Differential and Integral Equations, Volume 6, Number 3, May 1993, pp. 643–653.

A BOUNDARY VALUE PROBLEM WHOSE JUMPING NONLINEARITY IS NEITHER SMOOTH NOR LIPSCHITZIAN

NGUYÊN PHUONG CÁC
Department of Mathematics, University of Iowa, Iowa City, Iowa 52242-1466

(Submitted by: V. Lakshmikantham)

Abstract. Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$ and $g : \mathbb{R} \to \mathbb{R}$ for which $\lim_{s\to\pm\infty} \frac{g(s)}{s}$ exist. For a given smooth function h_1 on Ω and a smooth function φ positive on Ω, we are concerned with the number of solutions for large t of the problem $-\Delta u = g(u) + t\varphi + h_1$ on Ω, $u = 0$ on $\partial\Omega$. We shall assume only that $g(\cdot)$ is continuous, in contrast with many other works which require $g(\cdot)$ to be continuously differentiable and one-sided Lipschitzian.

I. Introduction. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. We denote by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$$

the sequence of distinct eigenvalues of the eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Let φ_1 be the eigenfunction corresponding to λ_1 with $\varphi_1 > 0$ on Ω and $\int_\Omega \varphi_1^2\, ds = 1$.
The following theorem is proved by Lazer and McKenna in [9], Theorem 2.4.

Theorem 2.4 of [9]. *Suppose that $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is of class C^1, that for some $k > 1$ and some $b_1 < \lambda_{k+1}$,*

$$g'(s) \leq b_1 \quad \textit{for each} \quad s \in \mathbb{R}. \tag{2}$$

Then for each $b \in (\lambda_k, \lambda_{k+1})$ there exists $a^*(b) \in (\lambda_{k-1}, \lambda_k)$ with the property that if

$$\lim_{s\to-\infty} g'(s) = a \in (a^*(b), \lambda_k), \quad \lim_{s\to+\infty} g'(s) = b, \tag{3}$$

then for every smooth h_1, $h_1 \perp \varphi_1$ in $L^2(\Omega)$, there exists $t_0 > 0$ such that when $t \geq t_0$ the boundary value problem (abbreviated to BVP in the sequel)

$$-\Delta u = g(u) + t\varphi_1 + h_1 \textit{ in } \Omega, \quad u = 0 \textit{ on } \partial\Omega \tag{4}$$

has at least three solutions.

It seems to us that three questions might come to mind concerning Theorem 2.4 of [9].

Received for publication January 1992.
AMS Subject Classification: 35J20, 35J25, 49J05.

First, since that number $a^*(b)$ in [9] arises from a rather rough estimate, one might ask what is the optimal value for it and how does this optimal value relate to the intrinsic ingredients of the problem like $-\Delta$ and the number b. In fact, if we denote by V_1 the span of the eigenfunctions of (1) corresponding to the eigenvalues λ_i with $i \leq k$ and by $\|\cdot\|_0$ the $L^2(\Omega)$-norm, then it is proven in Lemma 2.2 of [9] that there exists $a^*(b) \in (\lambda_{k-1}, \lambda_k)$ such that for any $a \in (a^*(b), \lambda_k)$ we have

$$v \in V_1 \implies \int_\Omega \{|\nabla v|^2 - b(v^+)^2 - a(v^-)^2\}\, dx \leq -\delta\|v\|_0^2,$$

where $\delta = \delta(a)$ is a positive number independent of v and $v^+(x) = \max\{v(x), 0\}$, $v^-(x) = v^+(x) - v(x)$, $x \in \Omega$.

Second, although the proof of Theorem 2.4 in [9] makes use of the continuous differentiability of $g(\cdot)$ at crucial steps, the nature of the result makes us wonder if this differentiability hypothesis is really necessary.

Third, one might ask how essential is the technical condition (2). This condition, which is somewhat contrived, is indispensable for carrying out a certain saddle point reduction as described in [4], [1] in the proof of Theorem 2.4 of [9].

We have addressed the first question in [3]. Roughly speaking, using the results of [2] (cf. also [7]), it turns out that the optimal value for $a^*(b)$ can be related in a natural manner to the extended spectrum Σ of $-\Delta$ which is defined to be the set of all ordered pairs (c, d) such that the BVP

$$-\Delta u = du^+ - cu^- \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5}$$

has a nontrivial solution. It was proven in [2], [7] that given $b \in (\lambda_k, \lambda_{k+1})$ there exists $\bar{a}(b) \in [\lambda_{k-1}, \lambda_k)$ such that for every $a \in (\bar{a}(b), \lambda_{k+1})$, $(a, b) \notin \Sigma$ and $\bar{a}(b) \leq a^*(b)$. Furthermore, if $\bar{a}(b) > \lambda_{k-1}$, which is the case if $b - \lambda_k$ is sufficiently small [2], then the pair $(\bar{a}(b), b) \in \Sigma$. It is shown in [3] that the optimal value for $a^*(b)$ in the context of Theorem 2.4 of [9] is $\bar{a}(b)$.

