Differential and Integral Equations, Volume 6, Number 5, September 1993, pp. 1137–1143.

ON THE ZEROS OF ASSOCIATED POLYNOMIALS OF CLASSICAL ORTHOGONAL POLYNOMIALS*

Árpád Elbert
Mathematical Institute of the Hungarian Academy of Sciences
Budapest, P.O.B. 127, H-1364, Hungary

Andrea Laforgia
Dipartimento di Matematica, Università degli Studi della Basilicata
Via N. Sauro, 85–85100 Potenza, Italy

(Submitted by: W.N. Everitt)

Abstract. We establish some general properties of the associated polynomials $r_{n-1}(x)$ of classical orthogonal polynomials $p_n(x)$. As a consequence of our results we prove a conjecture recently formulated by A. Ronveaux on the location of the zeros of $r_{n-1}(x)$ and $p_n'(x)$.

1. Introduction. For $n = 0, 1 \dots$ we denote by $p_n(x)$ any classical orthogonal polynomial (Jacobi, Hermite, and Laguerre) of degree n, on the interval (a, b) with the weight $\mu(x)$ where

$$\mu(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta, \ -1 < x < 1, \ \alpha > -1, \ \beta > -1 & \text{in Jacobi case,} \\ x^\alpha e^{-x}, \ 0 < x < \infty, \ \alpha > -1 & \text{in Laguerre case,} \\ e^{-x^2}, \ -\infty < x < \infty & \text{in Hermite case.} \end{cases}$$

It is well known that the classical orthogonal polynomials satisfy the second-order differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0, \tag{1.1}$$

where $\sigma(x)$ is a polynomial in x of the second degree at most, $\tau(x)$ is a polynomial of the first degree, and λ_n is a constant depending on n.

We define the associated polynomial $r_{n-1}(x)$ of $p_n(x)$ by means of the integral

$$r_{n-1}(x) = \frac{1}{c_0}\int_a^b \frac{p_n(x) - p_n(t)}{x - t}\mu(t)\,dt, \quad n = 1, 2, \dots, \tag{1.2}$$

where

$$c_0 = \int_a^b \mu(t)\,dt.$$

Received January 1992, in revised form June 1992.
*Work sponsored by Ministero della Ricerca Scientifica e Tecnologica of Italy.
AMS Subject Classification: 33C25.

A. Ronveaux proved the following identity [3]

$$\begin{aligned}\frac{1}{2}Kp_n^2(x) + \sigma(x)[p_n'(x)r_{n-1}(x) - p_n(x){r'}_{n-1}(x)]\\ + [\tau(x) - \sigma'(x)]p_n(x)r_{n-1}(x) = A(n),\end{aligned} \tag{1.3}$$

where

$$K = \sigma''(x) - 2\tau'(x)$$

is constant and $A(n)$ is a constant depending on n.

We introduce now the following notation. Let x_ν, $\nu = 1, 2, \ldots, n$, be the zeros of $p_n(x)$, in decreasing order. Let x'_ν, $\nu = 1, 2, \ldots, n-1$, be the zeros of $p_n'(x)$ in decreasing order, and let ξ_ν, $\nu = 1, 2, \ldots, n-1$, be the zeros of $r_{n-1}(x)$ in decreasing order. The dependence on n of such zeros is omitted.

The most important result which will be proved in the next section is the formula

$$r_{n-1}(x) = A(n) \sum_{\nu=1}^{n} \frac{1}{\sigma(x_\nu)} \frac{1}{[p_n'(x_\nu)]^2} \frac{p_n(x)}{x - x_\nu}. \tag{1.4}$$

By this it is possible to derive an expression for the constant $A(n)$ in (1.3). Indeed, the polynomial $p_n(x)$ has the form

$$p_n(x) = k_n x^n + \cdots$$

so by (1.2) it follows immediately that

$$r_{n-1}(x) = k_n x^{n-1} + \cdots$$

and

$$\lim_{x\to\infty} \frac{x r_{n-1}(x)}{p_n(x)} = 1.$$

Hence by (1.4) we find

$$\frac{1}{A(n)} = \sum_{\nu=1}^{n} \frac{1}{\sigma(x_\nu)} \frac{1}{[p_n'(x_\nu)]^2} \tag{1.5}$$

which gives the desired expression of $A(n)$.

