TRANSVERSAL AND NONTRANSVERSAL INTERSECTIONS
OF STABLE AND UNSTABLE MANIFOLDS IN
REACTION DIFFUSION EQUATIONS ON SYMMETRIC DOMAINS

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To the memory of Peter Hess

Abstract. Scalar reaction-diffusion equations on a ball in $\mathbb{R}^N$, $N \geq 2$, with radially symmetric nonlinearities and Dirichlet boundary condition are considered. If the nonlinearity is nonincreasing in the radial variable (in particular if it is independent of it) it is proved that the stable and unstable manifolds of any two nonnegative equilibria intersect trasversally. The crucial property used in the proof is that the unstable manifold of a positive equilibrium consists of radially symmetric functions. Then, an equation is constructed that admits two radially symmetric equilibria whose invariant manifolds intersect nontransversally. In the appendix, examples of spatially homogeneous equations with positive equilibria with high Morse indices are given.

1. Introduction. Consider the following semilinear parabolic problem

$$u_t = \Delta u + f(u) \quad \text{on } \Omega$$

(1.1)

$$u|_{\partial \Omega} = 0,$$

(1.2)

where $\Omega$ is the unit ball $\{x \in \mathbb{R}^N : |x| < 1\}$ in the Euclidian space $\mathbb{R}^N$, $N \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ is of class $C^2$.

This problem defines a local semiflow on an appropriate Banach space $X$. Specifically, we choose $X$ to be the Sobolev space $X = W_0^{1,p}(\Omega)$ with $p > N$. Then $X$ imbeds continuously in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, hence (1.1), (1.2) is well posed on $X$ by the theory of [21]. For $u_0 \in X$, we denote by $u(t, \cdot, u_0)$ the maximal solution of (1.1), (1.2) (in the sense of [21], hence a classical solution) satisfying the initial condition $u(0) = u_0$. Let $S(t), t \geq 0,$ be the local semiflow of (1.1), (1.2); that is, $S(t)u_0 = u(t, \cdot, u_0)$ when the latter is defined.

Though problems of the form (1.1), (1.2) present one of the basic classes of infinite-dimensional dynamical systems and have been widely studied, their dynamics have not been fully understood. On one hand, it is known that $S(t)$ has the gradient-like structure, the usual energy functional being its Lyapunov function. Hence all bounded trajectories of (1.1), (1.2) approach a set of equilibria. On the other hand, it has not been decided whether, in general, each such trajectory must converge to just
a single equilibrium. An exception is the case of analytic nonlinearity $f$, where the convergence has been established by Simon [33]. Another interesting open problem is how “frequent” are the Morse-Smale systems among these gradient systems when $f$ (or perhaps also $\Omega$) is varied.

In contrast, for problem (1.1), (1.2) in one space dimension, the answers to the above questions (and much more) have been known for some time. Thus, if $N = 1$ then each bounded trajectory of (1.1), (1.2) converges to a single equilibrium [40, 28]. Further, any two hyperbolic equilibria $e_1, e_2$ have the property that if the stable manifold of $e_1$ intersects the unstable manifold of $e_2$ then the intersection is automatically transversal (see [1, 22]). In particular, if a one-dimensional problem (1.1), (1.2) is dissipative [18] and has only hyperbolic equilibria (which happens generically with respect to $f$, see [6], [22], [30]), then it defines a Morse-Smale system and is thus structurally stable in the sense of [19].

It is quite natural to attempt to use the radial symmetry of $\Omega$ and one-dimensional techniques, in order to obtain information on the dynamics of (1.1), (1.2). This idea stems of course from the fact that radially symmetric solutions of (1.1), (1.2) satisfy the one-dimensional problem

\[
\begin{align*}
  u_t &= u_{rr} + \frac{N-1}{r} u_r + f(u), \quad r \in (0, 1) \\
  u_r(0) &= u(1) = 0,
\end{align*}
\]

where $r = |x|$ is the radial variable. Ignoring for the moment the singularity in (1.3), the results in one-dimension have strong consequences on the dynamics of (1.1), (1.2) in the positively invariant space $X_r \subset X$ consisting of radially symmetric functions. In particular, the invariant manifolds of hyperbolic equilibria of the restricted semiflow $S(t)|_{X_r}$ intersect transversally and any bounded trajectory in $X_r$ is convergent.

Though the description of the dynamics outside $X_r$ seems to be out of reach in general, symmetry can still be employed when nonnegative solutions of (1.1), (1.2) are discussed. The basic observation is that any bounded nonnegative solution approaches a set of equilibria which, being obviously nonnegative, are radially symmetric by a well known result of Gidas et al. [16] (see also [9]). This symmetrization property has been used in [20] in the proof of convergence of nonnegative solutions of (1.1), (1.2). (For a similar result for equations with time-dependent periodic nonlinearities see [23]).

In the present paper we consider stable and unstable manifolds of two hyperbolic nonnegative equilibria. We prove that they intersect transversally in $X$ (not just in $X_r$). This result has an immediate consequence: that a connecting (heteroclinic) orbit between two nonnegative hyperbolic equilibria of (1.1), (1.2), if it exists, persists if the equation (or the domain) is perturbed slightly, even if the perturbation breaks the symmetry invariance. The transversality property also allows us to prove certain structural stability results for equations with nonnegative nonlinearities or for restrictions of the semiflows to the cone of nonnegative functions. The theorems are formulated and proved in Section 2. The transversality theorem is proved there for
a slightly more general problem, where the nonlinearity is allowed to depend on the radial variable $r$, but it is required to be nonincreasing in $r$.

A natural question arises if the stable and unstable manifolds of two hyperbolic symmetric equilibria which are not nonnegative may not intersect transversally (even though they intersect transversally in $X_r$). We believe that this can indeed happen but are not able to prove it. Instead we present a weaker but still instructive result for the following equation

$$u_t = \Delta u + g(x, u), \quad x \in \Omega,$$

where $g$ is smooth and radially symmetric in $x$: $g = \tilde{g}(|x|, u)$. We find such a nonlinearity with the property that (1.5), (1.2) has two radially symmetric hyperbolic equilibria whose stable and unstable manifolds have a transversal intersection in $X_r$, but this intersection is not transversal in the whole space $X$. This is, up to our knowledge, the first example of a nontransversal intersection of the invariant manifolds for equations of this form. The example is presented in Section 3.

Let us remark, that in one space dimension, the transversality of stable and unstable manifolds of positive equilibria of (1.1), (1.2) can be proved quite easily using the relation between Morse indices and nodal properties of equilibria [7]. For positive equilibria this relation says that the Morse index can be at most one. This property, as well as any a priori bound on the Morse index, cannot be obtained in higher space dimension. We give counter-examples in the appendix.

