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SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract. Singularly perturbed couples of second-order differential equations are studied. The existence of solutions is shown by a Galerkin approximation method together with the Leray-Schauder degree theory. Limit behavior of solutions is investigated for specific problems. Discontinuous versions of such problems are studied as well.

1. Introduction. The purpose of this paper is to study the existence of classical solutions for problems of the type

$$\begin{aligned} -\varepsilon^2 u'' + g_1(u,v,x,\varepsilon) &= 0,\\ v'' + g_2(u,v,x,\varepsilon) &= 0,\\ v(0) = v(1) &= u(0) = u(1) = 0,\\ x \in [0,1], \quad \varepsilon &\neq 0, \end{aligned} \tag{1.1}$$

where g_1, g_2 are continuous functions, ε is a small parameter. The author was stimulated by [5, 6], but we shall be interested in the existence of solutions of (1.1). Our approach is quite different, since we can not apply the implicit function theorem due to the nondifferentiability of g_1, g_2. Instead of this, we follow a method from [3]. Thus we use a Galerkin approximation method together with the Leray-Schauder degree theory. We also study asymptotic behavior of solutions as ε tends to zero for some problems; we show the boundary layer phenomenon of u.

The plan of our paper is as follows. In Section 2, we study (1.1) when the second equation has a general form for v, but is a case of perturbations of two independent equations. We show asymptotic behavior of solutions as $\varepsilon \to 0$ for that case. Knowing that result, in Section 3, we apply it for several examples. In Section 4, we investigate (1.1) partly with a different function g_1 as in the previous sections and partly for the case when g_2 has a specific discontinuity in v. We can show only the existence of solutions for such problems, not their asymptotic behavior.

2. General problems. Let us consider the couple of equations

$$\begin{aligned} &-\varepsilon^2 u'' + f(u,x) + \varepsilon \cdot \phi(u,v,x) = 0,\\ &v = G(v) + \varepsilon \cdot H(u,v),\\ &u(0) = u(1) = 0, \end{aligned} \tag{2.1}$$

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where $f\colon [0,1] \to \mathbb{R}$, $\phi\colon \mathbb{R} \times \mathbb{R}^m \times [0,1] \to \mathbb{R}$, $G\colon Z \to Z$, $H\colon X \times Z \to Z$, $Z \subset C^0([0,1], \mathbb{R}^m)$ is a closed, linear subset, $X = C^0([0,1], \mathbb{R})$, ϕ satisfies Carathèodory continuity conditions (see [2]) and there are $r \in L^2(0,1)$, $\tilde{\phi}$ continuous such that

$$|\phi(u,v,x)| \le \tilde{\phi}(u,v) + r(x).$$

Furthermore, the operators G, H are compact, continuous and f is C^2-smooth.

Moreover, let us assume $|f_y| < \beta$, where $f = f(y,x)$, $f_y = \frac{\partial f}{\partial y}$, and

i) there is a C^2-function $h\colon [0,1] \to \mathbb{R}$ such that $f(h(x),x) = 0$ and $f_y(h(x),x) > \alpha > 0$;

ii) for each $\varepsilon \ne 0$ small, there is h_ε such that

$$\begin{aligned}
&|-\varepsilon^2 h_\varepsilon'' + f(h_\varepsilon, x)|_{C^0} = O(\varepsilon),\\
&h_\varepsilon(0) = h_\varepsilon(1) = 0,\ h_\varepsilon = h_{-\varepsilon},\\
&|h_\varepsilon|_{C^0} \le M,\ |\varepsilon h_\varepsilon'|_{C^0} \le M, \quad M \text{ is a constant},\\
&h_\varepsilon \to h \text{ uniformly on each } [\delta, 1-\delta],\ 0 < \delta < 1/2 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Conditions which guarantee the existence of h_ε can be found in [5].

We note, if it is possible, we shall use the notation C^0 instead of $C^0([0,1], \mathbb{R}^m)$ throughout this paper. The same holds for other functional spaces.

Theorem 2.1. *Assume there is a bounded, open subset $\Omega \subset Z$ such that*

$$\begin{aligned}
&i) \quad v \ne G(v) \text{ on } \partial\Omega,\\
&ii) \quad \mathrm{ind}\,(v - G(v), 0, \Omega) \ne 0.
\end{aligned}$$

Then there is a generalized solution $(u_\varepsilon, v_\varepsilon) \in W^{2,2}(0,1) \times C^0([0,1], \mathbb{R}^m)$ of (2.1) for each $\varepsilon \ne 0$ small satisfying

(i) $\lim_{\varepsilon \to 0} u_\varepsilon \to h$ *uniformly on each* $[\delta, 1-\delta]$, $0 < \delta < 1/2$;

(ii) $|\varepsilon u_\varepsilon'|_{C^0} \le M$ *for a constant* M;

(iii) $v_\varepsilon \in \Omega$ *and* $\lim_{\varepsilon \to 0} |v_\varepsilon - \{v \in Z \mid v = G(v)\}|_Z = 0$, *where* $|\cdot|_S$ *is a norm in the space* S.

