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AN EXAMPLE OF A BLOW-UP SEQUENCE FOR $-\Delta u = V(x)e^u$

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(Submitted by: Haim Brezis)

1. Introduction. In this paper we consider the equation

$$\begin{cases} -\Delta u = V(x)e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain and $V(x)$ is a given function in $L^\infty(\Omega)$.

In [1], H. Brezis and F. Merle study this equation and obtain the following uniform estimates for the solutions of (1).

Theorem 1. *Assume (u_n) is a sequence of solutions of* (1) *which satisfies*

$$\|V_n(x)\|_{L^\infty} < C, \tag{2}$$

$$V_n(x) \geq 0 \quad on\ \Omega, \tag{3}$$

and

$$\|e^{u_n}\|_{L^1} \leq C. \tag{4}$$

Then (u_n) is bounded in $L^\infty_{\text{loc}}(\Omega)$.

The aim of this paper is to answer a question raised by H. Brezis and F. Merle. Namely, we show that condition (3) is essential for this theorem.

2. Construction of the sequences (u_n) and (V_n). We shall prove the following theorem in this section:

Theorem 2. *There exist sequences (V_n) and (u_n) in $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ satisfying* (2) *and* (4) *(but not* (3)*) with $u_n \geq 0$ in Ω such that $u_n(0) \to +\infty$.*

For $t > 0$, let D_t be the domain $D_t = \{x \in \mathbb{R}^2 : |x| < t\}$ and for $1 \geq a \geq b > 0$, let $\sigma_{a,b}(x)$ be the function defined on D_1 by

$$\sigma_{a,b}(x) = \begin{cases} 1 & \text{on } D_a \setminus D_b, \\ 0 & \text{otherwise .} \end{cases}$$

Let $u = u_{a,b}$ be the solution of

$$\begin{cases} -\Delta u = -\sigma_{a,b}e^u + \frac{14}{3}\pi\delta & \text{in } D_1, \\ u = 0 & \text{on } \partial D_1, \end{cases} \tag{5}$$

where δ is the Dirac function at $x = 0$.

It is well-known that (5) admits an unique solution. One may, for example, use sub and super solutions to obtain the existence and use the maximum principle to obtain the uniqueness.

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Proposition 1. *Let a_0 be any number $a_0 \in (0,1)$. We have*

$$\|u_{a,b} - u_{a_0,b}\|_{L^\infty} \to 0 \quad \text{as } a \to a_0. \tag{6}$$

Proof. We assume $a > a_0$. (The argument is the same when $a < a_0$). Let $u_{a,b} = u_{a_0,b} + v_a$. Then v_a satisfies

$$\begin{cases} -\Delta v_a = -e^{u_{a_0,b}}(\sigma_{a,b}e^{v_a} - \sigma_{a_0,b}) & \text{in } D_1, \\ v_a = 0 & \text{on } \partial D_1. \end{cases} \tag{7}$$

Now let $\overline{v}_a$ be the solution of the equation

$$\begin{cases} -\Delta \overline{v}_a = -e^{u_{a_0,b}}(\sigma_{a,b} - \sigma_{a_0,b}) & \text{in } D_1, \\ \overline{v}_a = 0 & \text{on } \partial D_1. \end{cases} \tag{8}$$

By the maximum principle, $\overline{v}_a \geq 0$ on D_1. Hence, we have that $e^{\overline{v}_a} \geq 1$ on D_1 and $\overline{v}_a$ is a super solution for (7). Clearly, $\underline{v}_a \equiv 0$ is a sub solution for (7). Thus,

$$\|v_a\|_{L^\infty} \leq \|\overline{v}_a\|_{L^\infty} \to 0 \quad \text{as } a \to a_0.$$

Proposition 2. *For any $b \in (0,1)$, one may find an a_b with $1 \geq a_b \geq b$ such that*

$$u_{a_b,b} > 0 \quad \text{on } D_{a_b} \quad \text{and} \quad u_{a_b,b} \equiv 0 \quad \text{on } D_1 \setminus D_{a_b}, \tag{9}$$

and

$$\int_{D_1 \setminus D_b} e^{u_{a_b,b}}\, dx \leq \frac{14}{3}\pi. \tag{10}$$

Proof. Integrating (5), we obtain

$$-2\pi a \frac{\partial u}{\partial n}\Big|_{\partial D_a} = \frac{14}{3}\pi - \int_{D_a \setminus D_b} e^{u_{a,b}}\, dx, \tag{11}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative. By Proposition 1, the function

$$I(a) = \int_{D_a \setminus D_b} e^{u_{a,b}(x)}\, dx$$

is continuous on $(b,1)$. Thus, we may either choose $1 > a_b \geq b$ such that $I(a_b) = \frac{14}{3}\pi$ or choose $a_b = 1$ with $I(1) \leq \frac{14}{3}\pi$. By (11), in both cases, (9) and (10) are satisfied.

