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VIABILITY OF BOUNDARY OF THE VIABILITY KERNEL

PATRICK SAINT-PIERRE
Ceremade, Université Paris-Dauphine
Place du Maréchal de Lattre de Tassiny, 75775 Paris cedex 16, France

(Submitted by G. Da Prato)

Abstract. Viability theory allows the study of the dynamics of a system described through a set-valued map F and for which we look for trajectories which remain in a subset K. These are called *viable solutions* of the system. When F is upper semicontinuous, viability theorems state that, under some assumptions, for any initial state x_0, there exists a solution starting at x_0 and this solution is viable in K if and only if K is a viability domain; i.e., $\forall\, x \in K$, $F(x) \cap T_K(x) \neq \emptyset$. When K is not a viability domain, we study the largest viability domain of K, $Viab_K(F)$. We prove that its boundary enjoys the property of local viability at any point of the interior of K.

1. Introduction and definitions. If we know numerous results about local or global existence of solutions to differential inclusions and their viability in a subset (see Haddad [9], Aubin [2], [4], [5], Frankowska [4], [8]), and their approximation (see Aubin [6], Saint-Pierre [13], [14]), we are faced with new questions when we study the set of solutions, the target problem or the invariant and viability domains of a subset K associated to a system described by the differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \text{for almost all } t \geq 0, \\ x(0) = x_0 \in K, \\ x(t) \in K, & \forall\, t \geq 0, \end{cases} \tag{1}$$

where F is a set-valued map defined from a closed subset K of a finite dimensional vector space X to X and $K \subset \mathrm{Dom}\,(F)$, a closed subset of X.

We denote by $S_F(X)$ the set of solutions to the differential inclusion:

$$\dot{x}(t) \in F(x(t)), \quad \text{for almost all } t \geq 0, \quad x(0) = x; \tag{2}$$

$$S_F(X) = \left\{ x(\cdot) \in W^{1,1}(0, +\infty; X, e^{-bt}dt) : \;\; x(\cdot) \text{ is a solution of (2)} \right\},$$

where $W^{1,1}(0, +\infty; X, e^{-bt}dt)$ is the set of absolutely continuous functions defined by

$$\left\{ x(\cdot) \in L^1(0, +\infty; X, e^{-bt}dt) : \;\; \dot{x}(\cdot) \in L^1(0, +\infty; X, e^{-bt}dt) \right\}$$

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supplied with the topology for which any sequence $x_n(\cdot)$ converges to $x(\cdot)$ if and only if

$$\begin{cases} i) & x_n(\cdot) \text{ converges uniformly on any compact } [0,T] \text{ to } x(\cdot), \\ ii) & \dot{x}_n(\cdot) \text{ converges weakly in } L^1(0,+\infty; X, e^{-bt}dt) \text{ to } \dot{x}(\cdot). \end{cases} \quad (3)$$

The solutions we are looking for are those which are viable; i.e., which remain always in K. We want to investigate the subset of initial points in K from which there exists at least one viable solution.

Definition 1.1. (Viability Properties). A subset M is locally viable under F (or enjoys the local viability property for the map F) if for any initial point $x_0 \in M$, there exist $T > 0$ and at least one solution to the differential inclusion (1), viable in M; *i.e.*, $x(t) \in M$, $\forall t \in [0,T]$.

A subset M is viable under F (or enjoys the global viability property for the map F) if for any initial point $x_0 \in M$, there exists at least one solution issued from x_0 and viable in M, $\forall t \in [0,+\infty)$.

Let $T_K(x)$ be the contingent cone[1] of K at a point x.

Definition 1.2. (Viability Domain). A subset M is a viability domain of F if

$$\forall x \in M, \quad F(x) \cap T_M(x) \neq \emptyset. \quad (4)$$

Definition 1.3. (Marchaud map). A set-valued map F from X to Y is a Marchaud map if

$$\begin{cases} \forall x \in Y, \quad F(x) \text{ is a non empty convex compact set;} \\ F \text{ is upper semicontinuous;} \\ F \text{ has linear growth; i.e., } \exists\, c > 0 \ \|F(x)\| \leq c(\|x\| + 1), \quad \forall x \in X. \end{cases}$$

If the set-valued map F is a Marchaud map, a subset K enjoys the global viability property if and only if K is a viability domain of F. (See Haddad [9].)