On the other hand we obtain ([5, p. 55, Theorem 3.5.1; p. 57, Theorem 3.5.3])

$$\frac{r_{n-1}(x)}{p_n(x)} = \frac{1}{c_0} \sum_{\nu=1}^{n} \frac{\lambda_\nu}{x - x_\nu},$$

where λ_ν denotes the corresponding Christoffel numbers. For these we have ([5, pp. 352, 353])

$$\lambda_\nu = \begin{cases} 2^{\alpha+\beta+1} \dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} (1 - x_\nu^2)^{-1} [p_n^{(\alpha,\beta)\prime}(x_\nu)]^{-2}, \\ \qquad\qquad \nu = 1, 2, \ldots, n;\ \alpha > -1,\ \beta > -1, \\ \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} x_\nu^{-1} [\mathcal{L}_n^{(\alpha)\prime}(x_\nu)]^{-2}, \quad \nu = 1, 2, \ldots, n;\ \alpha > -1, \\ \pi^{1/2} 2^{n+1} n! [H_n'(x_\nu)]^{-2}, \quad \nu = 1, 2, \ldots, n. \end{cases}$$

Thus we obtain by (1.4)

$$A(n)=\begin{cases}\dfrac{\binom{n+\alpha}{n}\binom{n+\beta}{n}}{\binom{n+\alpha+\beta}{n}}(\alpha+\beta+1), & \text{for Jacobi polynomials,}\\ \binom{n+\alpha}{n}, & \text{for Laguerre polynomials,}\\ 2^{n+1}n!, & \text{for Hermite polynomials.}\end{cases}$$

The relations

$$\frac{p_n'(x)}{p_n(x)}=\sum_{\nu=1}^{n}\frac{1}{x-x_\nu}$$

and (1.4) suggest a comparison between the zeros ξ_ν of $r_{n-1}(x)$ and the zeros x_ν' of $p_n'(x)$.

Ronveaux observed that the zeros x_ν' of $p_n'(x)$ and the zeros ξ_ν of $r_{n-1}(x)$, $\nu=1,2,\dots,n-1$, $n=2,3,\dots$, coincide when $p_n(x)$ denotes the Chebyshev polynomial of the first kind and in the general case he has formulated the following conjecture ([4]):

Conjecture by Ronveaux. For $0\le x_{\nu+1}<x_\nu$ we have

$$x_{\nu+1}<x_\nu'<\xi_\nu<x_\nu,\quad \nu=1,2,\dots,[(n-1)/2].$$

Ronveaux verified his conjecture for $n=3$ in the case of Hermite polynomials, for $n=2$ in the case of Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x)$ with $\alpha>-1$, and finally in the ultraspherical case $P_n^{(\lambda)}(x)$ for $n=3$ and $\lambda>0$.

Moreover, F. Peherstorfer ([2]) proved the following result.

Theorem. *Let $A=\{(\alpha,\beta)\in(-1,-1/2)\times(-1/2,0):\alpha+\beta+1\le 0\}$ and let $D=\{(\alpha,\alpha):\alpha\in(-1,-1/2)\}$. Then the following relations hold for each $n=2,3,\dots$.*

(i) *if $(\alpha,\beta)\in A$, then*

$$\xi_\nu<x_\nu,\quad \nu=1,2,\dots,n-1.$$

This inequality must be reversed if α and β are exchanged.