Concerning the second question, it is also shown in [3] that Theorem 2.4 of [9] remains valid for nonlinearities $g(\cdot)$ that are merely continuous with (2) replaced by

$$\frac{g(r) - g(s)}{r - s} \leq b_1 < \lambda_{k+1} \quad \text{for all} \quad r,\ s \in \mathbb{R},\ r \neq s, \tag{6}$$

and (3) replaced by

$$\lim_{s \to -\infty} \frac{g(s)}{s} = a \in (\bar{a}(b), \lambda_k), \quad \lim_{s \to \infty} \frac{g(s)}{s} = b. \tag{7}$$

We note that (7) is more general and more stable under perturbations of $g(\cdot)$ than (3).

The main purpose of this paper is to address the third question raised above concerning the necessity of the technical condition (6) (or its predecessor (2)). In connection with this, we note that [9] proves that without (2) the BVP (4) still has at least three solutions for t large in the case $k = 2$. Because [9] uses the Morse index, for its result to hold it seems necessary to require that $g(\cdot)$ be of class C^1. Using more sophisticated machinery, [5] also establishes the existence of at least three solutions for (4) without (2) for large t when $g(\cdot)$ is of class C^1 for any $k > 1$. It should also be mentioned that if λ_k is simple then [13] proves that without (2) the BVP (4) has exactly three solutions for large t when $g(\cdot)$ is of class C^1.

Thus, to the best of our knowledge, without the technical condition (2) the existence of three solutions of (4) for large t has only been established for continuously differentiable nonlinearities $g(\cdot)$.

Roughly speaking, we shall assume that $g(\cdot)$ is merely continuous, not necessarily satisfying (6), and we shall show that for $k > 1$, $b \in (\lambda_k, \lambda_{k+1})$, $a \in (\bar{a}(b), \lambda_k)$ and any given smooth function h_1 on Ω, there are infinitely many φ that are arbitrary close to φ_1 in $C_0^1(\Omega)$ for which the BVP

$$-\Delta u = g(u) + t\varphi + h_1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has at least three solutions for all large t.

We shall prove a number of lemmas leading to this result; some of these, e.g., Lemma 4, seem to be of independent interest.

II. Main results. We recall that, as mentioned in the introduction, the extended spectrum Σ of $-\Delta$ is the set of all ordered pairs (c, d) such that the BVP

$$-\Delta u = du^+ - cu^- \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has only the trivial solution and given $b \in (\lambda_k, \lambda_{k+1})$ with $k > 1$ there exists $\bar{a}(b) \in [\lambda_{k-1}, \lambda_k)$ such that $(a, b) \notin \Sigma$ for all $a \in (\bar{a}(b), \lambda_{k+1})$, whereas $(\bar{a}(b), b) \in \Sigma$ if $\bar{a}(b) > \lambda_{k-1}$ (see [7], [2] for details). We shall need the following result proved by Lazer in [7; Theorem 3.1].

Lemma 1. *Consider the BVP*

$$-\Delta u = bu^+ - au^- + \varphi_1 + \psi \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{8}$$

where $\psi \in L^2(\Omega)$. Given $b \in (\lambda_k, \lambda_{k+1})$ with $k > 1$, there exists $\delta > 0$ with the property that if $\bar{a}(b) < a < \lambda_k$ and $\|\psi\|_0 < \delta$ then (8) has at least three solutions.

However, it should be emphasized that it is not known from the proof in [7] that these solutions are always isolated.

Let $N(\lambda_i)$, $i \geq 1$, be the eigenspace of λ_i. We have denoted by V_1 the sum $N(\lambda_1) \oplus \cdots \oplus N(\lambda_k)$. Let $V_2 = V_1^\perp$ in $L^2(\Omega)$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in the Sobolev spaces $H_0^1(\Omega)$ and $H^2(\Omega)$, respectively (we recall that $\|\cdot\|_0$ denotes the norm in $L^2(\Omega)$). The following theorem constitutes the major step toward proving the main result mentioned in the introduction.