2. Transversal intersection. In this section we first present precise statements of the transversality and structural stability theorems. Then we give the proofs. We start by introducing basic notation and definitions. Let $f : \tilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a $C^2$ function radially symmetric in $x$ (i.e., $f = f(r, u) = f(|x|, u)$) and satisfying everywhere the relation $f_r(r, u) \leq 0$. Consider the problem

$$u_t = \Delta u + f(r, u), \quad r = |x| < 1, \quad u|_{\partial \Omega} = 0. \quad (2.1)$$

Choose a $p > N$ and let $X = W_0^{1,p}(\Omega)$. For $u_0 \in X$, let $S(t)u_0 = u(t, \cdot, u_0)$ be the solution of (2.1), (2.2) with the initial condition $u_0$ on the maximal time interval. An equilibrium (stationary solution) $\psi$ of (1.1), (1.2) is said to be hyperbolic if $\mu = 0$ is not an eigenvalue of the problem

$$\Delta v + f_u(r, \psi(x))v + \mu v = 0, \quad x \in \Omega, \quad v|_{\partial \Omega} = 0. \quad (2.3)$$

For such equilibrium, the unstable manifold $W^u(\psi)$ is the set of all $u_0 \in X$ with the property that there is a curve $u(t) \in X$, $t \in (-\infty, 0)$, such that $u(0) = u_0$, $S(t)u(s) = u(t + s)$ for $-s \leq t \leq -s$ (that is, $u(t)$ is a negative orbit of $u_0$), and $u(t) \to \psi$ as $t \to -\infty$ (the first convergence is in $X$). The stable manifold of $\psi$ is
defined by \( W^s(\psi) = \{ u_0 \in X : S(t)u_0 \to \psi \text{ as } t \to \infty \} \). It is well known (see [21]), that \( W^u(\psi) \), \( W^s(\psi) \) are both imbedded submanifolds of \( X \) (gradient structure of the semiflow \( S(t) \) plays a role in the proof). Furthermore, the tangent spaces of \( W^s(\psi) \), \( W^u(\psi) \) at \( \psi \) are given by the eigenspaces of (1.5), (1.6) corresponding to positive and negative eigenvalues, respectively. In particular, \( \text{codim } W^s(\psi) = \dim W^u(\psi) < \infty \). We now state a transversality theorem for two radially symmetric equilibria. Recall that by [16, Theorem 2.1'] and [9] all nonnegative equilibria are radially symmetric. (Actually, minor modifications in [9] are needed in order to include \( f \) dependent on \( r \) and nonincreasing in \( r \).)

**Theorem A.** Let \( \psi^- , \psi^+ \in X_r \) be two hyperbolic equilibria of (2.1), (2.2). Then the following properties hold:

(i) If \( \psi^+ \geq 0 \), \( \psi^+ \neq 0 \) then at any \( \xi \in W^u(\psi^-) \cap W^s(\psi^+) \cap X_r \) the manifolds \( W^u(\psi^-) \) and \( W^s(\psi^+) \) intersect transversally: \( W^u(\psi^-) \cap \xi \subset W^s(\psi^+) \).

(ii) If \( \psi^+ \geq 0 \), \( \psi^- \geq 0 \) then \( W^u(\psi^-) \) and \( W^s(\psi^+) \) intersect transversally (everywhere): \( W^u(\psi^-) \cap W^s(\psi^+) \).

As remarked in the introduction, connecting orbits that come from a transversal intersection of the stable and unstable manifolds of hyperbolic equilibria occur in a "persistent manner". Statement (i) implies that radially symmetric connecting orbits to positive equilibria have such a persistence property. Small perturbations of (2.1), (2.2) will still have a connecting orbit of the perturbed hyperbolic equilibria.

Below we shall see that connecting orbits between positive equilibria are contained in \( X_r \). Thus statement (ii) is a consequence of (i), except for the special case when one of the equilibria is identical to zero. This case will be considered separately.

Theorem A can be used to show that for a generic nonnegative function \( f \) satisfying a dissipativity condition, problem (2.1), (2.2) defines a Morse-Smale dynamical system. We formulate and prove the theorem only for \( f \) independent of \( r \). This allows us to use a result of [8] directly and make the exposition simpler.

We introduce the class \( \mathcal{F} \) of all \( C^2 \) functions \( f : \mathbb{R} \to \mathbb{R} \) that satisfy the following two conditions:

\[(C1) \quad f(u) \geq 0 \quad \text{for } u \leq 0, \]
\[(C2) \quad \lim \sup_{u \to \infty} \frac{f(u)}{u} < \lambda_1, \]

where \( \lambda_1 \) is the first eigenvalue of the Dirichlet problem for the Laplacian on \( \Omega \).

Under these conditions, \( S(t) \) admits a global attractor, that is a compact set \( \mathcal{A} \subset X \), which is invariant (\( S(t)\mathcal{A} = \mathcal{A} \) for any \( t \geq 0 \)) and attracts any bounded set \( B \subset X \) in the following sense: for any neighborhood \( U \) of \( \mathcal{A} \) there is a \( T > 0 \) such that \( S(t)B \subset U \) for each \( t > T \). In particular, any trajectory is global and bounded. The fact that (C1), (C2) imply the existence of \( \mathcal{A} \) is a standard consequence of the abstract results on attractors (see [18, 5, 27]).

Let \( f \in \mathcal{F} \). We say, following [19], that the semiflow \( S(t) \) is Morse-Smale if the following properties are satisfied:

\[(M1) \quad \text{For each } t > 0 \text{ and } \varphi \in \mathcal{A}, \text{ the maps } S(t)\mathcal{A} \text{ and } d_\varphi S(t) \text{ are one-to-one;} \]
(M2) \( \mathcal{A} \) contains only finitely many equilibria, each of them hyperbolic;
(M3) for any equilibria of \( \psi^- \), \( \psi^+ \) of \( S(t) \) one has \( W^u(\psi^-) \cap W^s(\psi^+) \).

Assume that \( \mathcal{F} \) is endowed with either strong or weak \( C^2 \) topology (see [24], for definitions). We say that a property of \( f \) is generic in \( \mathcal{F} \) if it holds for each \( f \) in a residual subset of \( \mathcal{F} \).

**Theorem B.**

(i) Let \( f \in \mathcal{F} \). Suppose that all equilibria of (1.1), (1.2) are hyperbolic. Then the attractor \( \mathcal{A} \) consists of nonnegative radially symmetric functions and \( S(t) \) is a Morse-Smale system.

(ii) The hyperbolicity of all equilibria of (1.1), (1.2) is a generic property of \( f \in \mathcal{F} \).

Let us remark that if instead of (C1) one assumes that \( f \) only satisfies \( f(0) \geq 0 \), then results similar to those in Theorem B can be formulated for the restriction of the semiflow to the positively invariant set \( X_+ = \{ u \in X : u \geq 0 \} \). We leave this formulation (and the proof) to the reader.

It is well-known that the Morse-Smale property implies structural stability of the semiflow (cf. [19]). To illustrate that, we consider the following perturbation of (1.1)

\[
 u_t = \Delta u + f(u) + \varepsilon g(x, u),
\]

where \( g : \tilde{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function satisfying

(D1) \( g(x, u) \geq 0 \) for \( u \leq 0 \),

(D2) \( \limsup_{u \to \infty} \frac{g(x, u)}{u} \leq 0 \) uniformly for \( x \in \tilde{\Omega} \).

Let \( f \in \mathcal{F} \). Similarly as (1.1), (1.2), the problem (1.1)\( _\varepsilon \), (1.2) has for each \( \varepsilon \geq 0 \) a compact global attractor which we denote by \( \mathcal{A}_\varepsilon \). The family \( \mathcal{A}_\varepsilon \) is semicontinuous at \( \varepsilon = 0 \) : given any neighborhood \( U \) of \( \mathcal{A} = \mathcal{A}_0 \) one has \( \mathcal{A}_\varepsilon \subset U \) for all \( \varepsilon \geq 0 \) sufficiently small. This can be proved in a standard way (cf. [18, §4.10]) and the references therein; see also [11, §6]). Now Theorem B and the theory of [19] imply the following corollary.

