t)$ is a real-valued function in x, bounded and measurable on $\mathbb{R}^\infty$ such that $q(x,t) \geq C > 0$, C is a constraint. $A(t)$ is a bounded self-adjoint elliptic operator with an infinite number of variables mapping $W_0^1(\mathbb{R}^\infty)$ onto $W_0^{-1}(\mathbb{R}^\infty)$.

It has been proved (cf. [3-4]) that $A(t)$ satisfies the condition

$$\big(A(t)u, v\big)_{L_2(\mathbb{R}^\infty)} \geq \lambda \|u\|_{W_0^1(\mathbb{R}^\infty)}, \qquad \lambda > 0$$

for $u \in W_0^1(\mathbb{R}^\infty)$ (coercivity condition).

Also for all $u, v \in W_0^1(\mathbb{R}^\infty)$ the function $(A(t)u, v)$ is continuously differentiable with respect to t on $[0, T]$.

Under the above assumptions, in view of Lions' results (cf. [10] or Theorem 3.1 from [9]) we have that equation (2) with boundary condition (3) and with an initial condition (4) has a unique solution in the space $W(0, T)$, where

$$W(0,T) = \{f : f \in L_2(0,T; W_0^1(\mathbb{R}^\infty), \ \frac{df}{dt} \in L_2(0,T; W_0^{-1}(\mathbb{R}^\infty))\}.$$

In addition, the solution continuously depends on the right side of (2) and on the initial condition (4).

In addition we assume that

(i) there exist Fréchet derivatives $f_y(x,t,y,u)$ and $f_u(x,t,y,u)$,

(ii) f, f_y, f_u are continuous with respect to (y, u) for any $(x,t) \in \mathbb{R}^\infty \times (0,T)$ and measurable with respect to (x,t).

We will call the problem (1)-(6) under assumptions (i)-(ii), problem I. Denote by $Y = L_2(0,T; W_0^1(\mathbb{R}^\infty))$ the space of states.

Let us introduce an operator

$$F_2 : Y \times L_2(Q) \to L_2(0,T; W^{-1}(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty) \times L_2(0,T; W_0^1(\mathbb{R}^\infty))$$

of the form

$$F_2(y,u) = \big(\tfrac{\partial y}{\partial t} + A(t)y - u, \ y(x,0) - y_0(x), \ y(x,t)\big|_{x \in \Gamma}\big). \tag{7}$$

Then by F_3 let us denote an operator

$$F_3 : Y \times L_2(Q) \to L_2(\mathbb{R}^\infty)$$

of the form

$$F_3(y,u) = \big(y(x,T) - y_1(x)\big). \tag{8}$$

Then the Lagrange function for problem I will be of the form

$$L(y, u, \lambda_0, y_2^*, y_3^*) = \lambda_0 I(y,u) + \big(y_2^*, F_2(y,u)\big) + \big(y_3^*, F_3(y,u)\big),$$

where

$$y_2^* \in \big(L_2(0,T; W^{-1}(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty) \times L_2(0,T; W_0^1(\mathbb{R}^\infty))\big)^*, \ \ y_3^* \in L_2(\mathbb{R}^\infty), \ \lambda \in \mathbb{R}_0.$$

Now, proceeding analogously as in [13]; i.e., using the extension of the Dubovitskii-Milyutin method in the case of n equality constraints in the operator form from [12] we can prove the local extremum principle for problem I.

Theorem 1. (Local extremum principle). *If*

1) (y^0, u^0) *is an optimal process in problem I, and*

2) *an operator*

$$B : Y \times L_2(Q) \to L_2(0,T; W_0^{-1}(\mathbb{R}^\infty) \times L_2(\mathbb{R}^\infty) \times L_2(0,T; W_0^1(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty)$$

of the form

$$B(\bar{y}, \bar{u}) = \left(\tfrac{\partial \bar{y}}{\partial t} + A\bar{y} - \bar{u},\ \bar{y}(x,0), \bar{y}(x,t)\big|_{x\in\Gamma}, \bar{y}(x,T)\right)$$

has a closed image in

$$L_2(0,T; W^{-1}(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty) \times L_2(0,T; W_0^1(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty),$$

then there exists $\lambda_0 \geq 0$ *and a function* $p(\cdot)$ *satisfying the adjoint equation*

$$\frac{\partial p}{\partial t} = -\lambda_0 f_y + A^*(t)p \quad \text{for } x \in \mathbb{R}^\infty,\ \ t \in (0,T) \tag{9}$$

with the boundary condition

$$p(x,t) = 0 \quad \text{for } x \in \Gamma,\ \ t \in (0,T) \tag{10}$$

such that the extremum condition

$$\int_Q (p + \lambda_0 f_u)(u - u^0) d\rho(x)\, dt \geq 0 \tag{11}$$

holds for any $u \in U$.