Moreover,

$$|u_\varepsilon - h_\varepsilon|_{C^0} = O(|\varepsilon|^{1/2}) \quad \text{and } |u_\varepsilon - h_\varepsilon|_{L^2} = O(\varepsilon) \text{ as } \varepsilon \to 0.$$

Proof. We put $u = \varepsilon \cdot w + h_\varepsilon$ and modify (2.1) as

$$\begin{aligned}
&-\varepsilon^2 \varepsilon w'' + f_y(h_\varepsilon, x)\varepsilon w\\
&+ \big(f(\rho(|\varepsilon w|_{C^0})\varepsilon w + h_\varepsilon, x) - f(h_\varepsilon, x) - f_y(h_\varepsilon, x) \cdot \rho(|\varepsilon w|_{C^0}) \cdot \varepsilon w\big),\\
&+ \big\{\varepsilon\phi(\rho(|\varepsilon w|_{C^0})\varepsilon w + h_\varepsilon, v, x) + (-\varepsilon^2 h_\varepsilon'' + f(h_\varepsilon, x))\big\} = 0 \qquad (2.1+)\\
&v = G(v) + \varepsilon H(\rho(|\varepsilon w|_{C^0}) \cdot \varepsilon w + h_\varepsilon, v),\\
&w(0) = w(1) = 0,
\end{aligned}$$

where $\rho : \mathbb{R} \to [0,1]$ is a continuous function such that $\rho/[-\delta, \delta] = 1$, $\rho/(-\infty, \infty) \setminus (-2\delta, 2\delta) = 0$ for a constant $\delta > 0$ small.

Our assumptions imply that the term in the angle bracket has a L^2-norm over $[0,1]$ approximately $|\varepsilon|$ for $v \in \Omega$. Applying partly this estimate and partly the Taylor expansion formula and the fact $|\rho(x) \cdot x| \le 2 \cdot \delta$, $\forall x \in \mathbb{R}$ to the second term in (2.1+), equation (2.1+) has the form, for ε small,

$$\begin{aligned} & -\varepsilon^2 w'' + f_y(h_\varepsilon, x)w + \rho^2(|\varepsilon w|_{C^0}) \cdot \varepsilon w \cdot d_1(\varepsilon, w, v, x)w + d_2(\varepsilon, w, v, x) = 0, \\ & v = G(v) + \varepsilon H(\rho(|\varepsilon w|_{C^0}) \cdot \varepsilon w + h_\varepsilon, v), \\ & w(0) = w(1) = 0, \end{aligned} \tag{2.2}$$

where $|d_1(\varepsilon, w, v, \cdot)|_{C^0} = O(1)$, $|d_2(\varepsilon, w, v, \cdot)|_{L^2} = O(1)$ uniformly by w, v and $\varepsilon \to 0$. We consider (2.2) on the space $W_0^{1,2}(0,1) \times Z$, when the first equation has the form

$$\begin{aligned} & \big(F_\varepsilon(w,v), z\big)_{W_0^{1,2}} = \\ & \int_0^1 \big(\varepsilon^2 w' z' + f_y(h_\varepsilon, x)wz + \rho^2(|\varepsilon w|_{C^0})\varepsilon w d_1(\varepsilon, w, v, x)wz + d_2(\varepsilon, w, v, x)z\big)\,dx \\ & = 0, \end{aligned} \tag{2.3}$$

where $(\cdot,\cdot)_{W_0^{1,2}}$ is the scalar product on $W_0^{1,2}(0,1)$. Using [10, Lemma 6.2], $|\rho^2(x) \cdot x| \le 2\delta$, $\forall x \in \mathbb{R}$ and $|d_1|_{C^0} = O(1)$, $|d_2|_{L^2} = O(1)$, we obtain for fixed ε small and δ small,

$$\big(F_\varepsilon(w,v), w\big)_{W_0^{1,2}} \ge \tilde{\alpha} \cdot |w|_{L^2}^2 - c \cdot |w|_{L^2} \tag{2.4}$$

for constants $c > 0$, $\tilde{\alpha} > 0$.

We take finite dimensional subspaces H_N, $N \in \mathcal{N}$ such that $\overline{\cup H_N} = W_0^{1,2}(0,1)$ and H_N contain the usual functions, with the orthogonal projections P_N. Let us solve

$$P_N F_\varepsilon(w,v) = 0, \quad v = G(v) + \varepsilon H(\rho(|\varepsilon w|_{C^0}) \cdot \varepsilon w + h_\varepsilon, v) \tag{2.5}$$

for $w \in \{z \in H_N : |z|_{L^2} \le (c+1)/\tilde{\alpha}\}$, $v \in \Omega$. By (2.4) and the assumptions for G, H, we can solve (2.5) and this equation has a solution

$$(w_{N,\varepsilon}, v_{N,\varepsilon}).$$

From (2.5) and $|w_{N,\varepsilon}|_{L^2} \le (c+1)/\tilde{\alpha} = K_1$, it follows that $|\varepsilon w'_{N,\varepsilon}|_{L^2} \le K_2$ for a constant K_2 independent of ε, N. Hence we can assume

$$w_{N,\varepsilon} \to w_\varepsilon,\ v_{N,\varepsilon} \to v_\varepsilon \text{ uniformly on } [0,1],$$

and by (2.3) we have that $(w_\varepsilon, v_\varepsilon)$ is a solution of (2.1+). Moreover, we know

$$v_\varepsilon \in \Omega,\ |w_\varepsilon|_{L^2} \le M,\ |\varepsilon w'_\varepsilon|_{L^2} \le M,$$

and by (2.2) we obtain $|\varepsilon^2 \cdot w''_\varepsilon|_{L^2} \le M$ for a constant M. Now, applying the Sobolev inequalities we arrive at

$$|\varepsilon \cdot w_\varepsilon|_{C^0} \to 0,\ \ |\varepsilon^2 \cdot w'_\varepsilon|_{C^0} \to 0.$$

Finally, since $\rho/[-\delta,\delta]=1$, we see that $u_\varepsilon=\varepsilon\cdot w_\varepsilon+h_\varepsilon$ is a solution of our equation (2.1) for ε small and u_ε, v_ε possess the desired properties. To show the last assertion of Theorem 2.1, we compute

$$w_\varepsilon^2(x)=\int_0^x \left(w_\varepsilon^2(s)\right)'ds=2\int_0^x w_\varepsilon'(s)w_\varepsilon(s)ds\le 2|w_\varepsilon'|_{L^2}|w_\varepsilon|_{L^2}\le 2M^2/|\varepsilon|.$$

Hence

$$|\varepsilon w_\varepsilon|_{C^0}=O(|\varepsilon|^{1/2}).$$

Thus

$$|u_\varepsilon-h_\varepsilon|_{C^0}=|\varepsilon w_\varepsilon|_{C^0}=O(|\varepsilon|^{1/2}).$$

The proof is finished.