We can define now the viability kernel of a subset K.

Definition 1.4. (Viability Kernel). Let K be a subset of the domain of a set-valued map F. The viability kernel of K under F is the largest subset of K enjoying the global viability property and we denote it by $Viab_F(K)$.

We recall the following characterisation of the viability kernel:

If $\mathcal{K} \subset W^{1,1}(0,+\infty; X, e^{-bt}dt)$ is the closed subset of functions viable in K, then

$$Viab_F(K) = \{x \in K \text{ such that } S_F(x) \cap \mathcal{K} \neq \emptyset\}.$$

Since $Viab_F(K)$ is the largest subset of K enjoying the global viability property, any solution of (1) starting from any initial point $x_0 \in K \backslash Viab_F(K)$ never meets the viability kernel $Viab_F(K)$.

Moreover any solution of (1) which is not in $Viab_F(K)$ must leave K in a finite time.

[1] the contingent cone at a point z to a set $K \subset X$ is defined by:
$T_K(z) := \{w \in X \text{ such that } \liminf_{h \to 0^+} \frac{d_K(z+hw)}{h} = 0\}$

Let us consider the boundary $\partial Viab_F(K)$ of $Viab_F(K)$. When the set valued map F is Marchaud and Lipschitzian, the set $S_F(x)$ of solutions starting from any point $x \in \partial Viab_F(K) \cap \overset{\circ}{K}$ is locally contained in $\partial Viab_F(K)$ (see Quincampoix [11] Theorem 3.1). Quincampoix has proved a "semi-permeability" property of the boundary $\partial Viab_F(K)$ of the viability kernel $Viab_F(K)$. This means that for any $x_0 \in K$, every solution viable in K starting from the boundary of the viability kernel remains in the viability kernel.

Our purpose in this paper is to show that, when F is a Marchaud set-valued map, the boundary of the viability kernel is viable under F; that is to say that, for any $x \in \partial Viab_F(K)$, there exists a solution which remains in $\partial Viab_F(K)$. But there may also exists solution which enters the interior of the viability kernel:

$$S_F(\partial Viab_F(K)) \cap W^{1,1}(0,+\infty; \partial Viab_F(K), e^{-bt}dt) \neq \emptyset.$$

For any function $x(\cdot) \in W^{1,1}(0,+\infty; X, e^{-bt}dt)$ we define the *egress time* of $x(\cdot)$ on ∂K :

$$\theta_K(x(\cdot)) = \inf_{t\geq 0}\{t \text{ such that } x(t) \notin \overset{\circ}{K}\},$$

and the *exit time* of $x(\cdot)$ from K:

$$\tau_K(x(\cdot)) = \inf_{t\geq 0}\{t \text{ such that } x(t) \notin K\};$$

and for any point $x \in K$, we consider the time functions defined from X to $\mathbb{R} \cup \{+\infty\}$:

$$\theta_K^\flat(x) = \inf_{x(\cdot)\in S(x)}\{\theta_K(x(\cdot))\}; \quad \tau_K^\sharp(x) = \sup_{x(\cdot)\in S(x)}\{\tau_K(x(\cdot))\}.$$

The function $\theta_K^\flat(\cdot)$ is lower semicontinuous and the function $\tau_K^\sharp(\cdot)$ is upper semicontinuous from K to $[0,+\infty)$ (see Aubin [6], Picard [10]). We always have

$$\forall x \in K, \quad \theta_K^\flat(x) \leq \tau_K^\sharp(x).$$

2. Properties of the viability kernel.

The viability kernel enjoys many properties and we recall some of them we are interested in (see Aubin [6], Picard [10])