Proof: We shall make use of the extension of the Dubovitskii-Milyutin method from [12] in a way similar to [13]. Then we have to define an inequality constraint set

$$Z_1 = \{(y,u) \in Y \times L_2(Q) : u \in U\} \tag{12}$$

and two equality constraint sets

$$Z_i = \{(y,u) \in Y \times L_2(Q) : F_i(y,u) = 0\} \quad \text{for } i = 2,3, \tag{13}$$

where the operators F_2, F_3 are given by (7) and (8) respectively.

According to the Dubovitskii-Milyutin formalism we have to find the conical approximations for Z_i, $i = 1,2,3$, as well as a certain cone related to the functional I.

We have to evaluate the following cones: $DC(I(y^0,u^0))$, the cone of directions of decrease of the functional I at the point (y^0,u^0); $FC(Z_1,(y^0,u^0))$, the cone of feasible directions for the set Z_1 at the point (y^0,u^0); $TC(Z_i,(y^0,u^0))$, the cone of tangent directions to the set Z_i at the point (y^0,u^0) for $i = 2,3$; and the adjoint (or dual) cones to them (cf. [5]).

In view of assumptions (i) and (ii), functional I given by (1) is Frèchet differentiable and proceeding in a way similar to [5] Section 7, we can show that the Frèchet derivative of I is of the form

$$I'(y^0,u^0)(\bar{y},\bar{u}) = \int_Q (f_y\bar{y} + f_u\bar{u}) d\rho(x)\, dt.$$

Then, in view of [5] Section 7, we have that

$$DC(I,(y^0,u^0)) = \{(\bar{y},\bar{u}) : \textstyle\int_Q (f_y\bar{y} + f_u\bar{u})d\rho(x)dt < 0\}$$

and if $DC(I(y^0,u^0) \neq \emptyset$ the adjoint cone will be of the form (cf. [5], Section 10):

$$\begin{gathered}\big(DC(I,(y^0,u^0))\big)^* = \\ \{f_0 \in (Y \times L_2(Q))^* : f_0(\bar{y},\bar{u}) = -\lambda_0 \textstyle\int_Q (f_y\bar{y} + f_u\bar{u})d\rho(x)dt, \ \ \lambda_0 \geq 0\}.\end{gathered} \tag{14}$$

By using the same arguments as in [5] Section 10 we get that

$$\big(FC(Z_1,(y^0,u^0))\big)^* = \{f_1 \in (Y \times L_2(Q))^* : f_1(\bar{y},\bar{u}) = f_1'(\bar{u})\} \tag{15}$$

where f_1' is a functional supporting the set U at the point u^0. Then let us consider an equality constraint Z_2 given by (13) with an operator F_2 in the form (7). The operator F_2 is Frèchet differentiable and its derivative has the form:

$$F_2'(y^0,u^0)(\bar{y},\bar{u}) = \Big(\frac{\partial \bar{y}}{\partial t} + A(t)\bar{y} - \bar{u}, \bar{y}(x,0), \bar{y}(x,t)\big|_{x\in\Gamma}\Big). \tag{16}$$

From [9], Theorem 3.1, in view of assumptions on the operator $A(t)$ we have that all the assumptions of the Lusternik theorem are satisifed for the operator F_2 at the point (y^0,u^0) and then the tangent cone will be given by the following subspace:

$$\begin{gathered}TC\big(Z_2,(y^0,u^0)\big) = \\ \{(\bar{y},\bar{u}) \in Y \times L_2(Q) : \tfrac{\partial \bar{y}}{\partial t} + A(t)\bar{y} - \bar{u} = 0, \ \bar{y}(x,0) = 0, \ \bar{y}(x,t)\big|_{x\in\Gamma} = 0\}.\end{gathered} \tag{17}$$

Then we shall consider the set Z_3 given by (13) with the operator F_3 given by (8). The operator F_3 is Frèchet differentiable, satisfies assumptions of the Lusternik theorem and the tangent cone will have the form of the following subspace

$$TC\big(Z_3,(y^0,u^0)\big) = \{(\bar{y},\bar{u}) \in Y \times L_2(Q) : \bar{y}(x,T) = 0\}. \tag{18}$$

The procedure of finding the adjoint cone to (18) will be similar to [13].