Proposition 3. *If b is small enough, we have*

$$e^{u_{a_b,b}(x_b)} < \frac{1}{b^2}, \quad \text{where } |x_b| = b. \tag{12}$$

Proof. Assume by contradiction that there exists (b_n) with $b_n > 0$ and $b_n \to 0$ as $n \to \infty$ such that $e^{u_{a_{b_n},b_n}(x_n)} \geq \frac{1}{b_n^2}$.

One may check that $\phi_n(r) = \ln(b_n^{\frac{1}{3}}/r^{\frac{7}{3}})$ is the solution of the equation

$$\begin{cases} \phi_n'' + \phi_n'/r = 0 & \text{on } (b_n, b_n^{\frac{1}{3}}), \\ \phi_n(b_n) = \ln \frac{1}{b_n^2}, \\ \phi_n'(b_n) = -\frac{7}{3b_n}. \end{cases} \tag{13}$$

On the other hand, we have

$$u_{a_{b_n}, b_n}(x_{b_n}) \geq \ln \frac{1}{b_n^2} = \phi_n(b_n) \tag{14}$$

and

$$\frac{\partial}{\partial n} u_{a_{b_n}, b_n}(x_{b_n}) \mid_{\partial D_{b_n}} = -\frac{7}{3b_n}. \tag{15}$$

Let $w_n(r) = u_{a_{b_n}, b_n}(x_r) - \phi_n(r)$, where $|x_r| = r$. Then by (5) and (13), $w_n(r)$ satisfies

$$w_n'' + \frac{w_n'}{r} = \sigma_{a_{b_n}, b_n} e^{u_{a_{b_n}, b_n}(x_r)} \geq 0 \quad \text{on } (b_n, b_n^{\frac{1}{3}}). \tag{16}$$

We claim that

$$w_n(r) \geq 0 \quad \text{on } (b_n, b_n^{\frac{1}{3}}). \tag{17}$$

In fact, we have $w_n(b_n) = w_n'(b_n) = 0$ and $(rw_n')' \geq 0$. Thus, rw_n' is nondecreasing on $(b_n, b_n^{\frac{1}{3}})$. Hence, $w_n' \geq 0$ on $(b_n, b_n^{\frac{1}{3}})$ and so $w_n \geq 0$ on $(b_n, b_n^{\frac{1}{3}})$. We may deduce from (10) and (17) the following:

$$\frac{14}{3}\pi \geq \int_{D_1 \setminus D_{b_n}} e^{u_{a_{b_n}, b_n}} \, dx \geq \int_{b_n}^{b_n^{\frac{1}{3}}} 2\pi e^{\phi_n} r \, dr = \int_{b_n}^{b_n^{\frac{1}{3}}} 2\pi b_n^{\frac{1}{3}} r^{-\frac{4}{3}} \, dr = 6\pi(1 - b_n^{\frac{2}{9}}). \tag{18}$$

Clearly, (18) does not hold for small b_n. □

Let, for $\varepsilon < b$, $f_{b,\varepsilon}$ be the function

$$f_{b,\varepsilon}(x) = \begin{cases} -e^{u_{a_b, b}} & \text{on } D_{a_b} \setminus D_b \\ \frac{14}{3\varepsilon^2} & \text{on } D_\varepsilon \\ 0 & \text{on the rest of } D_1. \end{cases} \tag{19}$$

Proposition 4. *Let $u_{b,\varepsilon}$ be the solution of*

$$\begin{cases} -\Delta u_{b,\varepsilon} = f_{b,\varepsilon} & \text{in } D_1, \\ u_{b,\varepsilon} = 0 & \text{on } \partial D_1. \end{cases} \tag{20}$$

Then for sufficiently small b, we may find an ε_b with $\varepsilon_b < b$ such that u_{b,ε_b} satisfies the estimates

$$\int_{D_1} e^{u_{b,\varepsilon_b}} \, dx \leq C \tag{21}$$

and

$$\left\| \frac{f_{b,\varepsilon_b}}{e^{u_{b,\varepsilon_b}}} \right\|_{L^\infty} \le C, \tag{22}$$

where C is a constant independent of b.