(ii) *if $(\alpha,\alpha)\in D$, then*

$$\xi_\nu<x_\nu,\quad \nu=[n/2],\dots,n$$

$$\xi_\nu>x_\nu,\quad \nu=1,2,\dots,[\frac{n-1}{2}].$$

Clearly we have

$$x_{\nu+1}<x_\nu'<x_\nu$$

and ([5, p. 57])

$$x_{\nu+1}<\xi_\nu<x_\nu.$$

Thus the conjecture will be proved if we show the inequality

$$x_\nu'<\xi_\nu,\quad \nu=1,2,\dots,[(n-1)/2]. \tag{1.6}$$

In this paper we prove the Ronveaux conjecture in the cases of ultraspherical and Hermite polynomials. We remark that it is not possible to extend (1.6) to the general case of Jacobi polynomials. However in the next section we will present a case where (1.6) holds in Jacobi case, for any $\nu = 1, 2, \ldots, n-1$.

2. Proof of the main result. We consider only the case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$ because by [5, p. 107]

$$\frac{H_n(x)}{n!} = \lim_{\lambda\to\infty} \lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right),$$

where $P_n^{(\lambda)}(x) = P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)$, $\alpha = \beta = \lambda - 1/2$, is the ultraspherical polynomial, the formulas (1.2) and (1.4) remain valid also in the Hermite case.

Similar considerations hold for the Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x)$ in view of the formula ([5, p. 103])

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2\beta^{-1}x) = \mathcal{L}_n^{(\alpha)}(x).$$

Now we are going to prove formula (1.4) in the Jacobi case. The Jacobi polynomials satisfy the differential equation (1.1) with

$$\sigma(x) = 1 - x^2, \quad \tau(x) = \beta - \alpha - (\alpha+\beta+2)x,$$
$$\lambda_n = n(n+\alpha+\beta+1), \quad K = 2(\alpha+\beta+1);$$

i.e.,

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{2.1}$$

We need the following decompositions into partial fractions:

$$\frac{1}{p_n^2(x)} = \sum_{\nu=1}^{n} \frac{1}{[p_n'(x)]^2} \frac{1}{(x-x_\nu)^2} - \sum_{\nu=1}^{n} \frac{(\alpha+\beta+2)x_\nu + \alpha - \beta}{(1-x_\nu^2)[p_n'(x_\nu)]^2} \frac{1}{x - x_\nu} \tag{2.2}$$

and

$$\begin{aligned}\frac{1}{(1-x^2)p_n^2(x)} &= \frac{1}{2p_n^2(1)} \frac{1}{1-x} + \frac{1}{2p_n^2(-1)} \frac{1}{1+x} \\ &+ \sum_{\nu=1}^{n} \frac{1}{1-x_\nu^2} \frac{1}{[p_n'(x_\nu)]^2} \frac{1}{(x-x_\nu)^2} \\ &- \sum_{\nu=1}^{n} \frac{(\alpha+\beta)x_\nu + \alpha - \beta}{(1-x_\nu^2)^2[p_n'(x_\nu)]^2} \frac{1}{x-x_\nu},\end{aligned} \tag{2.3}$$

where now $p_n(x)$ denotes $P_n^{(\alpha,\beta)}(x)$.

First we show that the polynomial $r_{n-1}(x)$ given by (1.4) satisfies (1.3). To this end we write (1.3) in the Jacobi case

$$\frac{\alpha+\beta+1}{1-x^2} - \left[\frac{r_{n-1}(x)}{p_n(x)}\right]' + \frac{\beta - \alpha - (\alpha+\beta)x}{1-x^2} \frac{r_{n-1}(x)}{p_n(x)} = \frac{A(n)}{(1-x^2)p_n^2(x)}. \tag{2.4}$$