Corollary 2.2. *If $\Omega\cap\{v\in Z: v=G(v)\}=\{v_1,\dots,v_k\}$ in Theorem (2.1), then (2.1) possesses solutions $(u_\varepsilon,v_\varepsilon)$ such that $|v_\varepsilon-v_j|_{C^0}\to 0$ as $\varepsilon\to 0$ for some j fixed.*

Proof. Since ind $(v-G(v),0,\Omega)\neq 0$, the index of some v_j is nonzero. Now we take a small ball near this point instead of Ω and apply the above theorem.

Corollary 2.3. *If $\Omega\cap\{v\in Z: v=G(v)\}=\{v_1\}$ in Theorem (2.1). Then $|v_\varepsilon-v_1|_{C^0}\to 0$ as $\varepsilon\to 0$.*

Now we give some remarks concerning the approximate function h_ε. The above function h_ε is strong in the sense that

$$|-\varepsilon^2h_\varepsilon''+f(h_\varepsilon,x)|_{C^0}=O(\varepsilon)\text{ as }\varepsilon\to 0,$$

because more natural approximation can be chosen in the following way. Let $\theta:\mathbb{R}\to[0,1]$ be a C^∞-smooth function such that $\theta/(-\infty,0]=0$, $\theta/[\tilde\delta,\infty)=1$, $\tilde\delta>0$, and put $\tilde h_\varepsilon(x)=\psi_\varepsilon(x)h(x)$, where $\psi_\varepsilon(x)=\theta(x/|\varepsilon|)\theta((1-x)/|\varepsilon|)$. For such a function $\tilde h_\varepsilon$, we have

$$\begin{aligned}&|-\varepsilon^2\tilde h_\varepsilon''+f(\tilde h_\varepsilon,x)|_{C^0}=O(1),\quad \tilde h_\varepsilon(0)=\tilde h_\varepsilon(1)=0,\\&\tilde h_\varepsilon(x)=h(x)\quad\text{for}\quad \tilde\delta|\varepsilon|\le x\le 1-\tilde\delta|\varepsilon|,\\&|-\varepsilon^2\tilde h_\varepsilon''+f(\tilde h_\varepsilon,x)|_{L^2}=O(|\varepsilon|^{1/2}).\end{aligned}\tag{2.6}$$

Now put $u=|\varepsilon|^{1/2}w+\tilde h_\varepsilon$ and follow the above procedure for $2\beta\tilde\delta^2<1$. For this case, a corresponding equation of a form like (2.1+) has a variable $|\varepsilon|^{1/2}w$ in the function ρ, i.e., take $\rho(|\varepsilon|^{1/2}w)$ instead of $\rho(|\varepsilon w|_{C^0})$ in (2.1+). We can repeat the above proof of Theorem 2.1 to show the existence of solutions $(w_\varepsilon,v_\varepsilon)$ of this equation for $\varepsilon\neq 0$ small such that

$$|w_\varepsilon|_{L^2}\le\tilde K,\quad |\varepsilon w_\varepsilon'|_{L^2}\le\tilde K,\quad |\varepsilon^2w_\varepsilon''|_{L^2}\le\tilde K,$$

for a constant $\tilde K>0$. Indeed, it is sufficient to show

$$\int_0^1\left(\varepsilon^2(v')^2+f_y(\tilde h_\varepsilon,x)v^2\right)dx\ge c|v|_{L^2}^2,\quad c>0.$$

For this purpose, we use the following inequalities [2, p. 203]:

$$\int_0^l v^2(x)dx \le l^2 \int_0^l \left(v'(x)\right)^2 dx, \quad \int_l^1 v^2(x)dx \le (1-l)^2 \int_l^1 \left(v'(x)\right)^2 dx$$

for any $v \in C^1$, $v(0) = v(1) = 0$, $0 < l < 1$, and compute, by applying also (2.6),

$$\begin{aligned}
&\int_0^1 \left(\varepsilon^2(v')^2 + f_y(\tilde{h}_\varepsilon, x)v^2\right)dx \\
&\ge \int_0^1 f_y(h,x)v^2 dx + \int_{1-\tilde{\delta}\varepsilon}^1 \Big(\varepsilon^2(v')^2 + \big(f_y(\tilde{h}_\varepsilon, x) - f_y(h,x)\big)v^2\Big)dx \\
&\quad + \int_0^{\tilde{\delta}\varepsilon} \Big(\varepsilon^2(v')^2 + \big(f_y(\tilde{h}_\varepsilon, x) - f_y(h,x)\big)v^2\Big)dx \\
&\ge \alpha|v|_{L^2}^2 + \int_{1-\tilde{\delta}\varepsilon}^1 \varepsilon^2(v')^2 dx - 2\beta \int_{1-\tilde{\delta}\varepsilon}^1 v^2 dx + \varepsilon^2 \int_0^{\tilde{\delta}\varepsilon} (v')^2 dx - 2\beta \int_0^{\tilde{\delta}\varepsilon} v^2 dx \\
&\ge \alpha|v|_{L^2}^2 + \varepsilon^2(1 - 2\beta\tilde{\delta}^2)\Big(\int_0^{\tilde{\delta}\varepsilon} (v')^2 dx + \int_{1-\tilde{\delta}\varepsilon}^1 (v')^2 dx\Big).
\end{aligned}$$