Since the tangent cone given by (18) is a subspace, the adjoint cone will be an annihilator of it; i.e.,

$$\big(TC\big(Z_3,(y^0,u^0)\big)\big)^* = \big(TC\big(Z_3,(y^0,u^0)\big)\big)^\perp. \tag{19}$$

An operator $\wedge : Y \times L_2(Q) \to L_2(\mathbb{R}^\infty)$ in the form

$$\wedge(\bar{y},\bar{u}) = \bar{y}(x,T) \tag{20}$$

satisifes the assumptions of the annihilator lemma (cf. [6] Section 0.2) and then in view of (18) and (19)

$$\big(TC\big(Z_3,(y^0,u^0)\big)\big)^* = \operatorname{Im}\wedge^*, \tag{21}$$

where $\wedge^*$ denotes an adjoint operator to $\wedge$; i.e., $\wedge^* : L_2(\mathbb{R}^\infty) \to (Y \times L_2(Q))^*$.

Consider arbitrary $f_3 \in \big(TC\big(Z_3, (y^0, u^0)\big)\big)^*$. Then in view of (21) and from the form of a linear and continuous functional on $L_2(\mathbb{R}^\infty)$ we have that there exists a function $\phi \in L_2(\mathbb{R}^\infty)$ such that

$$f_3(\bar{y}, \bar{u}) = (\wedge^*\phi, (\bar{y}, \bar{u})) = (\phi(x), \bar{y}(x,T)) = \int_\Omega \phi(x)\bar{y}(x,T)d\rho(x). \tag{22}$$

Finally we define an operator

$$F : Y \times L_2(Q) \to L_2(0,T; W^{-1}(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty) \times L_2(0,T; W_0^1(\mathbb{R}^\infty)) \times L_2(\mathbb{R}^\infty)$$

of the form

$$F(y,u) = (F_2(y,u), F_3(y,u)) \tag{23}$$

where F_2, F_3 are given by (7) and (8).

The operator F is Frèchet differentiable and its Frèchet derivative is operator B from assumption 2 of the theorem.

Thus all the assumptions of Theorem 1 from [12] are satisifed. Then the Euler-Lagrange equation for problem I will have a form: there exist not vanishing simultaneously functionals:

$$f_0 \in \big(DC(I, (y^0, u^0))\big)^*, \quad f_1 \in \big(FC(Z_1, (y^0, u^0))\big)^*,$$
$$f_i \in \big(TC(Z_i, (y^0, u^0))\big)^*, \quad i = 2, 3$$

such that

$$f_0 + f_1 + f_2 + f_3 = 0. \tag{24}$$

Applying the formulas (14), (15) and (22) we will get (24) in the form

$$-\lambda_0 \int_Q (f_y\bar{y} + f_u\bar{u})d\rho(x)dt + f_1'(\bar{u}) + f_2(\bar{y}, \bar{u}) + \int_{\mathbb{R}^\infty} \phi(x)\bar{y}(x,T)d\rho(x) = 0 \tag{25}$$

for every $(\bar{y}, \bar{u}) \in Y \times L_2(Q)$.

Consider the above equation for a pair $(\bar{y}, \bar{u}) \in TC(Z_2, (y^0, u^0))$; i.e., $f_2(\bar{y}, \bar{u}) = 0$, then applying in (25) adjoint equation and (14) we get

$$f_1'(\bar{u}) = \lambda_0 \int_Q f_y\bar{y}d\rho(x)dt + \lambda_0 \int_Q f_u\bar{u}d\rho(x)dt - \int_{\mathbb{R}^\infty} \phi(x)\bar{y}(x,T)d\rho(x) \tag{26}$$

where

$$\begin{aligned}
\lambda_0 \int_Q f_y\bar{y}d\rho(x)dt &= \int_Q \big(-\frac{\partial p}{\partial t} + A^*(t)p\big)\bar{y}d\rho(x)dt \\
&= -\int_Q \frac{\partial p}{\partial t}\bar{y}d\rho(x)dt + \int_Q pA(t)\bar{y}d\rho(x)dt \\
&= \int_{\mathbb{R}^\infty} \big(p(x,0)\bar{y}(x,0) - p(x,T)\bar{y}(x,T)\big)d\rho(x) \\
&+ \int_Q p(x,t)\frac{\partial \bar{y}}{\partial t}d\rho(x)dt + \int_Q p(x,t)A(t)\bar{y}d\rho(x)dt \\
&= -\int_{\mathbb{R}^\infty} p(x,T)\bar{y}(x,T)d\rho(x) + \int_Q p(x,t)\bar{u}d\rho(x)dt.
\end{aligned}$$