Theorem 1. *Suppose that a and b are as in Lemma 1. Then there are infinitely many ψ in every neighborhood of 0 in V_1 such that (8) has only finitely many (and at least three) solutions, each of which has index ± 1.*

Proof. Since $\varphi_1 > 0$ on Ω and $\frac{\partial \varphi_1}{\partial \vec{n}} < 0$ on $\partial\Omega$, where $\vec{n}$ is the outward pointing normal vector to $\partial\Omega$, we can find $\delta^* > 0$ suitably small such that $\psi \in V_1$, $\|\psi\|_{C_0^1} < \delta^* \implies \varphi_1 + \psi > 0$ on Ω. Because V_1 is of finite dimension, we can and *shall throughout the remaining of the paper* assume that the number δ has been chosen such that $\psi \in V_1$, $\|\psi\|_0 < 2\delta \Longrightarrow \varphi_1 + \psi > 0$ on Ω and (8) has at least three solutions.

Let

$$\mathcal{S} = \{u \in L^2(\Omega) \mid u \text{ is a solution of (8) with } \psi \in V_1,\ \|\psi\|_0 < \delta\}. \tag{9}$$

Because $\varphi_1 + \psi > 0$ on Ω for such a ψ, each $u \in \mathcal{S}$ is equal to 0 only on a set of measure 0 [6].

To solve (8), we recall (cf. [4], [1]) that for each $v \in V_1$ there exists a unique $w \in V_2 \cap H_0^1(\Omega) \cap H^2(\Omega) := W_2$ such that (note that $\psi \in V_1$ from here on)

$$-\Delta w = (I - P)\{b(v+w)^+ - a(v+w)^-\}, \tag{10}$$

where P is the projection on V_1 in $L^2(\Omega)$. Let $w = \theta(v)$. It is well known ([4], [1]) that the mapping $v \to \theta(v)$ from V_1 into V_2 is continuous. From the theory of elliptic partial differential equations, it follows that $v \to \theta(v)$ is also continuous when $V_2 \cap H_0^1(\Omega) \cap H^2(\Omega) := W_2$ is equipped with the $H^2(\Omega)$-norm. We note that W_2 is a closed subspace of the Hilbert space $H^2(\Omega)$; $v + \theta(v)$ is a solution of (8) if

$$-\Delta v = P\{b(v + \theta(v))^+ - a(v + \theta(v))^-\} + \varphi_1 + \psi. \tag{11}$$

Conversely, every solution of (8) is of the form $v + \theta(v)$ with $v \in V_1$ being a solution of (11).

We now prove the following.

Lemma 2. *The set $\mathcal{S}$ defined by* (9) *is relatively compact in* $H^2(\Omega)$.

Proof. Since $(a, b) \notin \Sigma$, it can be proven by a standard contradiction argument that there exists a constant $c > 0$ (here and in the sequel c, c_1, c_2, etc.... denote generic positive constants, not always the same) such that $\|u\|_0 < c$ for all $u \in \mathcal{S}$. Let $\{u_n\}_{n=1}^{\infty} \subset \mathcal{S}$. Then there exists $\{\psi_n\} \subset V_1$, $\|\psi_n\|_0 < \delta$, $\forall n$, such that

$$-\Delta u_n = bu_n^+ - au_n^- + \varphi_1 + \psi_n \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega, \tag{12}$$

and we can extract from $\{\psi_n\}$ a subsequence, still denoted by $\{\psi_n\}$, converging to ψ in say $C_0^1(\Omega)$ with $\psi \in V_1$, $\|\psi\|_0 \le \delta$. Since $\|u_n\|_0 < c$, $n = 1, 2, \dots$, it follows from (12) that $\|u_n\|_2 < c$, $n = 1, 2, \dots$. Thus, we can extract from $\{u_n\}$ a subsequence, still denoted by $\{u_n\}$, such that as $n \to \infty$,

$$u_n \rightharpoonup u \text{ in } H^2(\Omega),\ u_n \to u \text{ in } H_0^1(\Omega),$$

where $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. Then

$$-\Delta u = bu^+ - au^- + \varphi_1 + \psi \tag{13}$$

in the distribution sense and therefore in the pointwise sense. The right-hand side of (12) converges to the right-hand side of (13) in $L^2(\Omega)$; hence, it follows from the theory of linear elliptic differential equations that $u_n \to u$ in $H^2(\Omega)$.