Note that we can not derive the relation $|\varepsilon|^{1/2} w_\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly on $[0,1]$ from the above inequalities for w_ε, but we have

$$|\varepsilon|^{1/2}|w_\varepsilon|_{L^2} \le \tilde{K}|\varepsilon|^{1/2}.$$

Thus the Lebesque measure of the set $\mathcal{M}_\varepsilon = \{x \in [0,1] : |\varepsilon|^{1/2}|w_\varepsilon(x)| \ge \delta\}$ satisfies

$$\mu(\mathcal{M}_\varepsilon) \le \tilde{K}^2|\varepsilon|/\delta^2,$$

where $\tilde{K}$ is a constant and $\delta > 0$ is small. On the other hand, the functions $u_\varepsilon = |\varepsilon|^{1/2} w_\varepsilon + \tilde{h}_\varepsilon$, v_ε satisfy the equation (2.1) on the set $[0,1] \setminus \mathcal{M}_\varepsilon$. Summing up we obtain

Theorem 2.4. *Generally, without the existence of h_ε for $2\beta\tilde{\delta}^2 < 1$, there are functions $u_\varepsilon, v_\varepsilon$ and a set $\mathcal{M}_\varepsilon \subset [0,1]$ for each $\varepsilon \ne 0$ small such that*

i) $\mu(\mathcal{M}_\varepsilon) = O(|\varepsilon|)$,
ii) $v_\varepsilon \in \Omega$, $|u_\varepsilon - \tilde{h}_\varepsilon|_{L^2} = O(|\varepsilon|^{1/2})$,
iii) $(u_\varepsilon, v_\varepsilon)$ *are generalized solutions of* (2.1) *on* $[0,1] \setminus \mathcal{M}_\varepsilon$.

Finally, we note that similarly we can solve equation (2.1) with the boundary value conditions

$$\begin{aligned}
&\text{a) } u(0) = \gamma,\ u(1) = \omega, \quad \text{b) } u(0) = \gamma,\ u'(1) = \omega, \\
&\text{c) } u'(0) = \gamma,\ u(1) = \omega, \quad \text{d) } u'(0) = \gamma,\ u'(1) = \omega,
\end{aligned}$$

respectively.

3. Examples. Since we can modify a lot of differential equations in the form $v - G(v) = 0$, where G has the properties of the previous section, we give several examples and remarks in this section. We consider $g_1(u,v,x,\varepsilon) = f(u,x) + \varepsilon\phi(u,v,x)$, where f, ϕ have the above properties and vary only the second equation of (1.1). For instance, let $-g_2(u,v,x,\varepsilon) = g(v,x) + \varepsilon h(u,v,x)$, where g, h are continuous, g is bounded, i.e., $|g| < K$ for a constant K. Consider

$$\begin{aligned} &-\varepsilon^2.u'' + g_1(u,v,x,\varepsilon) = 0,\\ &v'' = g(v,x) + \varepsilon h(u,v,x),\\ &u(0) = u(1) = v(0) = v(1) = 0. \end{aligned} \tag{3.1}$$

Then put

$$\begin{aligned} Z &= \{v \in C^0([0,1],\mathbb{R}) : v(0) = v(1) = 0\},\\ G(v) &= \mathcal{K}g(v,\cdot),\ H(u,v) = \mathcal{K}h(u,v,\cdot), \end{aligned}$$

where $\mathcal{K}$ is the inverse of $v \to v''$, $v(0) = v(1) = 0$. Since g is bounded, the operator G is bounded as well and for $\Omega = B_r = \{v \in Z : |v|_Z \le r\}$, r sufficiently large, we have $v \ne \lambda G(v)$ for any $v \in \partial\Omega$, $\lambda \in [0,1]$. Thus G satisfies the properties of Theorem 2.1. Moreover, assume that g is increasing in v; then

$$v'' = g(v,x), \quad v(0) = v(1) = 0 \tag{3.2}$$

has at most one solution [7]. Thus G has precisely one fixed point. Applying Corollary 2.3, we obtain

Theorem 3.1. *Under the above conditions, equation* (3.1) *has a solution* $(u_\varepsilon, v_\varepsilon)$ *for each* $\varepsilon \ne 0$. *The solution* u_ε *has the properties of Theorem* 2.1 *and the sequence* $\{v_\varepsilon\}$ *tends uniformly to the only solution of* (3.2) *as* $\varepsilon \to 0$.

We obtain the same result also for the boundary value conditions $u(0) = u(1) = v(0) = v'(1) = 0$ and $u(0) = u(0) = v'(0) = v(1) = 0$.

Now we consider the case $u(0) = u(1) = v'(0) = v'(1) = 0$ and modify (3.1) as

$$\begin{aligned} &-\varepsilon^2 u'' + g_1(u, v+c, x, \varepsilon) = 0,\\ &v = \mathcal{K}(I-P)\big(g(v+c,\cdot) + \varepsilon h(u,v+c,\cdot)\big),\\ &c = c + P\big(g(v+c,\cdot) + \varepsilon h(u,v+c,\cdot)\big), \end{aligned}$$

where $c \in \mathbb{R}$, $\mathcal{K}$ is the inverse of $v \to v''$, $v'(0) = v'(1) = 0$, $\int_0^1 v(x)dx = 0$ and $Pv = \int_0^1 v(x)dx$. Hence we have carried out the standard decomposition [8] and we put, for $\lambda \in [0,1]$,

$$G_\lambda(v+c) = \lambda\mathcal{K}(I-P)g(v+c,\cdot) + c + Pg(\lambda v + c,\cdot).$$