Proof. The solution of (20) is

$$u_{b,\varepsilon} = \begin{cases} u_{a_b,b} & \text{on } D_1 \setminus D_b \\ u_{a_b,b}(x_b) + \frac{7}{3}\ln\frac{b}{r} & \text{on } D_b \setminus D_\varepsilon \\ \frac{7}{6}(1 - \frac{r^2}{\varepsilon^2}) + u_{a_b,b}(x_b) + \frac{7}{3}\ln\frac{b}{\varepsilon} & \text{on } D_\varepsilon, \end{cases} \tag{23}$$

where $r = |x|$. By proposition 3, we have

$$e^{u_{b,\varepsilon}(x_b)} = e^{u_{a_b,b}(x_b)} < \frac{1}{b^2} \quad \text{since } b \text{ is small enough.} \tag{24}$$

In order to estabilsh (21) and (22), we choose ε_b satisfying

$$e^{u_{b,\varepsilon_b}(\varepsilon_b)} = \frac{1}{\varepsilon_b^2}. \tag{25}$$

In view of (23) and (24), we may achieve this provided b is small enough.

Verification of (21).

$$\int_{D_1} e^{u_{b,\varepsilon_b}}\,dx = \int_{D_{a_b}\setminus D_b} e^{u_{b,\varepsilon_b}}\,dx + \int_{D_b\setminus D_{\varepsilon_b}} e^{u_{b,\varepsilon_b}}\,dx + \int_{D_{\varepsilon_b}} e^{u_{b,\varepsilon_b}}\,dx. \tag{26}$$

By (10), the first term in the right side of (26) is less than $\frac{14}{3}\pi$. The other two terms in the right side of (26) will be respectively estimated as

$$\begin{aligned} \int_{D_b\setminus D_{\varepsilon_b}} e^{u_{b,\varepsilon_b}}\,dx &= \int_{\varepsilon_b}^{b} 2\pi e^{u_{a_b,b}(x_b)} b^{\frac{7}{3}} r^{-\frac{4}{3}}\,dr \le 6\pi b^{\frac{7}{3}} e^{u_{a_b,b}(x_b)} \Big(\frac{1}{\varepsilon_b^{\frac{1}{3}}} - \frac{1}{b^{\frac{1}{3}}}\Big) \\ &\le 6\pi e^{u_{b,\varepsilon_b}(\varepsilon_b)} \varepsilon_b^2 \le C \quad \text{(by (23) and (25))} \end{aligned} \tag{27}$$

and

$$\int_{D_{\varepsilon_b}} e^{u_{b,\varepsilon_b}}\,dx \le \pi e^{u_{b,\varepsilon_b}(\varepsilon_b) + \frac{7}{6}} \varepsilon_b^2 \le C \quad \text{(by (23) and (25))}. \tag{28}$$

Combining (26), (27) and (28), we obtain (21).

Verification of (22). By (19) and (23), we have

$$\left\| \frac{f_{b,\varepsilon_b}}{e^{u_{b,\varepsilon_b}}} \right\|_{L^\infty} \le \max\Big\{ \Big| \frac{f_{b,\varepsilon_b}(\varepsilon_b)}{e^{u_{b,\varepsilon_b}(\varepsilon_b)+\frac{7}{6}}} \Big|, 1 \Big\} \le C.$$

Proof of Theorem 2. Let (b_n) be a positive sequence which tends to 0 as n tends to ∞. By Proposition 4, $u_{b_n,\varepsilon_{b_n}}$ satisfies (20). Let $u_n = u_{b_n,\varepsilon_{b_n}}$. Then u_n satisfies (1) with $V_n = \frac{f_{b_n,\varepsilon_{b_n}}}{e^{u_n}}$. By (21), $\|e^{u_n}\|_{L^1}$ is bounded and by (22), $\|V_n\|_{L^\infty}$ is also bounded. But $u_n(0)$ tands to ∞ as n tends to ∞.

REFERENCES

[1] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solution of $-\Delta u = V(x)e^u$ in two dimensions*, (to appear).