If $u_0 \in \mathcal{S}$, i.e., u_0 is a solution of (8) with $\psi \in V_1$, $\|\psi\|_0 < \delta$, then as has been pointed out above, $\operatorname{mes}\{x \in \Omega \mid u_0(x) = 0\} = 0$ and by a result of Solimini [13], given $\epsilon > 0$ we can find $\tilde{\eta}(u_0) > 0$ such that with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$,

$$\begin{aligned} & u_i \in L^{2^*}, \ \|u_i - u_0\|_{L^{2^*}} < \tilde{\eta}(u_0), \ i = 1,2 \\ & \quad \Longrightarrow \|u_1^+ - u_2^+ - \chi(u_0)(u_1 - u_2)\|_0 < \epsilon \|u_1 - u_2\|_{L^{2^*}}, \end{aligned} \tag{14}$$

where $\chi(u_0)(x) = 1$ if $u_0(x) > 0$, $\chi(u_0)(x) = 0$ if $u_0(x) \leq 0$, $x \in \Omega$. It follows that given $\epsilon > 0$ we can find $\eta(u_0) > 0$ such that

$$\begin{aligned} & u_i \in H_0^1(\Omega), \ \|u_i - u_0\|_1 < \eta(u_0), i = 1,2 \\ & \quad \Longrightarrow \|u_1^+ - u_2^+ - \chi(u_0)(u_1 - u_2)\|_0 < \epsilon \|u_1 - u_2\|_1. \end{aligned} \tag{15}$$

This implies that as a mapping from $H_0^1(\Omega)$ into $L^2(\Omega)$, the mapping $u \to u^+$ is Fréchet differentiable at every $u_0 \in \mathcal{S}$. It further implies that although the above mapping is not of class C^1 in a neighborhood of such an u_0, still under appropriate conditions, suitable mappings involving u^+ may be locally invertible at such an u_0.

We shall need the following lemma; its proof is simple and is therefore omitted.

Lemma 2. *Let u_0 be a measurable function on Ω with* $\operatorname{mes}\{x \in \Omega \mid u_0(x) = 0\} = 0$. *Suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on Ω with $u_n(x) \to u_0(x)$ for almost all* (a.a.) *$x \in \Omega$ as $n \to \infty$. Then* $\lim_{n\to\infty} \chi(u_n) = \chi(u_0)$ *in $L^p(\Omega)$ for any p with $1 \leq p < \infty$.*

We next prove the following.

Lemma 3. *Let $v_0 \in V_1$ be such that $v_0 + \theta(v_0)$ is equal to* 0 *only on a subset of Ω of measure* 0. *Then, considered as a mapping from V_1 into $W_2 := V_2 \cap H_0^1(\Omega) \cap H^2(\Omega)$ equipped with the $H^2(\Omega)$-norm, the mapping $v \to \theta(v)$ is Fréchet differentiable at v_0.*

Note. From [11; Theorem 1.1], which is proven with an entirely different proof using monotone operators, it can be deduced that the mapping $v \to \theta(v)$ considered as a mapping from V_1 into $H_0^1(\Omega)$ is Fréchet differentiable at v_0. The draft of this paper was written before the appearance of [11]; in any case, for the proof of our main result, it seems that it is more transparent to consider mappings into $H^2(\Omega)$.

Proof of Lemma 3. Let $\hat{g}(t) = bt^+ - at^-$, $t \in \mathbb{R}$, $u_0 = v_0 + \theta(v_0)$. For $v \in V_1$, let

$$F(v,w) := -\Delta w - (I - P)\hat{g}(v + w) : V_1 \times W_2 \to V_2. \tag{16}$$

We note that $\theta(v)$ is the solution of the equation $F(v,w) = 0$. By the result of Solimini [11] recalled in (15), given $\epsilon > 0$ there exists $\eta(u_0)$ such that with

$$z(k) := [v_0 + \theta(v_0) + k]^+ - [v_0 + \theta(v_0)]^+ - \chi(u_0)k, \ \ k \in W_2,$$

we have

$$\|k\|_2 < \eta(u_0) \Longrightarrow \|z(k)\|_0 < \epsilon \|k\|_2.$$

Thus, for $\|k\|_2 < \eta(u_0)$,

$$\|(I-P)[v_0+\theta(v_0)+k]^+ - (I-P)[v_0+\theta(v_0)]^+ - (I-P)\chi(u_0)k\|_0$$
$$= \|(I-P)z(k)\|_0 \le \|z(k)\|_0 < \epsilon\|k\|_2.$$

Thus, the mapping $w \to (I-P)[v_0+w]^+$ from W_2 into V_2 is Fréchet differentiable at $\theta(v_0)$ and its derivative is the mapping $k \to (I-P)\chi(u_0)k$ ($k \in W_2$). Consequently, $F(v,w)$ is Fréchet differentiable with respect to the second variable at the point $(v_0, \theta(v_0)) \in V_1 \times W_2$ and

$$D_2F(v_0,\theta(v_0))k = -\Delta k - (I-P)a(u_0)k, \quad k \in W_2,$$

where

$$a(u_0)(x) = b\chi(u_0)(x) + a\chi(-u_0)(x), \quad x \in \Omega.$$

We shall first show that $D_2F(v_0,\theta(v_0)) : W_2 \to V_2$ is invertible.