**Corollary C.** Let \( f \) be as in Theorem B(i), and let \( g \) satisfy (D1), (D2). Then for all sufficiently small \( \varepsilon > 0 \) the semiflow \( S_\varepsilon(t) \) of (1.1)\( _\varepsilon \), (1.2) on \( \mathcal{A}_\varepsilon \) is equivalent to the semiflow \( S_0(t) = S(t)|_{\mathcal{A}} \).

Recall that \( S_\varepsilon(t) \) and \( S_0(t) \) are equivalent if there is a homeomorphism \( h : \mathcal{A} \to \mathcal{A}_\varepsilon \) which maps the trajectories of \( S(t) \) onto trajectories of \( S_\varepsilon(t) \) preserving the orientation by time.

We remark, that similar result can be proved if the domain \( \Omega \) is perturbed as well, (so it may lose the symmetry). Because of the technicalities involved, we do not consider such perturbations here.

We are now going to prove Theorem A. First we study properties of stable and unstable manifolds of nonnegative equilibria.
Lemma 2.1. Let \( \psi = \psi(|x|) \neq 0 \) be a nonnegative equilibrium of (1.1), (1.2). Then all negative eigenvalues of (2.3), (2.4) are simple and have radially symmetric eigenfunctions.

Proof. By [16, 9], \( \psi(r) > 0 \) for \( r \in [0, 1) \) and, moreover, \( \psi_r(r) < 0 \) for any \( r \in (0, 1) \). Differentiating the radial equation for \( \psi \), and using the assumption \( f_r(r, u) \leq 0 \), we see that \( w := u_r \) is a negative solution of the inequality

\[
w_{rr} + \frac{N-1}{r} w_r + (f_u(r, \psi(x)) - \frac{N-1}{r^2}) w \geq 0, \quad r \in (0, 1).
\]

This inequality, in conjunction with a comparison argument, can be now used to exclude nonradial eigenfunctions of (2.3), (2.4) with negative eigenvalues. For \( f \) independent of \( r \), in which case the inequality becomes an equality, the details have been given in [20] (see also [15], [35], [10]). The same arguments prove the result in the present case.

By uniqueness of solutions, it is easy to see that solutions with radially symmetric initial conditions remain radially symmetric. In other words, the subspace \( X_r \subset X \) of all radially symmetric functions is positively invariant. In the sequel, we denote by \( S_r(t) \) the restriction of \( S(t) \) to \( X_r \). Clearly, \( S_r(t) \) is the semiflow generated by the problem

\[
\begin{align*}
    u_t &= u_{rr} + \frac{N-1}{r} u_r + f(r, u), \quad r \in (0, 1), \quad u_r(0) = u(1) = 0.
\end{align*}
\]

For an equilibrium \( \psi \in X_r \), we denote by \( W^u_r(\psi) \), \( W^s_r(\psi) \) the unstable and stable manifolds of \( \psi \) associated with the semiflow \( S_r(t) \). Obviously, \( W^u_r(\psi) = W^u(\psi) \cap X_r \) and similarly for the stable manifolds.

Lemma 2.2. Let \( \psi \neq 0 \) be a nonnegative hyperbolic equilibrium of (2.1), (2.2). Then \( W^u(\psi) \subset X_r \), hence \( W^u(\psi) = W^u_r(\psi) \).

Proof. First observe that we have the following equality for the tangent spaces

\[
T_\psi W^u(\psi) = T_\psi W^u_r(\psi). \tag{2.5}
\]

Indeed, these spaces are the eigenspaces corresponding to the negative eigenvalues of the linearization (2.3), (2.4) considered on \( X \) and \( X_r \), respectively. But Lemma 2.1 tells us that these eigenspaces coincide.

Now, \( W^u_r(\psi) \subset W^u(\psi) \). Since these are both imbedded submanifolds of \( X \) (by [21, Theorem 6.1.10]) \( W^u_r(\psi) \) is a submanifold of \( W^u(\psi) \). Therefore (2.3) implies that in a neighborhood of \( \psi \) these two manifolds coincide.

Finally, by the definition of \( W^u(\psi) \), for any \( y \in W^u(\psi) \) one has \( y = S(t)z \), where \( t > 0 \) and \( z \in W^u(\psi) \) can be chosen arbitrarily close to \( \psi \). We can thus choose \( z \in W^u_r(\psi) \subset X_r \) and conclude, by the positive invariance of \( X_r \), that \( y \in X_r \).
Lemma 2.3. Let \( \psi \neq 0 \) be a nonnegative hyperbolic equilibrium of (2.1), (2.2). Let \( \xi \in W^s_r(\psi) \) and let \( Y \subset X_r \) be a complement of \( T_\xi W^s_r(\psi) \) in \( X_r \), that is,
\[
Y \oplus T_\xi W^s_r(\psi) = X_r.
\]
Then \( Y \) is also a complement of \( T_\xi W^s_r(\psi) \) in \( X \).

**Proof.** We have \( \text{codim} \ W^s_r(\psi) = \dim \ W^u_r(\psi) \) and \( \text{codim} \ W^s_r(\psi) = \dim \ W^u_r(\psi) \), hence, by Lemma 2.2, \( \text{codim} \ W^s_r(\psi) = \text{codim} \ W^s_r(\phi) \) (here the codimensions are understood in \( X \) and \( X_r \), respectively). It follows that for any \( \xi \in W^s_r(\psi) \), the tangent spaces \( T_\xi W^s_r(\psi) \) and \( T_\xi W^s_r(\phi) \) have the same (finite) codimensions. Thus, the lemma will be proved if we show that \( Y \cap T_\xi W^s_r(\psi) = \{0\} \). We proceed by contradiction. Suppose there is a \( v \in Y \setminus \{0\} \) with \( v \in T_\xi W^s_r(\psi) \). Consider the derivatives
\[
T(t) := d_\xi S(t), \quad T_r(t) := d_\xi S_r(t)
\]
of \( S(t) \) and \( S_r(t) \) at the point \( \xi \). Obviously, \( T_r(t) = T(t)|_{X_r} \). We use the characterization of tangent spaces of the stable manifolds given in [11, Appendix C]. First, by [11, Theorem C.2], \( v \in T_\xi W^s_r(\psi) \) implies
\[
\limsup_{t \to \infty} \| T(t)v \|_1^\frac{1}{2} < 1.
\]
Now, since \( v \in Y \subset X_r \), we can replace in this relation \( T(t) \) by \( T_r(t) \). Applying [11, Theorem C.2] (for the semiflow \( S_r(t) \)), we see that the new relation implies that \( v \in T_\xi W^s_r(\phi) \), contradicting the fact that \( Y \) is the complement of \( T_\xi W^s_r(\psi) \). \( \square \)

From Lemmas 2.2, 2.3 we immediately obtain the following corollary.

**Corollary 2.4.** Let \( \psi^-, \psi^+ \in X_r \) be hyperbolic equilibria, \( \psi^+ \geq 0 \) and \( \psi^+ \neq 0 \). Then \( W^u_r(\psi^-) \cap W^s_r(\psi^+) \) (in \( X_r \)) implies \( W^u_r(\psi^-) \cap W^s_r(\psi^+) \) at any \( \xi \in W^u_r(\psi^-) \cap W^s_r(\psi^+) \). In particular, if \( \psi^- \geq 0 \) and \( \psi^- \neq 0 \) then \( W^u_r(\psi^-) \cap W^s_r(\psi^+) \).