Proposition 2.1. *Let $F : X \rightsquigarrow X$ be a Marchaud set-valued map and K a closed subset of Dom(F). Then the following properties hold:*

$$\begin{cases} i) & Viab_F(K) = \{x \in X \text{ such that } \tau_K^\sharp(x) = +\infty\}, \\ ii) & \text{if } K \text{ and Graph}(F) \text{ are convex then } Viab_F(K) \text{ is convex,} \\ iii) & \text{if } \partial K \text{ is viable, then } K \text{ is viable.} \end{cases} \tag{5}$$

Other properties of the viability kernel which may be useful for a numerical approch of the viability kernel are the following:

Lemma 2.1. *Let $F : X \rightsquigarrow X$ be a Marchaud set-valued map and K a closed subset contained in Dom(F). Then*

$$\begin{cases} \text{Let } K_1 \text{ be such that } Viab_F(K) \subset K_1 \subset K; \text{ then} \\ \quad Viab_F(K_1) = Viab_F(K). \end{cases} \tag{6}$$

Proposition 2.2 (M. Quincampoix & P. Saint-Pierre [12]). *Let $F : X \rightsquigarrow X$ be a Marchaud set-valued map and K a closed subset contained in Dom(F). Then*

$$\text{If } X\backslash K \text{ is connected then } X\backslash Viab_F(K) \text{ is connected.} \tag{7}$$

3. Properties of the boundary of the viability kernel. We need now to define another concept of tangent cone.

Definition 3.1. (Dubovitsky-Milliutin cone). The Dubovitsky-Milliutin tangent cone $D_K(x)$ to K at x is defined as follows:

$$D_K(x) = \{v \in X \text{ such that } \exists\, \alpha > 0,\ \exists\, \epsilon > 0 : x+]0, \alpha](v + \epsilon B) \subset K\}.$$

It is quite easy to verify that, for any $x \in \partial K$, the Dubovitsky-Milliutin tangent cone $D_K(x)$ is the complement of the contingent cone $T_{X\backslash K}(x)$:

$$\forall\, x \in \partial K, \quad D_K(x) = X\backslash T_{X\backslash K}(x).$$

In particular, since $D_K(x) \subset T_K(x)$, for any $x \in \partial K$, we have $D_K(x) \cup T_{X\backslash K}(x) \subset T_K(x) \cup T_{X\backslash K}(x)$ and then

$$\forall\, x \in \partial K, \quad T_K(x) \cup T_{X\backslash K}(x) = X. \tag{8}$$

For any $x \in \partial K$, we characterize the contingent cone at x to the boundary ∂K thanks to the Quincampoix' theorem (see [11]):

Theorem 3.1. *Let K be a closed subset of X and $X\backslash K$ the closure of its complement. Then*

$$\forall\, x \in \partial K, \quad T_{\partial K}(x) = T_K(x) \cap T_{X\backslash K}(x)$$

and we can deduce the folowing property:

Corollary 3.1. *Let $x \in \partial K$, $v \in T_K(x)$ and $\hat{v} \in T_{X\backslash K}(x)$. Then the segment $[v, \hat{v}] = \{\lambda v + (1 - \lambda)\hat{v} : \lambda \in [0, 1]\}$ satisfies:*

$$[v, \hat{v}] \cap T_{\partial K}(x) \neq \emptyset.$$

Proof: From (8) we have

$$(T_K(x) \cap [v, \hat{v}]) \cup (T_{X\backslash K}(x) \cap [v, \hat{v}]) = [v, \hat{v}].$$

On the other hand, since $T_K(x)$ and $T_{X\backslash K}(x)$ are closed,

$$(T_K(x) \cap [v, \hat{v}]) \cap (T_{X\backslash K}(x) \cap [v, \hat{v}]) \neq \emptyset;$$

otherwise the closed interval $[v, \hat{v}]$ could be partitioned into two closed subintervals, which is impossible. The result follows from Theorem 3.1.