Now by (1.4) and (2.3) we find

$$\frac{\alpha+\beta+1}{1-x^2}\frac{1}{A(n)} = \frac{1}{2p_n^2(1)}\frac{1}{1-x} + \frac{1}{2p_n^2(-1)}\frac{1}{1+x}$$
$$+\frac{1}{1-x^2}\sum_{\nu=1}^{n}\frac{1}{(1-x_\nu^2)[p_n'(x_\nu)]^2}\frac{(\alpha+\beta)(1+xx_\nu)+(\alpha-\beta)(x+x_\nu)}{1-x_\nu^2}.$$

By (1.5) and (2.2) we obtain

$$(\alpha+\beta+1)\sum_{\nu=1}^{n}\frac{1}{1-x_\nu^2}\frac{1}{[p_n'(x_\nu)]^2} = (1+x)\frac{1}{2}\Bigg\{\sum_{\nu=1}^{n}\frac{1}{[p_n'(x_\nu)]^2}\frac{1}{(1-x_\nu)^2}$$
$$-\sum_{\nu=1}^{n}\frac{(\alpha+\beta+2)x_\nu+\alpha-\beta}{(1-x_\nu^2)[p_n'(x_\nu)]^2}\frac{1}{1-x_\nu}\Bigg\} + (1-x)\frac{1}{2}\Bigg\{\sum_{\nu=1}^{n}\frac{1}{[p_n'(x_\nu)]^2}\frac{1}{(1+x_\nu^2)}$$
$$+\sum_{\nu=1}^{n}\frac{(\alpha+\beta+2)x_\nu+\alpha-\beta}{(1-x_\nu^2)[p_n'(x_\nu)]^2}\frac{1}{1+x_\nu}\Bigg\} + \sum_{\nu=1}^{n}\frac{(\alpha+\beta)(1+xx_\nu)+(\alpha-\beta)(x+x_\nu)}{(1-x_\nu^2)^2[p_n'(x_\nu)]^2}.$$

By straightforward calculations we get that this equation is an identity and this shows that the polynomial $r_{n-1}(x)$ defined by (1.4) satisfies the identity (1.3). It remains to prove that $r_{n-1}(x)$ is the only polynomial of degree $n-1$ which satisfies the identity (1.3). For the proof suppose the contrary and let $\bar{r}_{n-1}(x)$ be another polynomial satisfying (2.4) (2.4 is (1.3) in the Jacobi case). Then for the difference

$$s = r_{n-1}(x) - \bar{r}_{n-1}(x)$$

we find

$$\frac{p_n'(x)s - p_n(x)s'}{p_n^2(x)} + \frac{\beta-\alpha-(\alpha+\beta)x}{1-x^2}\frac{s}{p_n(x)} = 0$$

and by this we obtain

$$s = C\,p_n(x)(1+x)^\beta(1-x)^\alpha,$$

where C is some constant. Since s is a polynomial of degree at most $n-1$, this relation is possible only for $C=0$ and this shows the uniqueness.

3. Consequences of formula (1.4). We are now in the position to prove inequality (1.6) in the ultraspherical case $\alpha=\beta=\lambda-1/2$.

We know that

$$0 \le x_{\mu+1} < x_\mu' < x_\mu$$

and we wish to prove the inequality

$$\frac{r_{n-1}(x_\mu')}{p_n(x_\mu')} > 0.$$

First we observe that using the symmetry of ultraspherical polynomials

$$P_n^{(\lambda)}(-x) = (-1)^n P_n^{(\lambda)}(x)$$

in (1.4) we find

$$\begin{aligned}\frac{r_{n-1}(x)}{p_n(x)} &= \sum_{\nu=1}^{n} \frac{1}{1-x_\nu^2} \frac{1}{[p_n'(x_\nu)]^2} \frac{1}{x-x_\nu} \\ &= x\sum_{\nu=1}^{n} \frac{1}{1-x_\nu^2} \frac{1}{[p_n'(x_\nu)]^2} \frac{1}{x^2-x_\nu^2}.\end{aligned} \tag{3.1}$$