Thus, in order to show the desired transversality for two positive equilibria, we only need to show the transversality in \( X_r \).

**Lemma 2.5.** Let \( \psi^-, \psi^+ \in X_r \) be any two hyperbolic equilibria. Then \( W^u_r(\psi^-) \cap W^s_r(\psi^+) \) in \( X_r \).

The proof can be done in much the same way as in the one-dimensional (nonsingular) case [1, 22]. Because of the singularity in (1.3), however, we cannot immediately apply the results on the zero number used in [1, 22]. This is remedied in the next lemma.

Given a function \( \phi \in C[0, 1] \), we denote by \( z(\phi) \) the supremum of all indices \( k \) with the property that there exist numbers \( 0 \leq r_1 < r_2 < \cdots < r_k \leq 1 \) such that \( \phi(r_j)\phi(r_{j+1}) < 0, \ j = 1, 2, \ldots, k \). This is the number (possibly infinite) of sign changes of \( \phi \). The following lemma is a substitute for a similar result used in [1, 22] (cf. [29]).
Lemma 2.6. Let \( v(t, r) \) be a classical solution of
\[
\frac{v_t}{v} = \frac{N-1}{r} v_r + c(t, r) v, \quad 0 < r < 1, \quad 0 < t \leq T,
\]
\[
v_r(t, 0) = v(t, 1) = 0, \quad 0 \leq t \leq T,
\]
where \( c(t, r) \) is continuous on \([0, T] \times [0, 1]\). Then \( t \mapsto z(v(t, \cdot)) \) is nonincreasing on \([0, T]\).

Proof. For \( c(t, r) \) of class \( C^\infty \), this result has been proved by Angenent (see [3, §3.4]). In fact, a stronger version of the result, similar to that in [2], follows from [3]. For a general continuous \( c(t, r) \), the result is obtained by approximating \( c \) by smooth functions and using a continuity argument. The lemma can also be derived directly from the maximum principle in the same way as in the proof of Matano’s result on the lap number [29].

Another property that is needed for extension of the proof of [1] to the present case is the relation \( z(\phi_n) = n \) for the eigenfunction corresponding to the \( n \)-th eigenvalue of
\[
\frac{v_{rr}}{v} + \frac{N-1}{r} v_r + c(r) v + \mu v = 0, \quad v_r(0) = v(1) = 0.
\]
Here, \( c \) is a continuous function. This Sturm-Liouville property can be proved as in the nonsingular case using comparison arguments and elementary properties of solutions of second order equations (see e.g. [27]).

Now one can easily prove Lemma 2.5 by a straightforward modification of the arguments given in [1]. We omit the details.

Proof of Theorem A. Statement (i) follows directly from Corollary 2.4 and Lemma 2.5. So does statement (ii) if both \( \psi^+, \psi^- \) are different from 0. We only have to consider the possibility \( \psi^- \equiv 0 \) or \( \psi^+ \equiv 0 \). First assume that \( \psi^- \equiv 0 \). Since the case \( \psi^- \equiv 0 \equiv \psi^+ \) is trivial, we may assume that \( \psi^+(r) > 0 \) for \( r \in (0, 1) \). Moreover we have \( \psi^+_r(1) < 0 \), because \( \psi^+_r(1) = 0 \) would imply \( \psi^+ \equiv 0 \) (\( \psi^+ \) and 0 satisfy the same second-order equation and \( \psi^+(1) = 0 \)). If \( W^u(0) \cap W^s(\psi^+) = \emptyset \), then the transversality holds trivially. Suppose the intersection is nonempty. We claim that \( \psi^+ \) is linearly asymptotically stable. Since in such a case \( \text{codim} W^s(\psi^+) = 0 \), the transversality holds trivially, too.

We prove the claim. Since \( W^u(0) \) intersects \( W^s(\psi^+) \), there is a connecting orbit from 0 to \( \psi^+ \). Specifically, there exists a function \( t \mapsto y(t) : \mathbb{R} \rightarrow X \) such that \( S(t) y(s) = y(t + s) \) for any \( t, s \in \mathbb{R} \) and \( y(t) \rightarrow \psi^+ \) as \( t \rightarrow \infty \) and \( y(t) \rightarrow 0 \) as \( t \rightarrow -\infty \). This convergence can be understood in \( C^1[0, 1] \), by the smoothing property of \( S(t) \). Now, the fact that \( \psi^+ \gg 0 \) (i.e., \( \psi^+(x) > 0 \) for \( x \in \Omega \) and \( \frac{\partial \psi^+}{\partial \nu} < 0 \) on \( \partial \Omega \), where \( \nu \) is the unit outward normal) implies the relation \( \psi^+ \gg y(t_0) \) for sufficiently large negative \( t_0 \). We have thus found a trajectory, namely that of \( y(t_0) \), that starts below \( \psi^+ \) and converges to \( \psi^+ \). It is a well-known property, due to the comparison principle (see e.g. [31, Lemma 3.4]), that such a trajectory can exist only if the first eigenvalue of (2.1), (2.2), with \( \psi = \psi^+ \), is nonnegative. By hyperbolicity of \( \psi^+ \), this eigenvalue must be positive, hence \( \psi^+ \) is indeed linearly asymptotically stable.
The case $\psi^- > 0$ and $\psi^+ \equiv 0$ is similar. This time one shows that 0 is asymptotically stable from above if $W^s(0) \cap W^u(\psi^-) \neq \phi$. The proof of Theorem A is complete. \(\square\)

**Proof of Theorem B.** Let $f \in \mathcal{F}$ and suppose that all equilibria are hyperbolic. Due to the gradient-like structure of (1.1), (1.2), we have the following characterization of the attractor $\mathcal{A}$ (see [18, 5]):

$$\mathcal{A} = \bigcup_{\psi} W^u(\psi),$$

where the union is over all equilibria.

We first show that $\mathcal{A}$ consists of nonnegative functions. Choose a number $\eta_0 < 0$ such that

$$\eta_0 \leq \varphi(x), \quad \text{for any } \varphi \in \mathcal{A} \text{ and any } x \in \Omega. \quad (2.7)$$

Such an $\eta_0$ exists as $\mathcal{A}$ is compact in $X \hookrightarrow C^u(\Omega)$. Let $\eta(t)$ be the solution of

$$\dot{\eta} = f(\eta), \quad \eta(0) = \eta_0.$$ 

We have $\eta(t) < 0$ on an interval $[0, \tau]$. Applying on this interval a standard comparison principle [32], we obtain that for any $\varphi \in \mathcal{A}$,

$$\eta(\tau) < S(\tau)(\varphi)(x), \quad x \in \Omega.$$ 

Finding for each $\varphi \in \mathcal{A}$ a $\bar{\varphi} \in \mathcal{A}$ such that $S(\tau)\bar{\varphi} = \varphi$ (which is possible by the invariance of $\mathcal{A}$), we see that for each $\varphi \in \mathcal{A}$,

$$\eta(\tau) < \varphi(x), \quad x \in \Omega.$$ 

Since $\eta_0 \leq \eta(\tau)$, by (C1), we conclude that each $\eta_0 < 0$ that satisfies (2.7) also satisfies the strict relation $\eta_0 < \varphi(x)$, for any $x \in \Omega$ and $\varphi \in \mathcal{A}$. By compactness of $\mathcal{A}$, we must therefore have

$$\eta_0 < \inf_{x \in \Omega, \varphi \in \mathcal{A}} \varphi(x).$$

It follows that the supremum of all $\eta_0$ that satisfy (2.5) must be equal to 0. Hence, $\varphi(x) \geq 0$ on $\Omega$ for any $\varphi \in \mathcal{A}$.