We take $Z = \{v \in C^0([0,1],\mathbb{R}) : \int_0^1 v(x)dx = 0\} \oplus \mathbb{R}$ and $\Omega = \{v + c : |v|_{C^0} \le r_1,\ |c| \le r_2\}$ for r_1, r_2 sufficiently large. Assume

$$\liminf_{z\to\infty} g(z,x) > \tilde\beta > 0 > \tilde\gamma > \limsup_{z\to-\infty} g(z,x) \tag{3.3}$$

uniformly for $x \in [0,1]$. Then utilizing (3.3), we fix r_1 sufficiently large and choose r_2 such that

$$\int_0^1 g(\lambda v + c, x)dx > \tilde\beta/2 > 0, \qquad \int_0^1 g(\lambda v + c, x)dx < \tilde\gamma/2 < 0$$

for each λ, $|v|_{C^0} \le r_1$, $c = r_2$, $c = -r_2$, respectively. Then it is clear that G_1 has the desired properties of Theorem 2.1. We note that this approach is also well-known [9] and (3.3) is a type of Landesman-Lazer conditions. Furthermore, if g is increasing in v then $v'' = g(v,x)$, $v'(0) = v'(1) = 0$ has at most one solution. Summing up we *obtain*

Theorem 3.2. *Let us consider* (3.1) *with the boundary value condition* $u(0) = u(1) = v'(0) = v'(1) = 0$ *and assume that* g *is increasing in* v *and bounded. If there is a continuous function* $\tilde{r} : [0,1] \to \mathbb{R}$ *such that* $g(\tilde{r}(x), x) = 0$, $\forall x \in [0,1]$, *then this equation has a solution* $(u_\varepsilon, v_\varepsilon)$ *for each* $\varepsilon \neq 0$ *small, where* u_ε *has the properties of Theorem* 2.1 *and* v_ε *tends uniformly to the unique solution of* $v'' = g(v, x)$, $v'(0) = v'(1) = 0$ *as* $\varepsilon \to 0$.

Proof. Since $g(\tilde{r}(x), x) = 0$, $\forall x \in [0,1]$ and g is increasing in v, condition (3.3) is satisfied. According to the above results, we can apply Corollary 2.3 to finish the proof.

4. Special cases. First, we return to (2.1), when g_1, g_2 depend only on u, v. Let assume that g_1 is C^2-smooth and there is a C^2-function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ such that

i) $g_1(\tilde{h}(v), v) = 0$, for each $v \in \mathbb{R}$;

ii) $g_{1u}(\tilde{h}(v), v) > \alpha > 0$, $|g_{1u}(.,.)| < \tilde{\alpha}$, where α, $\tilde{\alpha}$ are constants.

Moreover, we assume that g_2 is globally Lipschitz in $u \in \mathbb{R}$ with a constant M. Hence we consider the system of equations

$$\begin{aligned} &-\varepsilon^2 u'' + g_1(u, v) = 0, \\ &v'' + g_2(u, v) = 0, \\ &u(0) = u(1) = v(0) = v(1) = 0, \end{aligned} \tag{4.1}$$

where g_1, g_2 have the above properties.

For any $y \in C^0([0,1], \mathbb{R})$, we put $v = \mathcal{K}y \in Z = \{z \in C^0([0,1], \mathbb{R}) : z(0) = z(1) = 0\}$, where $\mathcal{K}$ is the inverse of $v \to v''$, $v(0) = v(1) = 0$. Hence (4.1) has the form

$$\begin{aligned} &-\varepsilon^2 u'' + g_1(u, \mathcal{K}y) = 0, \\ &y + g_2(u, \mathcal{K}y) = 0, \quad y \in C^0([0,1], \mathbb{R}). \end{aligned} \tag{4.2}$$

Let us assume

$$\int_{\tilde{h}(0)}^{k} g_1(u, 0)\, du > 0 \text{ for } k \neq \tilde{h}(0) \text{ in the closed interval between } \tilde{h}(0) \text{ and } 0.$$

For each $y \in C^0$, $u \in \mathbb{R}$, we take $g_1(u, \mathcal{K}y) = f(u, x)$ and we construct by following [5, p. 500] the approximate function

$$h_{\varepsilon,y}(x) = \tilde{h}(\mathcal{K}y(x)) + z_0(x, \varepsilon) + z_1(x, \varepsilon)$$

(see (2.8) in [5]). We note that

$$\begin{aligned} f\big(\tilde{h}(\mathcal{K}y(x)), x\big) &= g_1\big(\tilde{h}(\mathcal{K}y(x)), \mathcal{K}y(x)\big) = 0, \\ f_u\big(\tilde{h}(\mathcal{K}y(x)), x\big) &= g_{1u}\big(\tilde{h}(\mathcal{K}y(x)), \mathcal{K}y(x)\big) > 0 \end{aligned}$$

and we also see that for $i = 0, 1$,

$$\int_{\tilde{h}(\mathcal{K}y(i))}^{k} f(u, i)\, du = \int_{\tilde{h}(0)}^{k} g_1(u, 0)\, du > 0$$

for $k \neq \tilde{h}(\mathcal{K}y(i)) = \tilde{h}(0)$ in the closed interval between $\tilde{h}(0)$ and 0. Hence the assumptions (2.3-5) from [5] are satisfied and we have been able to perform the construction of $h_{\varepsilon,y}$.

Directly we can check that

$$|-\varepsilon^2 h''_{\varepsilon,y} + g_1(h_{\varepsilon,y}, \mathcal{K}y)|_{C^0} = O(\varepsilon),$$

$$h_{\varepsilon,y} \rightrightarrows \tilde{h}(\mathcal{K}y) \quad \text{on} \quad [\delta, 1-\delta],\, 0 < \delta < 1/2 \quad \text{as} \quad \varepsilon \to 0$$

uniformly on each bounded subset of $C^0([0,1],\mathbb{R})$ according to $y \in C^0([0,1],\mathbb{R})$.