(i) *Its kernel is* $\{0\}$. In fact, let $k \in \ker D_2F(v_0,\theta(v_0))$. Then

$$-\Delta k - (I-P)a(u_0)k = 0.$$

Since $k \in W_2 := V_2 \cap H_0^1(\Omega) \cap H^2(\Omega)$ and $V_2 = (I-P)L^2(\Omega)$, we have

$$\int_\Omega \{|\nabla k|^2 - a(u_0)k^2\}\, dx = 0. \tag{17}$$

Since k is orthogonal to $V_1 := N(\lambda_1) \oplus \cdots \oplus N(\lambda_k)$ in $L^2(\Omega)$,

$$\int_\Omega |\nabla k|^2\, dx \ge \lambda_{k+1} \int_\Omega k^2\, dx. \tag{18}$$

It follows from (17) and (18) that $k \equiv 0$ on Ω because $a,\ b < \lambda_{k+1}$.

(ii) *The range of* $D_2F(v_0,\theta(v_0))$ *is the whole of* V_2. To show this, we find it more convenient to view $D_2F(v_0,\theta(v_0))$ as a mapping densely defined on V_2 into V_2. That it is densely defined is clear: it is defined for all eigenfunctions of $-\Delta$ corresponding to an eigenvalue $\ge \lambda_{k+1}$. Its range is closed in V_2: Let $\{k_n\}_{n=1}^\infty$, $\{f_n\}_{n=1}^\infty \subset V_2$ with $f_n \to f$ in $L^2(\Omega)$ as $n \to \infty$ and

$$-\Delta k_n - (I-P)a(u_0)k_n = f_n \text{ in } \Omega, \quad k_n = 0 \text{ on } \partial\Omega. \tag{19}$$

Then, by multiplying (19) by k_n and integrating, we deduce that $\{k_n\}$ is bounded in $H^2(\Omega)$. Thus, we can assume without loss of generality that $\{k_n\}$ converges to some k weakly in $H^2(\Omega)$ and strongly in $H_0^1(\Omega)$. Letting $n \to \infty$ in (19), we obtain

$$-\Delta k - (I-P)a(u_0)k = f$$

in the distribution sense and hence in the pointwise sense. It then follows that f belongs to the range of $D_2F(v_0,\theta(v_0))$ and thus this range is closed. A similar argument shows that the mapping $D_2F(v_0,\theta(v_0)) : V_2 \to V_2$ is closed. Furthermore, it is not difficult to see that it is also symmetric. Hence, by the closed range theorem [14; Theorem VII. 5, page 205], the range of $D_2F(v_0, \theta(v_0))$ is the whole of V_2.

It follows from (i) and (ii) that $D_2F(v_0, \theta(v_0)) : W_2 \to V_2$ is invertible. We note that the Fréchet derivative of $F(\cdot,\cdot) : V_1 \times W_2 \to V_2$ with respect to the first variable exists at the point $(v_0, \theta(v_0))$. In fact,

$$D_1F(v_0, \theta(v_0))h = -(I-P)a(u_0)h, \quad h \in V_1.$$

Consequently, by a familiar implicit function argument (cf. [12], page 62), $\theta(\cdot) : V_1 \to W_2$ is Fréchet differentiable at v_0.

From (16) (or (10)), we obtain for any $h \in V_1$,

$$-\Delta\theta'(v_0)h = (I-P)\{a(u_0)(h + \theta'(v_0)h)\}. \tag{20}$$

The following result seems to be of independent interest. It shows that beyond being differentiable at a point like v_0, $\theta(\cdot)$ inherits fully a characteristic of the map $u \to u^+$ as described by (14).

Lemma 4. *Let $v_0 \in V_1$ be such that $u_0 = v_0 + \theta(v_0)$ is equal to 0 only on a subset of measure 0 of Ω. Then given any $\epsilon > 0$ there exists $\delta(v_0) > 0$ such that*

$$\begin{aligned} &v_i \in V_1, \ \|v_i - v_0\|_0 < \delta(v_0), \ i = 1,2 \\ &\quad \Longrightarrow \|\theta(v_2) - \theta(v_1) - \theta'(v_0)(v_2 - v_1)\|_2 < c\epsilon\|v_2 - v_1\|_0, \end{aligned} \tag{21}$$

where $c > 0$ is a constant independent of v_0 and ϵ.