Next we prove that $\mathcal{A} \subset X_r$. By Lemma 2.2, $W^u(\psi) \subset X_r$ for any positive equilibrium. In view of (2.6), we only need to prove that $W^u(0) \subset X_r$. But this is trivial because, by (C1), we have $f'(0) \leq 0$, hence 0 is linearly asymptotically stable and $W^u(0) = \{0\}$. To complete the proof of statement (i) of Theorem B, we note that, since the hyperbolic equilibria are isolated and lie in $\mathcal{A}$, there are only finitely many of them. Thus the property (M2) of the definition of Morse-Smale systems is satisfied. Property (M1) is a simple consequence of the backward uniqueness for (1.1), (1.2) and its variational equation. Finally, (M3) is a consequence of Theorem A. Statement (i) is proved.
For the proof of statement (ii), we recall the following result of [8]: Let $G$ be the set of all $C^2$ functions $f : \mathbb{R} \to \mathbb{R}$ endowed with a $C^2$-topology (either weak or strong). Then the set

$$G_n = \{ f \in G : \text{each equilibrium of (1.1), (1.2) that lies in } X_r \text{ and satisfies } |\psi|_{L^\infty} \leq n \text{ is hyperbolic} \}$$

is open and dense in $G$.

We prove that the set $F_n := G_n \cap F$, which is open in the relative topology of $F$, is dense in $F$. Statement (ii) will then follow, since the intersection of the $F_n$ is a residual set and, as shown above, for $f \in F$ all equilibria are radially symmetric.

Thus, given any $f_0 \in F$, we have to approximate it (in $F$) by an $f \in F_n$. We may assume, perturbing $f_0$ slightly if necessary, that $f_0(0) > 0$. We can then find an arbitrarily precise approximation $f \in G_n$ of $f_0$ which satisfies $f(0) > 0$, too. Since $f_0(u) \geq 0$ for $u \leq 0$, we can easily modify $f$, keeping its proximity to $f_0$ and without changing its values in $[0, n]$, so as to achieve that $f \in F$. Such a modification of course does not affect the hyperbolicity of equilibria $\psi$ satisfying $0 \leq \psi(r) \leq n$. Since for $f \in F$, all equilibria are nonnegative, we clearly have $f \in F_n$. This completes the proof of density of $F_n$. □

Remark. In one space dimension, assumptions (C1), (C2) would imply that $\dim A \leq 1$. This follows from (2.6) and the fact that the Morse index of a positive equilibrium is at most one (see [7]). Examples in the appendix show that as soon as $N \geq 2$, $\dim A$ can be arbitrarily big.

3. An example of a nontransversal intersection. In this section we consider equations of the following form:

$$u_t = \Delta u + g(r, u), \quad r = |x| < 1, \quad t > 1, \quad (3.1)$$

$$u|_{\partial \Omega} = 0, \quad (3.2)$$

where $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is smooth. This problem also defines a semiflow on $X$ for which $X_r$ is positively invariant. Similarly as for equations considered in the previous section, zero-number arguments show that for any hyperbolic equilibria $\psi^-, \psi^+ \in X_r$, one has $W^u_r(\psi^-) \cap W^s_r(\psi^+)$. Our aim here is to find a nonlinearity $g$ with the property that (3.1), (3.2) has two hyperbolic equilibria in $X_r$ such that $W^u_r(\psi^-)$ and $W^s_r(\psi^+)$, intersect nontransversally. It will be clear from the construction that once we have a $g$ with such property, the same property will be satisfied by any function $\bar{g}(r, u)$ sufficiently close to $g(r, u)$ in the $C^1$ topology (either weak or strong). Thus the nontransversal intersection cannot be "removed" by a small perturbation in the class of equivariant equations.

Our construction is based on a bifurcation analysis. We consider a one-parameter family of problems (3.1), (3.2) and arrange first that a nontrivial symmetric equilibrium bifurcates from the trivial one. This ensures that there is a connecting orbit between these two equilibria and, moreover, that their Morse indices in $X_r$ differ by
1. Then we consider the linearizations at these equilibria in the whole space $X$ and arrange, using perturbation arguments, that the Morse indices in $X$ are such that they prevent a transversal intersection of the corresponding stable and unstable manifolds.

Before we start with the details, we make a few remarks on the linearization of (3.1), (3.2) at symmetric equilibria.

Given a hyperbolic equilibrium $\psi \in X_r$ of (3.1), (3.2), the number of negative eigenvalues of
\begin{align}
  v_{rr} + \frac{N-1}{r} v_r + a(r)v + \mu v = 0, \\
  v_r(0) = v(1) = 0,
\end{align}
where $a(r) = g_u(r, \psi(r))$, will be called the \textit{symmetric Morse index} of $\psi$. Note that this is the number of negative eigenvalues of the linearization of (3.1), (3.2) that correspond to symmetric eigenfuctions. The Morse index of $\psi$ will refer, as before, to the number of all negative eigenvalues (counting multiplicities) of the linearization. Whether these two indices differ or not depends on the sign of the first eigenvalue that $a$ has a nonsymmetric eigenfuction. A separation of variables in the spherical coordinates implies that this eigenvalue is also the minimal eigenvalue of the problem
\begin{align}
  w_{rr} + \frac{N-1}{r} w_r + (a(r) - \frac{N-1}{r^2})w - \nu w = 0, \\
  w(1) = 0, \quad w \text{ regular at } r = 0
\end{align}
(cf. [13]). Below we shall also use the facts that the first eigenvalue of (3.5), (3.6) has an eigenfuction positive in $(0,1)$, and that its multiplicity as an eigenfuction of the linearization of (3.1), (3.2) is equal to $N$, the dimension of the space of spherical harmonics of the first order in $N$ variables [36].

We now consider the following one-parameter family of equations
\begin{align}
  u_t = \Delta u + (a(r) + \lambda c(r))u + b(r)u^2,
\end{align}
where $a, b, c$ are smooth functions to be determined is such a way that the following conditions are satisfied:

(E1) The second eigenvalue of the problem (3.3), (3.4) is equal to zero. We denote by $\tilde{v}$ the corresponding eigenfuction satisfying
\begin{align}
  \int_0^1 \tilde{v}^2(r)r^{N-1}dr = 1, \quad \tilde{v}(1) > 0;
\end{align}

(E2) The first eigenvalue of (3.5), (3.6) is equal to zero. We denote by $\tilde{w}$ the corresponding eigenfuction satisfying
\begin{align}
  \int_0^1 \tilde{w}^2(r)r^{N-1}dr = 1, \quad \tilde{w}(1) < 0;
\end{align}

(E3) $\int_0^1 c(r)\tilde{v}^2(r)r^{N-1}dr > 0$;

(E4) $\int_0^1 c(r)\tilde{w}^2(r)r^{N-1}dr < 0$;

(E5) $\int_0^1 b(r)\tilde{v}^2(r)r^{N-1}dr \neq 0$;

(E6) $\int_0^1 c(r)\tilde{v}^2(r)r^{N-1}dr - \int_0^1 c(r)\tilde{w}^2(r)r^{N-1}dr < 0$. 