Combining (25) and (26) we get

$$f_1'(\bar{u}) = \int_Q (p + \lambda_0 f_u)\bar{u}d\rho(x)dt - \int_{\mathbb{R}^\infty} (\phi(x) + p(x,T))\bar{y}(x,T)d\rho(x). \tag{27}$$

Define

$$\phi(x) = -p(x,T) \quad \text{for } x \in \mathbb{R}^\infty. \tag{28}$$

Combining (27), (28) and from the definition of the functional supporting the set we have the maximum condition.

The proof of the fact that multipliers λ_0 and function $p(\cdot)$ are not vanishing simultaneously is the same as in [13] or [14] Theorem 1.

During the proof we assumed that $DC(I(y^0,u^0)) \neq \emptyset$. Now assume that this condition is not satisfied; i.e., for any $(\bar{y},\bar{u}) \in Y \times L_2(Q)$

$$\int_Q (f_y\bar{y} + f_u\bar{u})d\rho(x)dt = 0.$$

Then it is enough to put $\lambda_0 = 1$, $p = 0$, $\phi = 0$, $f_1' = 0$ to get (25). Then the proposition can be obtained in the same way as in the previous case, which completes the proof.

IV. Formulation of the problem with nonoperator equality constraint. Consider the problem I without the operator equality constraint (5) but also without the assumption that the set U possesses a nonempty interior in $L_2(Q)$. Then the problem, let us call it problem II, will contain two equality constraints: the set Z_1 given by (12) and the set Z_2 given by (13). The set Z_1 is a nonoperator equality constraint, so in a way similar to [14] we can use some generalization of the Dubovitskii-Milyutin method for the case of n equality constraints in arbitrary form from [15].

Let C_0 denotes a cone contained in the cone of decrease of the functional I at x_0; C_i, $i = 1,2,\dots,k$, denote cones contained in feasible cones to sets Z_i at x_0, respectively; C_i, $i = k+1,\dots,n$, denote cones contained in tangent cones to sets Z_i at x_0, respectively.

Theorem 4.1. (Walczak [15]). *Assume that*

(i) *the cones C_i, $i = 0,1,\dots,k$, are open and covex;*

(ii) *the cones C_i, $i = k+1,\dots,n$, are convex and closed;*

(iii) *the cone $\tilde{C} = \bigcap_{i=k+1}^n C_i$ is contained in a cone tangent to the set $V = \bigcap_{i=k+1}^n Z_i$;*

(iv) *the cones C_i^*, $i = k+1,\dots,n$, are either of the same sense or of the opposite sense;*

(v) *x_0 is a solution of the problem*

$$\min \left\{ I(x),\ x \in \bigcap_{i=1}^n Z_i \right\};$$

then there exist functionals $f_i \in C_i^*$, $i = 0, 1, \dots, n$, *not vanishing simultaneously, such that*

$$f_0 + f_1 + \dots + f_n = 0.$$

The above generalization of the Dubovitskii-Milyutin theorem from [15] is based on the definitions of the cones of the same sense and of the opposite sense which are introduced there (see Definitions 2.1 and 2.2 from [15]). But for the purpose of our problem II we are going to use the following sufficient condition for two cones to be of the same sense.

Theorem 3.3. (Walczak [15]). *Let* C_1 *be a cone of the form* $C_1 = \{(x_1, y_1) \in X \times Y : x_1 = Ay_1\}$, $C_2 = X \times \tilde{C}_2$, *where* $\tilde{C}_2$ *is a cone in* Y (X, Y*-normed spaces*). *If the operator* A *is linear and continuous, then*

$$C_1^* = \{(x_1^*, y_1^*) \in (X \times Y)^* : y_1^* = -A^* x_1^*\}$$
$$C_2^* = \{(0, y_2^*) \in (X \times Y)^* : y_2^* \in \tilde{C}_2^*\}$$

and the cones C_1^*, C_2^* *are of the same sense.*

By using above results from [15] we can prove the extremum principle for the problem II.