Theorem 3.2. *Let $F : X \rightsquigarrow X$ a Marchaud set-valued map and K a closed subset of X contained in* Dom(F) *with non-empty interior. Then for all $x \in \partial Viab_F(K) \cap \mathring{K}$ there exists $x(\cdot) \in S(x)$ such that*

$$\forall t < \theta_K(x(\cdot)), \quad x(t) \in \partial Viab_F(K).$$

Proof: Set $K_\infty = Viab_F(K)$. We have to prove that, from any point $x \in \partial K_\infty \cap \mathring{K}$, there exists a viable solution which remains on ∂K_∞ as long as it belongs to $\mathring{K}$; that is, $\partial K_\infty \cap \mathring{K}$ is locally viable. From the local viability theorem, this holds if and only if

$$\forall x \in \partial K_\infty \cap \mathring{K}, \quad F(x) \cap T_{\partial K_\infty}(x) \neq \emptyset. \tag{9}$$

From Theorem 3.1, property (9) is equivalent to

$$\forall x \in \partial K_\infty \cap \mathring{K}, \quad F(x) \cap T_{K_\infty}(x) \cap T_{\hat{K}_\infty}(x) \neq \emptyset. \tag{10}$$

Let us assume $\mathring{K}_\infty \neq \emptyset$. If not, we would have $\partial K_\infty = K_\infty$ and the theorem becomes obvious.

Let $x_0 \in \partial K_\infty \cap \mathring{K}$ and $\eta > 0$ such that $B(x_0, 2\eta) \subset \mathring{K}$. We prove first that $F(x_0) \cap T_{K_\infty}(x_0) \neq \emptyset$. Since $x_0 \in \partial K_\infty \subset K_\infty$, from the definition of the viability kernel, there exists at least one solution $x_1(\cdot) \in S_F(x_0)$ viable in K_∞. Then the necessary condition of the local viability theorem implies that

$$\exists\, y_1 \in F(x_0) \cap T_{K_\infty}(x_0) \neq \emptyset.$$

To prove now that $F(x_0) \cap T_{X \setminus K_\infty}(x_0) \neq \emptyset$, we need to show the following results:

Lemma 3.1. *There exists $\tau_0 > 0$ such that*

$$\forall x \in B(x_0, \eta), \quad \theta^\flat_K(x) \geq \tau_0. \tag{11}$$

Proof: We have to prove that any solution starting from $B(x_0, \eta)$ and reaching the boundary of K in a finite time, i.e., such that $\theta^\flat_K(x) < +\infty$, stays in the interior of K during at least the nonempty time interval $[0, \tau_0]$. Since $B(x_0, 2\eta) \subset K$, it is sufficient to prove this for any solution starting from $B(x_0, \eta)$ and reaching the boundary of $B(x_0, 2\eta)$.

Let $x(\cdot)$ be any one solution in $S_F(x)$ such that $\theta_{B(x_0,2\eta)}(x(\cdot))$ is finite. Since for all $t > 0$ we have

$$x(t) = x + \int_0^t \dot{x}(\sigma)\, d\sigma,$$

in particular

$$\int_0^{\theta_{B(x_0,2\eta)}(x(\cdot))} \dot{x}(\sigma)\, d\sigma = x(\theta_{B(x_0,2\eta)}(x(\cdot))) - x.$$

Since F is compact valued, there exists a constant $c > 0$ such that

$$\sup_{x \in B(x_0,2\eta)} \sup_{z \in F(x)} \|z\| \leq c < +\infty.$$

Then

$$\|x(\theta_{B(x_0,2\eta)}(x(\cdot))) - x\| \leq c \times \theta_{B(x_0,2\eta)}(x(\cdot)).$$

But $x(\theta_{B(x_0,2\eta)}(x(\cdot))) \in \partial B(x_0, 2\eta)$; thus $\|x(\theta_{B(x_0,2\eta)}(x(\cdot))) - x\| > \eta$ and then

$$\begin{cases} \forall\, x \in B(x_0, \eta),\ \ \forall\, x(\cdot) \in S_F(x) \text{ such that } \theta_K(x(\cdot)) < +\infty, \\ \theta_K(x(\cdot)) \geq \theta_{B(x_0,2\eta)}(x(\cdot)) \geq \frac{\eta}{c} = \tau_0. \end{cases} \tag{12}$$

Obviously, (12) holds even if $\theta_{B(x_0,2\eta)}(x(\cdot))$ is equal to $+\infty$.