The fact that functions $a$, $b$, $c$ satisfying (E1)-(E6) actually exist will be proved later. Now we show that under these conditions the nontransversal intersection occurs for some $\lambda > 0$.

**Proposition 3.1.** Let $a$, $b$, $c$ satisfy (E1)-(E6). Then there exist an $\varepsilon$ and a neighborhood $U$ of $0$ in $X_r$ such that for each $|\lambda| < \varepsilon$, the set of radially symmetric equilibria of (3.7), (3.2) in $U$ consists of two functions $0$, $\psi(\lambda) \in X_r$ which satisfy the following properties:

(i) $\psi(0) = 0$ and $\psi(\lambda) \neq 0$ for $\lambda \neq 0$,

(ii) for $\lambda \in (0, \varepsilon)$ the symmetric Morse indices of $0$ and $\psi(\lambda)$ are $2$ and $1$, respectively, and there exists a connecting (heteroclinic) orbit in $X_r$ from $0$ to $\psi(\lambda)$,

(iii) for $\lambda \in (0, \varepsilon)$, the Morse indices of $0$ and $\psi(\lambda)$ are $2$ and $N+1$, respectively; in particular, $W^u(0)$ and $W^s(\psi(\lambda))$ have a nontransversal intersection.

**Proof.** Consider first the linearization of (3.7), (3.2) at the trivial equilibrium. The eigenvalues with eigenfunctions in $X_r$ (below such eigenvalues are called symmetric eigenvalues) are the eigenvalues of the following problem:

$$
\begin{align*}
v_{rr} + \frac{N-1}{r}v_r + (a(r) + \lambda c(r))v + \mu v &= 0, \\
v_r(0) &= v(1) = 0.
\end{align*}
$$

(3.10) (3.11)

We examine the second eigenvalue $\mu_2(\lambda)$ of this problem. Condition (E1) states that $\mu_2(0) = 0$. By standard perturbation theory [26], both $\mu_2(\lambda)$ and the corresponding eigenfunction normalized as in (3.8) depend smoothly on $\lambda$. An easy calculation (differentiate (3.10) with respect to $\lambda$, multiply by the eigenfunction and integrate with weight $r^{N-1}$) yields

$$
\mu_2'(0) = -\int_0^1 r^{N-1} c(r) \bar{\tilde{v}}^2(r) \, dr.
$$

Hence $\mu_2'(0) < 0$ by (E3). This in particular implies that for $\lambda > 0$ near $0$, the symmetric Morse index of the trivial solution is $2$.

To show further statements of (i), (ii), we now investigate the bifurcation from "simple eigenvalue" for symmetric equilibria of (3.7), (3.2). Our method is standard (similar calculations using the Lyapunov-Schmidt reduction can be found in [14], [34, §II.13.A,B], [38, §6.4.11], [17], [12]), we thus skip some details. Let $\beta \in (0, 1)$ and let $Y = X_r \cap C^{2,\beta}(\widetilde{\Omega})$, $Z = \{u \in C^\beta(\widetilde{\Omega}) : u$ is radially symmetric$\}$ with the usual $C^{2,\beta}$ and $C^\beta$ norms, respectively. Define $F : Y \times \mathbb{R} \rightarrow Z$ by

$$
F(u, \lambda) = \Delta u + (a(r) + \lambda c(r))u + b(r)u^2.
$$

The equilibria in $X_r$ of (3.7), (3.2) correspond to solutions of the equation $F(u, \lambda) = 0$. Clearly, $F$ is smooth and $F_u(0, 0)$ has a one-dimensional kernel spanned by $\bar{\tilde{v}}$. The Fredholm alternative gives that the range $R$ of $F_u(0, 0)$ consists of all $y \in Z$ that
are $L_2$-orthogonal to $\bar{v}$: $\int_0^1 y(r)\bar{v}(r) r^{N-1} dr = 0$ (we have used the spherical coordinates). Applying the Lyapunov-Schmidt procedure with the orthogonal projections (more precisely, with their restrictions to $Y$ and $Z$), we find that near $(0, 0)$ the set $F^{-1}(0)$ is equal to

$$\{(s\bar{v} + \xi(s, \lambda), \lambda) : s, \lambda \in \mathbb{R}, \beta(s, \lambda) = 0\}, \quad (3.12)$$

where $\xi$ and $\beta$ are smooth (in fact, analytic) functions defined near $(0, 0)$, taking values in $Y \cap R$ and $\mathbb{R}$, respectively, and assuming the zero values at $(0, 0)$. Since $F(0, \lambda) \equiv 0$, we obviously have $\xi(0, \lambda) \equiv 0$ and $\beta(0, \lambda) \equiv 0$. We shall also need formulas for a few derivatives of $\xi$ and $\beta$. The following can be obtained by implicit differentiation (see the formulas on pages 31, 33 of [17], simplified by the relation $F(0, \lambda) \equiv 0$):

$$\xi_s(0, 0) = 0, \quad \beta_s(0, 0) = 0, \quad \beta_{ss}(0, 0) = 2 \int_0^1 r^{N-1} b(r) \bar{v}^3(r) dr,$$

$$\beta_{s\lambda}(0, 0) = \int_0^1 r^{N-1} c(r) \bar{v}^2(r) dr. \quad (3.13)$$

Next we look for nontrivial solutions of $\beta(s, \lambda) = 0$. For that we define $\gamma$ by $s\gamma(s, \lambda) \equiv \beta(s, \lambda)$. Clearly $\gamma$ is smooth and

$$\gamma(0, 0) = 0, \quad \gamma_s(0, 0) = \frac{1}{2} \beta_{ss}(0, 0), \quad \gamma_\lambda(0, 0) = \beta_{s\lambda}(0, 0). \quad (3.14)$$

By (3.13) and (E.5), $\gamma_s(0, 0) \neq 0$. Hence all solutions of $\gamma(s, \lambda) = 0$ near 0 lie on a smooth curve $(s(\lambda), \lambda)$. This combined with (3.12) implies that near $(0, 0)$ the set $F^{-1}(0)$ consists of the trivial branch and of the curve $\{(\psi(\lambda), \lambda)\}$, where

$$\psi(\lambda) = s(\lambda)\bar{v} + \xi(s(\lambda), \lambda)).$$

Since $\psi(0) = 0$, $\psi(0) = s'(0)\bar{v}$ and $s'(0) \neq 0$, as we show in a moment, $\psi(\lambda)$ is a nontrivial solution for $\lambda \neq 0$ near 0. Statement (i) is proved.

Next we compute $s'(0)$:

$$s'(0) = -\frac{\gamma_s(0, 0)}{\gamma(0, 0)} = -2 \frac{\beta_{s\lambda}(0, 0)}{\beta_s(0, 0)}. \quad (3.15)$$

By (3.13) and (E3), we have $s'(0) \neq 0$, as claimed.