Theorem 2. (The extremum principle). *Let* (y^0, u^0) *be an optimal process in problem II. Then there exist some* $\lambda_0 \geq 0$ *and a function* $p(\cdot)$ *not vanishing simultaneously satisfying the adjoint equation*

$$\frac{\partial p}{\partial t} = -\lambda_0 f_y + A^*(t)p \quad \text{for } x \in \mathbb{R}^\infty, \ t \in (0, T)$$

with boundary condition

$$p(x, t) = 0 \quad \text{for } x \in \Gamma, \ t \in (0, T)$$

and with terminal condition

$$p(x, T) = 0 \quad \text{for } x \in \mathbb{R}^\infty$$

such that the maximum condition from Theorem 1 *holds.*

Proof: Let sets Z_1, Z_2 be defined as before by (12) and (13), respectively. According to Theorem 4.1 from [15], in the framework of the conical approximations for the Problem II we will have one open and convex cone (i.e., $k = 0$) precisely the cone of decrease of the functional I at the point (y^0, u^0), $DC(I, (y^0, u^0))$ with the adjoint one given by (14), and two closed and convex cones (i.e., $n = 2$); i.e., tangent cones to the sets Z_1 and Z_2 respectively, where $TC(Z_2, (y^0, u^0))$ given by formula (17).

According to (12) the tangent cone to the set Z_1 at (y^0, u^0) can be expressed as

$$TC(Z_1, (y^0, u^0)) = Y \times \tilde{C}_1, \tag{29}$$

where $\tilde{C}_1 \subset L_2(Q)$ denotes the tangent cone to the set U at the point u^0. On the base of Theorem 10.5 from [5], the adjoint cone to $TC(Z_1, (y^0, u^0))$ will be of the form (15).

Proceeding as in [14], we have to use Theorem 3.3 from [15], to show that $\big(TC(Z_1, (y^0, u^0))\big)^*$ given by (15) and $\big(TC(Z_2, (y^0, u^0))\big)^*$ are of the same sense. (Note that we do not need to determine the explicit form of $\big(TC(Z_2, (y^0, u^0))\big)^*$ in order to derive this conclusion.) It is enough to use the theorem about the existence and the uniqueness of the solution for the parabolic equation with infinitely many variables with the boundary condition and with the initial condition (cf. [10] or [9], Theorem 3.1) to the system of equations which determine $TC(Z_2, (y^0, u^0))$ in (17). According to this theorem the solution of such a system depends continuously on the right side; i.e., in our case on u, so we can rewrite the cone given by (17) in the form

$$TC(Z_2, (y^0, u^0)) = \big\{(\bar{y}, \bar{u}) \in Y \times L_2(Q) : \bar{y} = M\bar{u}\big\}, \tag{30}$$

where $M : L_2(Q) \to Y$ is a linear and continuous operator. Then, applying Theorem 3.3 from [15] to the cones given by (29) and (30), we get that the assumption (iv) of Theorem 4.1 from [15] is satisfied.

The proof of the fact that the assumption (iii) of Theorem 4.1 is satisfied is analogous to the one of Theorem 2 from [14] with, of course, different meanings of the spaces Y, $L_2(Q)$ and the operator M.

Since all the assumptions of Theorem 4.1 from [15] are satisfied, there exist functionals, not all zero,

$$f_0 \in \big(DC(I, (y^0, u^0))\big)^*, \quad f_i \in \big(TC(Z_i, (y^0, u^0))\big)^*, \quad i = 1, 2$$

such that

$$f_0 + f_1 + f_2 = 0. \tag{31}$$

Now, in a way similar to Theorem 1, it is enough to use (14), (15) and (17) in (31) to get

$$-\lambda_0 \int_Q (f_y \bar{y} + f_u \bar{u}) d\rho(x) dt + f_1'(\bar{u}) = 0 \tag{32}$$

for all $(\bar{y}, \bar{u}) \in TC(Z_2, (y^0, u^0))$. Then by using (26) we can rewrite (32) in the form

$$f_1'(\bar{u}) = \int_Q (p + \lambda_0 f_u) \bar{u} d\rho(x) dt - \int_{\mathbb{R}^\infty} p(x, T) \bar{y}(x, T) d\rho(x). \tag{33}$$

Then making use of the condition $p(x, T) = 0$ for all $x \in \mathbb{R}^\infty$ and applying the definition of the functional supporting the set in (33) we get the extremum condition of the proposition.