In particular, if $x \in B(x_0,\eta)\backslash K_\infty$, any solution $x(\cdot) \in S_F(x)$ encounters the boundary of K at a finite time: $\theta_K(x(\cdot)) < +\infty$; and then we have:

$$\forall\, x \in B(x_0, \eta)\backslash K_\infty, \quad +\infty > \theta_K^\flat(x) \geq \tau_0.$$

Lemma 3.2. *For all $x \in B(x_0,\eta)\backslash K_\infty$, for all $x(\cdot) \in S_F(x)$, the following property holds:*

$$\forall\, t \in [0, \theta_K^\flat(x)[, x(t) \notin K_\infty. \tag{13}$$

Proof: Assertion (13) is true; otherwise, if t_1 is the time when $x(\cdot)$ reaches ∂K_∞, we could build a new solution $\tilde{x}(\cdot)$ such that

$$\tilde{x}(t) = \begin{cases} x(t), & \forall\, t \leq t_1 \\ x_1(t - t_1), & \forall\, t \geq t_1, \end{cases}$$

where $x_1(\cdot)$ is a viable solution starting at $x(t_1) \in K_\infty$. Then x should belong to K_∞, which is impossible.

Let us consider now a sequence $x_n \in B(x_0,\eta)\backslash K_\infty$, which converges to x_0. Since $x_n \notin K_\infty$, $\theta_K^\flat(x_n)$ is finite and from Lemma 3.1 strictly positive. Since the solution map $S_F(\cdot)$ is upper semicontinuous, the map $x \rightsquigarrow \theta_K^\flat(x)$ is lower semicontinuous and from Lemma 3.1,

$$\forall\, \epsilon > 0,\ \exists\, N : \forall\, n \geq N, 0 < \tau_0 \leq \theta_K^\flat(x_0) \leq \theta_K^\flat(x_n) + \epsilon < +\infty.$$

On the other hand, from Lemma 3.2

$$\forall\, t < \tau_0, \quad x_n(t) \notin K_\infty.$$

Then, since F is a Marchaud map, $\dot{x}_n(\cdot)$ is bounded and the family $x_n(\cdot)$ is equicontinuous in $W^{1,1}(0, +\infty; X, e^{-bt}dt)$.

From the compactness of the set of solutions (see Aubin-Cellina [1]) there exists a subsequence of solutions which converges in $W^{1,1}(0, +\infty; X, e^{-bt}dt)$ to a solution $x_2(\cdot) \in S_F(x_0)$ such that

$$\forall\, t < \tau_0, \quad x_2(t) \in \overline{X\backslash K_\infty};$$

and by the necessary condition of the local viability theorem at x_0, this implies that

$$y_2 = \liminf_{t\to 0_+} \frac{x_2(t) - x_0}{t} \in F(x_0) \cap T_{X\backslash K_\infty}(x_0),$$

which ends the proof that $F(x_0) \cap T_{X\backslash K_\infty}(x_0)$ is not empty.

We apply Corollary 3.1, since there exist $y_1 \in T_{K_\infty}(x_0)$, $y_2 \in T_{X\backslash K_\infty}(x_0)$, there exists $y_0 \in [y_1, y_2] \cap T_{\partial K_\infty}(x_0)$ and $y_0 \in F(x_0)$ since both y_1 and y_2 belong to the convex set $F(x_0)$:

$$y_0 \in F(x_0) \cap T_{\partial K_\infty}(x_0) \neq \emptyset.$$

From Theorem 3.2 we deduce immediately the following theorem.

Theorem 3.3. *Let $F : X \rightsquigarrow X$ be a Marchaud set-valued map and K a closed subset of X contained in Dom (F) with non-empty interior.*

If $Viab_F(K) \subset \overset{\circ}{K}$, then its boundary is viable:

$$\partial Viab_F(K) = Viab_F(\partial Viab_F(K)).$$

That is to say that for any x_0 in $\partial Viab_F(K)$, there exists at least one solution which is viable in $\partial Viab_F(K)$.

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