We now find the symmetric Morse index of $\psi(\lambda)$. To this end, we compute the derivative with respect to $\lambda$ of the second eigenvalue $\mu_2(\lambda)$ of the problem

$$v_{rr} + \frac{N-1}{r} v_r + (a(r) + \lambda c(r) + 2b(r)\psi(\lambda(r)) v + \mu v = 0, \quad v_r(0) = v(1) = 0.$$

Similarly as for the trivial solution, the derivative is found to be

$$\mu'_2(0) = -\int_0^1 r^{N-1} (c(r) + 2b(r)\psi(\lambda)(0)) \bar{v}^2(r) dr$$

$$= -\int_0^1 r^{N-1} c(r) \bar{v}^2(r) dr - 2s'(0) \int_0^1 r^{N-1} b(r) \bar{v}^3(r) dr$$

$$= \int_0^1 r^{N-1} c(r) \bar{v}^2(r) dr > 0$$

$$= (3.16)$$
(we have used (3.15), (3.13) and (E3)). This implies that the symmetric Morse index of \( \psi(\lambda) \) is 1 if \( \lambda > 0 \) is close to 0.

To complete the proof of (ii), we need to establish the existence of a connecting orbit from 0 to \( \psi(\lambda) \). We use a simple dynamic bifurcation argument. For \( \lambda > 0 \) near 0, the semiflow of (3.7), (3.2) in \( X_r \) has a one-dimensional locally invariant manifold (the center manifold) that carries the essential information on the dynamics near 0 (see e.g. [21, §6.3], [39], [12]). In particular, this manifolds contains the equilibria 0 and \( \psi(\lambda) \). Since these are the only equilibria near 0, the one-dimensional flow on the invariant manifold must contain a connecting orbit between them. From what we know on the second symmetric eigenvalue of the linearization at 0 (it is negative for \( \lambda > 0 \)) it follows that 0 is unstable for the flow on the manifold, hence the direction of the connecting orbit is from 0 to \( \psi(\lambda) \).

Finally, we prove statement (iii). For that we compute the derivatives of the first nonsymmetric eigenvalues of the linearizations of (3.7), (3.2) at 0 and \( \psi(\lambda) \). In the case of the equilibrium 0 this is the first eigenvalue \( v_1(\lambda) \) of

\[
w_{rr} + \frac{N-1}{r} w_r + (a(r) + \lambda c(r) - \frac{N-1}{r^2}) w + \nu w = 0, \quad w(1) = 0, \quad w \text{ regular at } r = 0.
\]

By (E2), \( v_1(0) = 0 \). Similarly as for the problem (3.10), (3.11), we find that

\[
v_1'(0) = -\int_0^1 r^{-1} c(r) \bar{w}^2(r) \, dr,
\]

which is positive by (E4). Thus \( v_1(\lambda) > 0 \) for \( \lambda > 0 \) near zero and the Morse index of 0 is the same as its symmetric Morse index and equals 2.

For the linearization at \( \psi(\lambda) \), a calculation similar to that in (3.16) yields

\[
v_1'(0) = -\int_0^1 r^{-1} c(r) \bar{w}^2(r) \, dr - 2s'(0) \int_0^1 b(r) \bar{u}(r) \bar{w}^2(r) r^{N-1} \, dr,
\]

hence, by (3.15), (3.13) and (E6),

\[
v_1'(0) = \frac{\int_0^1 c(r) \bar{w}^2(r) r^{N-1} \, dr}{\int_0^1 b(r) \bar{w}^3(r) r^{N-1} \, dr} \int_0^1 r^{-1} b(r) \bar{u}(r) \bar{w}^2(r) \, dr - \int_0^1 r^{N-1} c(r) \bar{w}^2(r) \, dr < 0.
\]

It follows that \( v_1(\lambda) < 0 \) for \( \lambda > 0 \) near 0. Thus the linearization of (3.7), (3.2) at \( \psi(\lambda) \) has, in addition to the first symmetric eigenvalue, one more negative eigenvalue of multiplicity \( N \). This shows that the Morse index of \( \psi(\lambda) \) for \( \lambda > 0 \) near 0 is equal to \( N + 1 \).

We conclude that for small \( \lambda > 0 \), \( \dim W^u(0) = 2 \) and \( \text{codim } W^{s}(\psi(\lambda)) = N + 1 \). This readily implies that the intersection of \( W^u(0) \) and \( W^{s}(\psi(\lambda)) \) (which is nonempty by statement (iii)) is not transversal. \( \Box \)

To complete our example, we next prove the following proposition.
Proposition 3.2. There exist smooth functions \(a, b, c\) on \([0, 1]\), such that (E1)–(E6) hold true.

Proof. We first find an \(a\) such that the first eigenvalue of (3.3), (3.4) and the second eigenvalue of (3.5), (3.6) coincide. Denote these eigenvalues by \(\mu_2(a)\) and \(\nu_1(a)\), respectively. It suffices to find \(a_1, a_2\) such that \(\mu_2(a_1) < \nu_1(a_1)\) and \(\mu_2(a_2) > \nu_1(a_2)\).

A simple continuity argument then shows that \(\mu_2(sa_1+(1-s)a_2) = \nu_1(sa_1+(1-s)a_2)\) for some \(s \in (0, 1)\).

A smooth function satisfying \(\mu_2(a_1) < \nu_1(a_1)\) is found in the appendix as a step in the construction there. On the other hand, for \(a \equiv 0\) one has \(\mu_2(0) > \nu_1(0)\). This can be seen as follows. If \(a \equiv 0\) and \(v\) is the eigenfunction corresponding to the second eigenvalue of (3.3), (3.4), then \(w = v\) is a solution of (3.5) with \(v = \mu_2(0)\). Now \(v\) has a zero in \((0, 1)\), hence \(v_r\) has two zeros in \([0, 1]\). This and a comparison argument (see the remarks following (A.4) in the appendix) imply that any solution of (3.5) with \(v \geq \mu_2(0)\) has a zero in \((0, 1)\). Since to the first eigenvalue \(\nu_1(0)\) there corresponds a positive eigenfunction, we must have \(\mu_2(0) > \nu_1(0)\).

We conclude that \(\mu_2(a) = \nu_1(a)\) for some smooth function \(a\). Subtracting \(\mu_2(a)\) from \(a\), we obtain a function that satisfies (E1), (E2).

Next we find a smooth function \(c\) satisfying (E3), (E4). This is easy because the functions \(\tilde{v}_2^2\) and \(\tilde{w}_2^2\) are linearly independent over \((0, 1)\) (\(\tilde{v}\) has a zero in \((0, 1)\), while \(\tilde{w}\) is positive in \((0, 1)\)). By choosing \(c\) properly, we can thus assign any given values to the integrals in (E3), (E4).

Finally, we determine \(b\). Again, the functions \(\tilde{v}_2^3\) and \(\tilde{w}\tilde{w}_2^2\) are linearly independent. Thus the values of the integrals in (E5), (E6) involving \(b\) can prescribed arbitrarily by adjusting \(b\). In particular, we achieve (E5), (E6) easily. □

Remark. In this section we have not required that the nonlinearity in the equation (3.1) be nonincreasing in \(r\). It is therefore easy to transform the equation (e.g., by passing to the new unknown \(\bar{u} = u + \varepsilon(r)\), for an appropriate smooth \(\varepsilon(r)\)) so as to achieve that the equilibria whose stable and unstable manifold intersect nontransversally are positive.