Now we assume: There is an open, bounded subset $\emptyset \neq \Omega \subset C^0([0,1],\mathbb{R})$ such that

$$\begin{aligned} &\text{i)} \quad v'' + g_2\big(\tilde{h}(v), v\big) = 0, \quad v(0) = v(1) = 0 \quad \text{has no solution in } \partial\Omega, \\ &\text{ii)} \quad \operatorname{ind}\big(v + g_2(\tilde{h}(\mathcal{K}v), \mathcal{K}v), 0, \Omega\big) \neq 0. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Under the above condition* (4.3), *equation* (4.1) *has a solution for each* $\varepsilon \neq 0$ *small provided that* M *is sufficiently small.*

Proof. We put $u = \varepsilon w + h_{\varepsilon,y}$ and modify (4.2) as in the proof of Theorem 2.1:

$$\begin{aligned} &-\varepsilon^2 \varepsilon w'' + g_{1u}(h_{\varepsilon,y}, \mathcal{K}y)\varepsilon w + \big(g_1(\rho(|\varepsilon w|_{C^0})\varepsilon w + h_{\varepsilon,y}, \mathcal{K}y) - g_1(h_{\varepsilon,y}, \mathcal{K}y) \\ &\quad - g_{1u}(h_{\varepsilon,y}, \mathcal{K}y)\rho(|\varepsilon w|_{C^0})\varepsilon w\big) - \varepsilon^2 h''_{\varepsilon,y} + g_1(h_{\varepsilon,y}, \mathcal{K}y) = 0, \\ &y + g_2\big(\tilde{h}(\mathcal{K}y), \mathcal{K}y\big) + g_2\big(\rho(|\varepsilon w|_{C^0})\varepsilon w + h_{\varepsilon,y}, \mathcal{K}y\big) - g_2\big(\tilde{h}(\mathcal{K}y), \mathcal{K}y\big) = 0. \end{aligned} \tag{4.4}$$

Now since g_2 is globally Lipschitz in $u \in \mathbb{R}$ with the constant M, we have

$$|g_2\big(\rho(|\varepsilon w|_{C^0})\varepsilon w + h_{\varepsilon,y}, \mathcal{K}y\big) - g_2\big(\tilde{h}(\mathcal{K}y), \mathcal{K}y\big)| \leq M\big(|h_{\varepsilon,y} - \tilde{h}(\mathcal{K}y)| + 2\delta\big)$$

and for M sufficiently small, the perturbation of the second equation of (4.4) is small. Now we follow the proof of Theorem 2.1, since equation (4.4) is very similar to (2.1+).

Corollary 4.2. *If all solutions of* (4.3) *i) in Theorem* 4.1 *are isolated in* Ω, *then we can construct such solutions* $(u_\varepsilon, v_\varepsilon)$ *of* (4.1) *that*

i) $|v_\varepsilon - v_0|_{C^0} \to 0$ *as* $\varepsilon \to 0$, *where* v_0 *is a solution of* (4.3) *i) in* Ω;
ii) $u_\varepsilon \to \tilde{h}(v_0)$ *uniformly on* $[\delta, 1-\delta]$, $0 < \delta < 1/2$ *as* $\varepsilon \to 0$;
iii) $|\varepsilon u'_\varepsilon|_{C^0} \leq c$ *for a constant* c.

Proof. The proof is analogous to the proof of Corollary 2.2. □

Now we extend our method to other cases. First, we consider

$$\begin{aligned} &-\varepsilon^2 u'' + a(u,v,x)u + b(u,v,x) = 0, \\ &v = G(v) + \varepsilon H(u,v), \\ &u(0) = u(1) = 0, \end{aligned} \tag{4.5}$$

where a, b are continuous, bounded, $a(\cdot,\cdot,\cdot) > \alpha > 0$ for a constant α, and G, H have the properties of Theorem 2.1.

Theorem 4.3. *The equation* (4.5) *possesses a solution* $(u_\varepsilon, v_\varepsilon)$ *for each* $\varepsilon \neq 0$ *small such that* $v_\varepsilon \in \Omega$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Next we study the following version of (4.5):

$$\begin{aligned} &-\varepsilon^2 u'' + a(u,v,x)u + \varepsilon b(u,v,x) = 0,\\ &v = H(u,v),\\ &u(0) = u(1) = 0, \end{aligned} \tag{4.6}$$

where H has the properties of Theorem 2.1. Moreover, there is an open, bounded subset $\emptyset \neq \Omega \subset Z$ such that $v \neq H(0,v)$ on $\partial\Omega$ and $\operatorname{ind}\big(v - H(0,v), 0, \Omega\big) \neq 0$.

Theorem 4.4. *Under the above conditions, equation* (4.6) *has a solution* $(u_\varepsilon, v_\varepsilon)$ *for each* $\varepsilon \neq 0$ *small such that*

i) $u_\varepsilon \to 0$, $\varepsilon u'_\varepsilon \to 0$ *uniformly on* $[0,1]$ *as* $\varepsilon \to 0$;
ii) $v_\varepsilon \in \Omega$ *and* $|v_\varepsilon - \{v \in \Omega : v = H(0,v)\}|_Z \to 0$ *as* $\varepsilon \to 0$.

Proof. We put $u = \varepsilon w$ and (4.6) has the form

$$-\varepsilon^2 w'' + a(\varepsilon w, v, x)w + b(\varepsilon w, v, x) = 0, \quad v = H(\varepsilon w, v). \tag{4.7}$$

We follow the proof of Theorem 2.1 as in Theorem 4.3.