Proof. Taking $h = v_2 - v_1$ in (20), we obtain

$$-\Delta\theta'(v_0)(v_2 - v_1) = (I-P)\{a(u_0)(v_2 - v_1 + \theta'(v_0)(v_2 - v_1))\}. \tag{22}$$

Let

$$\begin{aligned} u_i &= v_i + \theta(v_i), \quad i = 1,2, \\ \zeta &= \hat{g}(u_2) - \hat{g}(u_1) - a(u_0)(u_2 - u_1), \\ k &= \theta(v_2) - \theta(v_1) - \theta'(v_0)(v_2 - v_1). \end{aligned}$$

Recall that from (10) we have

$$-\Delta\theta(v_i) = (I-P)\hat{g}(u_i), \quad i = 1,2. \tag{23}$$

From (22) and (23), we deduce that

$$-\Delta k = (I-P)\{a(u_0)k + \zeta\}. \tag{24}$$

By Solimini's result [13], given $\epsilon > 0$ we can find $\tilde{\delta}(v_0) > 0$ such that

$$\begin{aligned} &u_i \in H_0^1(\Omega), \ \|u_i - u_0\|_1 < \tilde{\delta}(v_0), \ i = 1,2 \\ &\quad \Longrightarrow \|\hat{g}(u_2) - \hat{g}(u_1) - a(u_0)(u_2 - u_1)\|_0 < \epsilon\|u_2 - u_1\|_1. \end{aligned} \tag{25}$$

From (23), we have with $i,\ j = 0, 1, 2$,

$$-\Delta[\theta(v_i) - \theta(v_j)] = (I - P)\{\hat{g}(u_i) - \hat{g}(u_j)\}. \tag{26}$$

Since

$$|\hat{g}(u_i) - \hat{g}(u_j)| \le b|v_i + \theta(v_i) - v_j - \theta(v_j)|,$$

we deduce from (26) that

$$(1 - \lambda_{k+1}^{-1} b)\|\theta(v_i) - \theta(v_j)\|_1 \le b\|v_i - v_j\|_0. \tag{27}$$

From (25) and (27), it follows that there exists $\delta(v_0) > 0$ such that

$$\begin{aligned} & v_i \in V_1,\ \|v_i - v_0\|_0 < \delta(v_0),\ i = 1, 2 \\ & \quad \Longrightarrow \|\hat{g}(u_2) - \hat{g}(u_1) - a(u_0)(u_2 - u_1)\|_0 < c_1\epsilon\|v_2 - v_1\|_0, \end{aligned} \tag{28}$$

where c_1 is a constant independent of ϵ and v_0.

Since $\lambda_{k-1} < a(u_0)(x) \le b$, $x \in \Omega$, from (24) we obtain

$$(1 - \lambda_{k+1}^{-1} b)\|k\|_1^2 \le \|k\|_0\|\zeta\|_0.$$

Hence, it follows from (28) that

$$v_i \in V_1,\ \|v_i - v_0\| < \delta(v_0),\ i = 1, 2 \Longrightarrow \|k\|_1 < c_2\epsilon\|v_2 - v_1\|_0, \tag{29}$$

where $c_2 > 0$ is a constant independent of ϵ and v_0. Now, (21) follows from the theory of elliptic differential equations, (24) and (29).

Now consider the set $P(\mathcal{S})$ of V_1 where, it might be recalled, P is the orthogonal projection on V_1 in $L^2(\Omega)$ and $\mathcal{S}$ is defined by (9); in other words, $v \in P(\mathcal{S})$ if and only if

$$-\Delta v = P\{\hat{g}(v + \theta(v))\} + \varphi_1 + \psi \tag{30}$$

for some $\psi \in v_1$, $\|\psi\|_0 < \delta$, where $\hat{g}(t) = bt^+ - at^-$, $t \in \mathbb{R}$. Since the mapping $v \to -\Delta v - P\{\hat{g}(v + \theta(v)\}$ of V_1 into itself is continuous, $P(\mathcal{S})$ is open in V_1.