The proof of the fact that the multipliers λ_0 and $p(\cdot)$ are not vanishing simultaneously, as well as the proof of the case $DC(I, (y^0, u^0)) = \emptyset$, is the same as in [14] Theorem 2.

IV. Sufficient condition of optimality. Let us consider problem I with the following additional assumptions:

(iii) $f(x, t, y, u)$ is continuous with respect to (x, t, y, u);

(iv) there exist $f_y(x,t,y,u)$ and $f_u(x,t,y,u)$ which are continuous with respect to (x,t,y,u);

(v) $f(x,t,\cdot,)$ is strictly convex with respect to the pair (y,u); i.e.,

$$f(x,t,\lambda y_1+(1-\lambda)y_2,\lambda u_1+(1-\lambda)u_2)<\lambda f(x,t,y_1,u_1)+(1-\lambda)f(x,t,y_2,u_2)$$

for all $y_1,y_2,u_1,u_2\in\mathbb{R}$, $(y_1,u_1)\neq(y_2,u_2)$, $\lambda\in(0,1)$;

(vi) there exists a control $\tilde{u}\in\operatorname{int}U$ and a trajectory $\tilde{y}\in Y$ satisfying parabolic equation (2) with constraints (3), (4) and (5).

We can prove the sufficient condition by using the sufficient condition of optimality via the Dubovitskii-Milyutin method in the case of n equality constraints in the operator form from [13].

Theorem 2. (cf. [13]). *Let*

1) *$G(x)$ be a convex and continuous functional on X;*
2) *$Z_1,Z_2,\dots,Z_{n+1},\dots,Z_{n+m}$ be convex sets in X and*
3) *there exists $\tilde{x}$ such that $\tilde{x}\in Z_i^0$, $i=1,2,\dots,n$, $\tilde{x}\in Z_{n+j}$ for $j=1,2,\dots,m$;*
4) *$x_0\in Z=\bigcap_{i=1}^{n+m}Z_i$;*
5) *C_0 be a cone of directions of decrease of the functional G at the point x_0;*
6) *$C_1,C_2,\dots,C_n$ be cones of feasible directions for the sets $Z_1,Z_2,\dots,Z_n$ at the point x_0;*
7) *the equality constraints Z_j, $j=n+1,\dots,n+m$, be of the form $Z_j=\{x\in X:F_j(x)=0\}$ for $j=n+1,\dots,n+m$, where $F_j:X\to Y_j$ are continuously differentiable operators regular and functionally independent at the point x_0.*

Then the necessary and sufficient condition for the fact that the point x_0 is the minimum point of the functional G on the set Z is the following: there exist linear and continuous functionals $f_i\in C_i^*$ for $i=0,1,\dots,n+m$, not vanishing simultaneously, such that

$$f_0+f_1+\ldots+f_{n+1}+\ldots+f_{n+m}=0. \tag{34}$$

Theorem 3. *Let the operator D of the form*

$$D(\bar{y},\bar{u})=\Big(\frac{\partial\bar{y}}{\partial t}+A\bar{y}-\bar{u},\bar{y}(x,0),\bar{y}(x,t)\big|_{x\in\Gamma},\bar{y}(x,T)\Big)$$

maps $Y\times L_2(Q)$ onto

$$L_2(0,T;W^{-1}(\mathbb{R}^\infty))\times L_2(\mathbb{R}^\infty)\times L_2(0,T;W_0^1(\mathbb{R}^\infty))\times L_2(\mathbb{R}^\infty).$$

Then the process (y^0,u^0) is optimal if and only if there exists a function p satisfying the equation

$$\frac{\partial p}{\partial t}=-f_y+A^*(t)p \quad \text{for } x\in\mathbb{R}^\infty,\ \ t\in(0,T)$$

with a boundary condition

$$p(x,t)=0 \quad \text{for } x\in\Gamma,\ \ t\in(0,T)$$

such that the maximum condition

$$\int_Q (p + f_u)(u - u^0) d\rho(x) dt \geq 0$$

holds for any $u \in U$.

Proof: In order to get the sufficient condition of optimality for problem I with additional assumptions (iii)–(vi) we have to show that all the assumptions of Theorem 2 from [13] are satisfied.