Theorem 4.5. *Let* a, b *be* C^2*-smooth and independent of* x. *Moreover, we assume* $Z = \{v \in C^0([0,1],\mathbb{R}) : v(0) = v(1) = 0\}$, H *is Lipschitz in* u *uniformly by* v *and the set* $\{v \in \Omega : v = H(0,v)\}$ *contains only isolated points. Then* (4.6) *has a solution* $(u_\varepsilon, v_\varepsilon)$ *for each* $\varepsilon \neq 0$ *small such that*

i) $u_\varepsilon = \varepsilon w_\varepsilon$, $|w_\varepsilon|_{C^0} \leq M$, $|\varepsilon w'_\varepsilon|_{C^0} \leq M$ *for a constant* M, *and* $w_\varepsilon \to \big(-b(0,v_0(x))/a(0,v_0(x)\big)$ *uniformly on* $[\delta, 1-\delta]$, *where* v_0 *is a solution of* $v = H(0,v)$ *in* Ω, $0 < \delta < 1/2$;
ii) $|v_\varepsilon - v_0|_{C^0} \to 0$ *as* $\varepsilon \to 0$.

Proof. It follows from (4.7) that

$$\begin{aligned} &-\varepsilon^2 w'' + a(0,v)w + b(0,v) + \{(a(\varepsilon w,v) - a(0,v))w + b(\varepsilon w,v) - b(0,v)\} = 0,\\ &v = H(0,v) + \{H(\varepsilon w,v) - H(0,v)\}. \end{aligned}$$

By our assumptions, the terms in the angle brackets are approximately $O(\varepsilon)$. Hence we can follow the proof of Theorem 4.1 (see also the proof of Corollary 4.2). □

In the last part of this section, we study the system of equations

$$\begin{aligned} &-\varepsilon^2 u'' + a(u,v,x)u + b(u,v,x) = 0,\\ &-v'' = g(v) + \varepsilon h(u,v,x),\\ &u(0) = u(1) = v(0) = v(1) = 0, \end{aligned} \tag{4.8}$$

where a, b have the properties of (4.5), $g \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R})$, $\beta > g'(x) > 1/M > 0$ for $x \neq 0$, β, M constants. Furthermore, we assume

$$\lim_{x \to 0_-} g(x) = g(0-) < g(0+) = \lim_{x \to 0_+} g(x),$$

h is C^1-smooth and $|h|$, $|\frac{\partial h}{\partial v}| < M$. We see that (4.8) is a discontinuous version of (3.1). The author was originally inspired by the paper [1] and we follow an approach of the paper [4].

It is clear that the mapping from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$,

$$z \to g(z) + \varepsilon h(u, z, x) = \Psi(u, z, x, \varepsilon),$$

is increasing for each $u \in \mathbb{R}$, x fixed and ε small. This map has a discontinuity at the point 0. We define for $u \in \mathbb{R}, x, \varepsilon$ fixed,

$$\Phi(z, u, x, \varepsilon) = \begin{cases} 0 & \text{if } z \in [\tilde{g}(0-), \tilde{g}(0+)], \\ s & \text{with } \Psi(u, s, x, \varepsilon) = z \text{ for } z \notin [\tilde{g}(0-), \tilde{g}(0+)], \end{cases}$$

where $\tilde{g}(0\pm) = g(0\pm) + \varepsilon h(u, 0, x)$. Hence $\Phi(z, u, x, \varepsilon) = d$ if and only if $z \in [\Psi(u, d-, x, \varepsilon), \Psi(u, d+, x, \varepsilon)]$ and Φ is continuous, where

$$\lim_{w \to d\pm} \Psi(u, w, x, \varepsilon) = \Psi(u, d\pm, x, \varepsilon).$$

Now we shall modify (4.8). We put $y(x) = \Psi(u, v, x, \varepsilon) = -v''$. We can not compute v from this equation since Ψ is discontinuous in v. Instead, we put

$$-\mathcal{K}y = v = \Phi(y, u, x, \varepsilon).$$

Hence we consider the problem

$$\begin{aligned} &-\varepsilon^2 u'' + a(u, -\mathcal{K}y, x)u + b(u, -\mathcal{K}y, x) = 0, \\ &\Phi(y, u, x, \varepsilon) + \mathcal{K}y = 0, \\ &u(0) = u(1) = 0,\ y \in L^2(0,1). \end{aligned} \tag{4.9}$$

Theorem 4.6. *Let $\beta < \pi^2$. Then for each $\varepsilon \neq 0$ small,* (4.9) *has a solution, i.e., the following relation*

$$\begin{aligned} &-\varepsilon^2 u'' + a(u, v, x)u + b(u, v, x) = 0, \quad v \in H^2(0,1), \\ &-v''(x) \in [g(v-) + \varepsilon h(u, v, x), g(v+) + \varepsilon h(u, v, x)] \quad \text{a.e. in } (0,1), \\ &u(0) = u(1) = v(0) = v(1) = 0 \end{aligned}$$

has a solution for each $\varepsilon \neq 0$ small.