Lemma 5. *The mapping $v \to \theta'(v)$ from $P(\mathcal{S})$ to $L(V_1, H^2(\Omega))$ is continuous, where $L(X, Y)$ is the Banach space of bounded linear mappings from a Banach space X into a Banach space Y.*

Proof. Let v_n, $v_0 \in P(\mathcal{S})$, $n = 1, 2, \ldots$, and $v_n \to v_0$ as $n \to \infty$. Let $h \in V_1$ and $w_i = \theta'(v_i)h$, $u_i = v_i + \theta(v_i)$, $i = 0, 1, 2, \ldots$. From (20), we have

$$-\Delta w_i - (I - P)\{a(u_i)w_i\} = (I - P)\{a(u_i)h\} \tag{31}$$

and

$$\begin{aligned} & -\Delta(w_n - w_0) - (I - P)\{a(u_0)(w_n - w_0)\} \\ & \qquad = (I - P)\{[a(u_n) - a(u_0)]h + [a(u_n) - a(u_0)]w_n\}. \end{aligned} \tag{32}$$

Since $\lambda_{k-1} < a(u_i) \le b < \lambda_{k+1}$, multiplying (31) by w_i and integrating, we obtain that

$$\|w_i\|_1 < c\|h\|_0, \quad i = 0, 1, 2, \ldots, \tag{33}$$

for some constant c.

Multiplying (32) by $(w_n - w_0)$ and integrating, we obtain, by using (33) and Lemma 2, that

$$\|w_n - w_0\|_1 < c\epsilon_n \|h\|_0, \quad n = 1, 2, \dots, \tag{34}$$

where $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. From (32), (34), Lemma 2 and the theory of elliptic differential equations, we obtain

$$\|w_n - w_0\|_2 < c\epsilon_n \|h\|_0, \quad n = 1, 2, \dots,$$

and thus the lemma is proved.

Referring to (11), let

$$E(v) := -\Delta v - P\{b(v + \theta(v))^+ - a(v + \theta(v))^-\}, \quad v \in V_1. \tag{35}$$

Lemma 6. *Let $\delta > 0$ be the number introduced in the line preceding* (9). *Then the subset of $\{\psi \in V_1 \colon\ \|\psi\|_0 < \delta\}$ consisting of those ψ for which there exists a solution v of $E(v) = \varphi_1 + \psi$ with the property that $E'(v) : V_1 \to V_1$ not being invertible is of measure zero in the Euclidean space V_1.*

Proof. Recall that $E(v) = \varphi_1 + \psi$, $\|\psi\|_0 < \delta$ implies $v \in P(\mathcal{S})$. By the last lemma, $E(\cdot)$ is continuously differentiable on the open set $P(\mathcal{S})$ of V_1. Furthermore, by Lemma 1 and the choice of δ, for every $\psi \in V_1$, $\|\psi\|_0 < \delta$, there are $v \in V_1$ with $E(v) = \varphi_1 + \psi$. Hence, Lemma 6 follows from Sard's theorem (cf., e.g., [10]).

A solution u_0 of (8) which is equal to 0 only on a subset of Ω of measure 0 is called *nondegenerate* if the BVP

$$-\Delta u = a(u_0)u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{36}$$

has only the trivial solution. Since $u \to -\Delta u - a(u_0)u$, $u \in H^2(\Omega)$, is the Fréchet derivative at u_0 of the mapping $u \to -\Delta u - (bu^+ - au^-)$, a nondegenerate solution of (8) is isolated and has index ± 1 (see [13]). We have the following.

Lemma 7. *A solution $u_0 = v_0 + \theta(v_0)$ of* (8) *that is equal to* 0 *only on a subset of Ω of measure* 0 *is nondegenerate if and only if $E'(v_0)$, which is defined from* (35), *is invertible.*

Proof. From (35), we obtain

$$E'(v_0)h = -\Delta h - P\{a(u_0)(h + \theta'(v_0)h)\}, \quad h \in V_1. \tag{37}$$

First suppose that u_0 is nondegenerate and $E'(v_0)h = 0$; i.e.,

$$-\Delta h = P\{a(u_0)(h + \theta'(v_0)h\}. \tag{38}$$

From (20), which characterizes $\theta'(v_0)$, and (38), we obtain

$$-\Delta\{h + \theta'(v_0)h\} = a(u_0)(h + \theta'(v_0)h).$$

Because u_0 is nondegenerate, it follows from this last equation that $h+\theta'(v_0)h=0$. Since $h \in V_1$, whereas $\theta'(v_0)h$ is orthogonal to V_1 in $L^2(\Omega)$, we deduce that $h=0$. Now suppose that $E'(v_0)$ is invertible and u is a solution of (36). Let $u=h+k$ with $h \in V_1$ and k orthogonal to V_1 in $L^2(\Omega)$. From (36), we then have

$$-\Delta h = P\{a(u_0)(h+k)\}, \tag{39}$$

$$-\Delta k = (I-P)\{a(u_0)(h+k)\}. \tag{40}$$

Subtract (20) from (40), multiply the result by $k-\theta'(v_0)h$ and integrate to obtain

$$\int_\Omega |\nabla(k-\theta'(v_0)h)|^2\,dx = \int_\Omega a(u_0)(k-\theta'(v_0)h)^2\,dx.$$