Similarly, we get

$$\frac{p_n'(x)}{p_n(x)} = \sum_{\nu=1}^{n} \frac{1}{x-x_\nu} = x\sum_{\nu=1}^{n} \frac{1}{x^2-x_\nu^2}. \tag{3.2}$$

It is convenient now to introduce the function

$$V(x) = (1-x^2)[p_n'(x)]^2 + n(n+2\lambda-1)p_n^2(x) \tag{3.3}$$

and to show that it is increasing for positive x and fixed $\lambda > 0$. Taking into account the differential equation (2.1) with $\alpha = \beta = \lambda - 1/2$ we obtain

$$V'(x) = 4\lambda[p_n'(x)]^2 x$$

which is positive for $\lambda > 0$ and $x > 0$ except at the zeros of $p_n'(x)$. Replacing in (3.1) x by x_μ' we have

$$\frac{r_{n-1}(x_\mu')}{p_n(x_\mu')} = x_\mu' \sum_{\nu=1}^{n} \frac{1}{V(x_\nu)} \frac{1}{(x_\mu')^2 - x_\nu^2} > \frac{1}{V(x_\mu')} x_\mu' \sum_{\nu=1}^{n} \frac{1}{(x_\mu')^2 - x_\nu^2} = 0,$$

where the last equality is a consequence of (3.2).

On the other hand by (1.4) the function $r_{n-1}(x)/p_n(x)$ decreases with $x > 0$ on the interval $(x_{\mu+1}, x_\mu)$. Therefore the zero ξ_μ of $r_{n-1}(x)$ in $(x_{\mu+1}, x_\mu)$ occurs after x_μ' leading to the desired result.

Remark 1. The same argument can be used to prove that the inequality (1.6) is true also in the case of the Hermite polynomials.

Remark 2. In the general case of Jacobi polynomials we do not have a general statement. For example when $\alpha \geq -1/2$ and $-1 < \beta \leq -1/2$, but not $\alpha = \beta = -1/2$, we have

$$x_\mu' < \xi_\mu, \quad \mu = 1, 2, \ldots, n-1$$

while, when $-1 < \alpha \leq -1/2$ and $\beta \geq -1/2$, but not $\alpha = \beta = -1/2$, we find

$$\xi_\mu < x_\mu', \quad \mu = 1, 2, \ldots, n-1.$$

Indeed, following the lines of the argument used in the ultraspherical case we get that the function (3.3) has the derivative

$$V'(x) = 2[\alpha - \beta + (\alpha+\beta+1)x][p_n'(x)]^2$$

which in the first case is positive and in the second is negative for $-1 < x < 1$.

Remark 3. Finally we observe that the formula (2.3) in the Legendre case reduces to

$$1 - p_n^2(x) = \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2} \left[\frac{p_n(x)}{p_n'(x_\nu)(x-x_\nu)} \right]^2$$

which is the known Egerváry-Turán formula ([1]).

Acknowledgments. The authors thank J. Szabados for valuable suggestions and comments. They also thank the referee for pointing out the reference [2].

REFERENCES

[1] E. Egerváry and P. Turán, *Notes on interpolation, V, On the stability of interpolation,* Acta Math. Scient. Hung., 9 (1958), 259–267.

[2] F. Peherstorfer, *A partial solution to a problem,* Proc. 1991 Granada Meeting on Orthogonal Polynomials, Open Problems Section, to appear.

[3] A. Ronveaux, *On a "first integral" relating classical orthogonal polynomials and their numerator polynomials,* Simon Stevin, A Quarterly Journal of Pure and Applied Mathematics, to appear.

[4] A. Ronveaux, *Zeros of associate versus zeros of derivatives of classical orthogonal polynomials,* Open problem section in "IMACS, Annals on computing and applied mathematics", vol. 9, Orthogonal Polynomials and Their Applications, C. Brezinski et al. Editors, J.C. Baltzer AG, Scient. Publ. Company, Basel, (1991), p. 419.

[5] G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloquium Publications, vol. 23, 4th ed., 1975.