Assumptions 1), 2) and 3) of Theorem 2 follow immediately for the assumptions (iii), (iv) and (vi) respectively of the problem.

Consider operators F_2, F_3 given by (7) and (8), respectively. The Frèchet derivative of F_2 is given by (16) and in view of Theorem 3.1 from [9] the operator F_2 is regular at the point (y^0, u^0). The Frèchet derivative of the operator F_3 of the form $\bar{y}(x,T) : Y \times L_2(Q) \to L_2(\mathbb{R}^\infty)$ is an operator "onto," so F_2 is regular at (y^0, u^0), too. Define an operator

$$F : Y \times L_2(Q) \to L_2\big(0,T; W^{-1}(\mathbb{R}^\infty)\big) \times L_2(\mathbb{R}^\infty) \times L_2\big(0,T; W_0^1(\mathbb{R}^\infty)\big) \times L_2(\mathbb{R}^\infty))$$

in the form

$$F(y,u) = \big(F_2(y,u), F_3(y,u)\big)$$

with F_2 and F_3 given by (7) and (8) respectively. The operator F is Frèchet differentiable at (y^0, u^0) and its derivative has a form of the operator D from the assumption. Thus F is regular at (y^0, u^0) and in view of Lemma 2 from [12] F_2 and F_3 are functionally independent at (y^0, u^0).

Then all the assumptions of Theorem 2 from [13] are satisfied. Making use of this theorem and proceeding indentically as in the proof of the Theorem 1 we obtain the proposition.

REFERENCES

[1] I.M. Gali and A.H. El-Saify, *Optimal control of a system governed by a hyperbolic operator with an infinite number of variables,* J. Math. Anal. Appl., 85 (1982), 24–30.

[2] I.M. Gali and A.H. El-Saify, *Distributed control of a system governed by Dirichlet or Neumann problems for self-adjoint elliptic operators with an infinite number of variables,* J. Opt. Theory Appl., 39 (1983), 293–298.

[3] I.M. Gali and A.H. El-Saify, *Distributed control of a system governed by the Dirichlet and Neumann problems for a self-adjoint elliptic operator with an infinite number of variables,* Proc. Fifth Annual Operations Research Conference, Tagasig University, 1978.

[4] I.M. Gali and A.H. El-Saify, *Time optimal control of a system governed by a second order elliptic or parabolic operator with an infinite number of variables,* Proc. Fourteenth Annual Conference in Statistics, Computer Science, Operations Research and Mathematics, Cairo University, Vol. 5, 1979.

[5] I.V. Girsanov, "Lectures on the Mathematical Theory of Extremum Problems," Springer-Verlag, New York, 1972.

[6] A.D. Ioffe and V.M. Tikhomirov, "Theory of Extremal Problems," North Holland 1979.

[7] A. Kowalewski and W. Kotarski, *Application of the Milyutin-Dubovicki's method to solving an optimal control problem of hyperbolic systems,* Problems of Control and Information Theory, 9 (1980), 183–193.

[8] W. Kotarski and A. Kowalewski, *An optimal control problem with initial state not a priori given,* Problems of Control and Information Theory, 12 (1983), 349–359.

[9] W. Kotarski, *Optimal control of a system governed by a parabolic equation with an infinite number of variables,* J. Opt. Theory Appl., 60 (1989), 33–41.

[10] J.L. Lions, "Optimal Control of Systems Governed by Partial Differential Equations," Springer Verlag, New York, 1972.

[11] J.L. Lions and E. Magenes, "Nonhomogeneous Boundary Value Problems and Applications," I, II, Springer-Verlag, New York, 1972.

[12] U. Ledzewicz-Kowalewska, *On some specification of the Dubovitskii-Milyutin method,* Nonlinear Anal., 12 (1986), 1367–1371.

[13] U. Ledzewicz-Kowalewska, *The necessary and sufficient condition for the problems of optimal control of parabolic systems with equality constraints,* Bulletin de La Société des Sciences et des Lettres de Lódź, XXXVI (1986), 1–12.

[14] U. Ledzewicz-Kowalewska, *Optimal control problems of hyperbolic systems with operator and nonoperator equality constraints,* Applicable Analysis, 27 (1988), 199–215.

[15] S. Walczak, *On some properties of cones in named spaces and their application to investigating extremal problems,* J. Opt. Theory Appl., 42 (1984), 561–582.