Proof. We define by using (4.9) an operator on $W_0^{1,2}(0,1) \times L^2(0,1) = X$ as

$$\big(F_\varepsilon(u,y), w\big)_{W_0^{1,2}} = \int_0^1 \big(\varepsilon^2 u'w' + a(u, -\mathcal{K}y, x)uw + b(u, -\mathcal{K}y, x)w\big)\,dx,$$

$$G_\varepsilon(u, y) = \Phi(y, u, \cdot, \varepsilon),$$

and solve $F_\varepsilon = 0$, $G_\varepsilon + \mathcal{K} = 0$. We have

$$\big(F_\varepsilon(u, v), u\big)_{W_0^{1,2}} \geq \alpha |u|_{L^2}^2 - c|u|_{L^2}$$

for a constant c. Since $\beta > g' > 0$ and $|\frac{\partial h}{\partial v}| < M$, we can choose ε_0 for each $\delta > 0$ small such that for all $0 < |\varepsilon| < \varepsilon_0$,

$$\Phi(y_1, u_1, x, \varepsilon) y_1 \geq \frac{1}{\beta + \delta} y_1^2 - c, \quad \forall y_1, u_1 \in \mathbb{R}, x \in [0,1]$$

for a constant c. Hence

$$\big(\Phi(y, u, \cdot, \varepsilon), y\big)_{L^2} \geq \frac{1}{\beta + \delta} |y|_{L^2}^2 - c.$$

Thus

$$\big(\Phi(y, u, \cdot, \varepsilon) + \mathcal{K} y, y\big)_{L^2} \geq \Big(\frac{1}{\beta + \delta} - \|\mathcal{K}\|_{L^2}\Big) |y|_{L^2}^2 - c.$$

Since $\|\mathcal{K}\|_{L^2} = 1/\pi^2$ and $\beta < \pi^2$, we have for δ small and fixed,

$$\begin{aligned} &\big(F_\varepsilon(u, y), u\big)_{W_0^{1,2}} \geq \alpha |u|_{L^2}^2 - c|u|_{L^2}, \\ &\big(G_\varepsilon(u, y) + \mathcal{K} y, y\big)_{L^2} \geq \gamma |y|_{L^2}^2 - c \end{aligned} \tag{4.10}$$

for constants $\gamma > 0$, c. Now we take finite dimensional subspaces $H_N \subset X$ such that $\overline{\cup H_N} = X$, H_N contains C^∞-smooth functions. Let us solve

$$P_N^1 F_\varepsilon(u, y) = 0, \quad P_N^2 \big(G_\varepsilon(u, y) + \mathcal{K} y\big) = 0, \ \ (u, y) \in H_N, \tag{4.11}$$

where $P_N = (P_N^1, P_N^1)$ are orthogonal projections onto H_N for each $N \in \mathcal{N}$. By using (4.10) we can solve this equation in $\Omega_N = H_N \cap \{(u, v) : |u|_{L^2} \leq \tilde{K},\ |y|_{L^2} \leq \tilde{K}\}$ for $\tilde{K}$ large and fixed. Thus (4.11) has a solution $(u_{N,\varepsilon}, y_{N,\varepsilon})$ in Ω_N for any N, ε. Following the proof of Theorem 2.1, we can easily check that

$$|\varepsilon u'_{N,\varepsilon}|_{L^2} \leq K_1, \quad |\varepsilon^2 . u''_{N,\varepsilon}|_{L^2} \leq K_1$$

for a constant K_1. Utilizing these inequalities and $|y_{N,\varepsilon}|_{L^2} \leq \tilde{K}$, we can assume that

$$\begin{aligned} &u_{N,\varepsilon} \to u_\varepsilon \quad \text{uniformly on} \quad [0,1] \quad \text{by} \quad N, \\ &y_{N,\varepsilon} \rightharpoonup y_\varepsilon \quad \text{weakly in} \quad L^2. \end{aligned}$$

Thus $\mathcal{K} y_{N,\varepsilon} \to \mathcal{K} y_\varepsilon$ uniformly on $[0,1]$. We have from the definition of F_ε,

$$F_\varepsilon(u_\varepsilon, y_\varepsilon) = 0.$$

Furthermore, we note that $G_\varepsilon(u, y)$ is monotone in y, i.e.,

$$\big(G_\varepsilon(u, y_1) - G_\varepsilon(u, y_2), y_1 - y_2\big)_{L^2} \geq 0$$

for each y_1, $y_2 \in L^2$. Hence for any $z \in L^2$,

$$0 \leq \big(G_\varepsilon(u_{N,\varepsilon}, y_{N,\varepsilon}) - G_\varepsilon(u_{N,\varepsilon}, z), y_{N,\varepsilon} - z\big)_{L^2}.$$

Utilizing

$$\begin{aligned} &P_N^2 G_\varepsilon(u_{N,\varepsilon}, y_{N,\varepsilon}) = -P_N^2 \mathcal{K} y_{N,\varepsilon},\, u_{N,\varepsilon} \to u_\varepsilon \quad \text{in} \quad C^0, \\ &\mathcal{K} y_{N,\varepsilon} \to \mathcal{K} y_\varepsilon \quad \text{in} \quad L^2, \\ &y_{N,\varepsilon} \rightharpoonup y_\varepsilon \quad \text{weakly in} \quad L^2, \end{aligned}$$

we have

$$0 \leq \big(-\mathcal{K} y_\varepsilon - G_\varepsilon(u_\varepsilon, z), y_\varepsilon - z \big)_{L^2}$$

for any $z \in L^2$. Using the Minty's method [2]: $z = y_\varepsilon - b z_1$, $b > 0$, we obtain

$$0 \leq \big(-\mathcal{K} y_\varepsilon - G_\varepsilon(u_\varepsilon, y_\varepsilon - b z_1), z_1 \big)_{L^2}$$

for any $z_1 \in L^2$. We have for $b \to 0_+$,

$$0 \leq \big(-\mathcal{K} y_\varepsilon - G_\varepsilon(u_\varepsilon, y_\varepsilon), z_1 \big)_{L^2}.$$

Hence

$$-\mathcal{K} y_\varepsilon = G_\varepsilon(u_\varepsilon, y_\varepsilon)$$

and the proof is finished. □

We must note that our approach to (4.8) can be applied to equations similar to (4.8), where we can consider the first equation of (2.1) instead of the first equation of (4.8). We can also vary the boundary value conditions for v.

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