Because $\lambda_{k-1} < a(u_0) \le b < \lambda_{k+1}$, we deduce that $k=\theta'(v_0)h$. Then it follows from (39) and the fact that $E'(v_0)$ is invertible that $h=0$ and $k=\theta'(v_0)h=0$. Thus, (36) has only the trivial solution.

Proof of Theorem 1 continued. It now follows from Lemmas 6 and 7 that in every neighborhood of 0 in V_1 there are infinitely many ψ for which each solution of (8) is nondegenerate (hence isolated) and has index ± 1. (This fact could also be deduced from Lemma 6 and a homotopy argument as in [8].) Since, by Lemma 2, the set of all solutions of (8) for each ψ is relatively compact, for every such $\psi \in V_1$, $\|\psi\|_0 < \delta$, there can only be finitely many solutions of (8).

We can now prove the main result mentioned in the introduction.

Theorem 2. *Let $g:\mathbb{R}\to\mathbb{R}$ be continuous, $\lim_{s\to-\infty}\frac{g(s)}{s}=a$, $\lim_{s\to+\infty}\frac{g(s)}{s}=b$ exist and $g_0(s):=g(s)-bs^+ + as^-$ be bounded on $\mathbb{R}$. Suppose that a, b and ψ are as in Theorem 1. Then given any smooth function h_1 on Ω, there exists a $t_0>0$ such that if $t\ge t_0$, the BVP $-\Delta u = g(u)+t(\varphi_1+\psi)+h_1$ in Ω, $u=0$ on $\partial\Omega$ has at least three solutions.*

Proof. The claim follows from Theorem 1 by perturbing $\hat{g}(s):=bs^+ - as^-$ of Theorem 1 with $g_0(\cdot)$, $\lim_{s\to\pm\infty}\frac{g_0(s)}{s}=0$, and by applying a homotopy argument. We refer the readers to the proof of Proposition 2.10 of [13] for details. We note that the boundedness of $g_0(\cdot)$, which simplifies somewhat the argument, can be replaced by a sublinear growth condition.

REFERENCES

[1] H. Amann, *Saddle points and multiple solutions of differential equations,* Math. Z., 169 (1979), 127–166.

[2] N.P. Các, *On nontrivial solutions of a Dirichlet problem whose jumping nonlinearity crosses a multiple eigenvalue,* J. Differential Equations, 80 (1989), 379–404.

[3] N.P. Các, *On a boundary value problem with nonsmooth jumping nonlinearity,* J. Differential Equations, 93 (1991), 238–259.

[4] A. Castro, *Hammerstein integral equations with indefinite kernel,* Internat. J. Math. Sci., 1 (1978), 187–201.

[5] E.N. Dancer, *Multiple solutions of asymptotically homogeneous problems,* Ann. Mat. Pura Appl., CL11 (1988), 63–78.

[6] D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inequalities and Their Applications," Academic Press, New York, 1980.

[7] A.C. Lazer, *Introduction to multiplicity theory for boundary value problems with asymmetric nonlinearities,* in "Partial Differential Equations," F. Cardoso et al., Editors, Lecture Notes in Mathematics, No. 1324, Springer-Verlag, New York, 1988, 137–165.
[8] A.C. Lazer and P.J. McKenna, *Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems,* J. Math. Analysis Appl., 107 (1985), 371–395.
[9] A.C. Lazer and P.J. McKenna, *Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues,* Comm. in P.D.E., 10 (1985), 107–150.
[10] N.G. Lloyd, "Degree Theory," Cambridge University Press, London, New York, 1978.
[11] C.A. Magalhães, *Semilinear elliptic problem with crossing of multiple eigenvalues,* Comm. in P.D.E., 15 (1990), 1265–1292.
[12] L. Nirenberg, "Topics in Nonlinear Functional Analysis," Courant Institute of Mathematical Sciences, New York University, 1974.
[13] S. Solimini, *Some remarks on the number of solutions of some nonlinear elliptic problems,* Analyse Non Linéaire, Ann. Inst. Henri Poincaré, 2 (1985), 143–156.
[14] K. Yosida, "Functional Analysis," Springer-Verlag, 3rd Ed., New York, 1971.