Appendix. In this section we examine the Morse index of positive equilibria of (3.1), (3.2). We show that, unlike in one space dimension, for \(N \geq 2\) the Morse index can be arbitrarily big.

For \(3 \leq N \leq 9\), there is an explicit example of an equation having positive equilibria whose Morse indices cover the whole set of positive integers. Specifically, the parameterized stationary problem

\[
\Delta u + \lambda e^u = 0, \quad \text{on } \Omega, \quad u|_{\partial \Omega} = 0
\]

is known to have a branch of positive solutions with infinitely many turning points. The bifurcation values \(\lambda\) accumulate at a \(\lambda_0\) for which the problem has infinitely many solutions (see [25]). As the author was shown by S. Angenent [4], the following property holds true: following the branch, as \(\max u\) increases, the Morse index of the corresponding solution \(u\) is increased by one at each turning point, otherwise it is constant. Consequently, for \(\lambda = \lambda_0\), the Morse indices of the (infinitely many) equilibria cover the set of all positive integers.

We now give an independent construction of equations with high Morse indices. The construction works for any \(N \geq 2\). However, it does not provide for an equation with equilibria of all possible Morse indices.
Proposition A.1. Let $N \geq 2$ and $n \geq 0$ be arbitrary. There exists a smooth function $f$ such that (1.1), (1.2) has an equilibrium of Morse index at least $n$.

Proof. The proof will be carried out in the following two steps:

Step 1. We find a smooth function $a(r)$, $r \in [0, 1]$, which is a constant for $r$ near $0$ and such that the following properties (a), (b) are satisfied:

(a) The equation
$$w_{rr} + \frac{N-1}{r} w_r + (a(r) - \frac{N-1}{r^2})w = 0,$$  \hspace{1cm} \text{(A.1)}

has a $C^\infty$ solution on $[0, 1]$, which is positive in $(0, 1]$ and satisfies $w(0) = 0$.

(b) The problem
$$w_{rr} + \frac{N-1}{r} w_r + (a(r) + \mu)w = 0,$$
$$w_r(0) = w(1) = 0,$$  \hspace{1cm} \text{(A.2)} \text{ and (A.3)}

has at least $n$ negative eigenvalues.

Step 2. We find a nonlinearity $f$ and a corresponding positive equilibrium in such a way that the symmetric eigenvalues of the linearization at this equilibrium are given by (A.2), (A.3).

In Step 1, the following preliminary remarks will be useful. First we note that the equation
$$w_{rr} + \frac{N-1}{r} w_r + (1 - \frac{N-1}{r^2})w = 0,$$  \hspace{1cm} \text{(A.4)}

has a (smooth) solution which is regular at $0$ and satisfies $w(0) = 0$, $w_r(0) = 1$. In fact $w(r) = r^{1-N/2} J_N(r)$, where $J_N$ is the Bessel function of index $N/2$. Of course, this solution is positive on an interval $(0, \delta)$. In our construction below, $a(r) \equiv 1$ for $r \geq 0$ near $0$. Hence there exists a solution $w$ of (A.1) which coincides with the above positive solution of (A.4) near $0$. We shall verify that $w$ remains positive in the whole of $(0, 1]$. In that we shall rely on the following comparison argument: if $w$ is a solution of (A.1) and $\tilde{w}$ is a regular solution of
$$w_{rr} + \frac{N-1}{r} w_r + b(r)w = 0,$$

with $b(r) \geq a(r) - (N-1)r^{-2}$, then, for any two zeros $r_1, r_2$ of $w$, either $(r_1, r_2)$ contains a zero of $\tilde{w}$ or else $\tilde{w} \equiv cw$ in $(r_1, r_2)$ for a constant $c$. This is a standard Sturm comparison, except for the singularities in the equations. However, it can be seen easily that these singularities do not cause any trouble (cf. [20]).

We now give a detailed construction of $a$. Let $a(r)$ be a $C^\infty$ function on $[0, 1]$ satisfying the following conditions:

(i) $a(r) \equiv \frac{m}{r^2}$, \hspace{1cm} for $\delta \leq r \leq 1$,
(ii) $a(r) \leq \frac{m}{r^2}$, \hspace{1cm} for $0 \leq r \leq 1$,
(iii) $a(r) \equiv 1$, \hspace{1cm} for $0 \leq r \leq \min\{\delta, \frac{\pi}{2}\}$,

where $\delta$ is as above and $m, \epsilon$ are positive constants we are going to determine. First we choose an $m$ such that the equation
$$w_{rr} + \frac{N-1}{r} w_r + \frac{m-N+1}{r^2}w = 0,$$  \hspace{1cm} \text{(A.5)}

has a positive solution on $(0, \infty)$, and, at the same time, the equation
$$w_{rr} + \frac{N-1}{r} w_r + \frac{m}{r^2}w = 0,$$  \hspace{1cm} \text{(A.6)}

has a solution with infinitely many positive zeros accumulating at $0$. A possible choice is $m = \frac{N^2}{4}$ for in this case we have the solutions
$$r^{1-N/2}, \hspace{1cm} r^{-1-N/2}\cos\ln(r^{\sqrt{2N-1}})$$

of (A.5), (A.6), respectively.
Now choose an \( \varepsilon > 0 \) such that the above solution of (A.6) has \( n + 1 \) zeros in the interval \((\varepsilon, 1)\). We verify that if \( a \) is as in (i)-(iii) then it satisfies the properties (a), (b). Indeed, by comparison with (A.5) (which has a positive solution and whose last coefficient is not less than \( a(r) - (N - 1)r^{-2} \)) we see that the solution of (A.1) satisfying \( w(0) = 0, w_r(0) = 1 \) cannot have another zero in \([0, 1]\). Thus (a) is satisfied. Property (b) is satisfied, too, for the following reason. If \( a \) is as in (i)-(iii) then it satisfies the properties (a), (b). Indeed, by comparison with (A.5) (which has a positive solution and whose last coefficient is not less than \( a(r) - (N - 1)r^{-2} \)) we see that the solution of (A.1) satisfying \( w(0) = 0, w_r(0) = 1 \) cannot have another zero in \([0, 1]\). Thus (a) is satisfied.

Property (b) is satisfied, too, for the following reason. If \( a \) is as in (i)-(iii) then it satisfies the properties (a), (b). Indeed, by comparison with (A.5) (which has a positive solution and whose last coefficient is not less than \( a(r) - (N - 1)r^{-2} \)) we see that the solution of (A.1) satisfying \( w(0) = 0, w_r(0) = 1 \) cannot have another zero in \([0, 1]\). Thus (a) is satisfied.

Further, \( u \) is of class \( C^\infty \) and \( u(r) > 0 \) for \( r \in [0, 1] \). By (A.7), \( u \) has a smooth inverse \( r = \xi(u) \) defined on \([0, u(0)]\). Let \( f \) be determined by

\[
f'(u) = a(\xi(u)), \quad \text{for } u \in [0, u(0)), \quad \text{and } f(0) = -u_r(1) - (N - 1)u_r(1). \tag{A.9}
\]

Note that, by (iii), \( f'(u) = 1 \) for \( u \) near \( u(0) \), hence \( f \) can be extended to a \( C^\infty \) function on \( \mathbb{R} \).

For this choice of \( f \) and \( u \) we show that

\[
u_{rr} + \frac{N - 1}{r} u_r + f(u) = 0. \tag{A.10}
\]

Indeed, by (A.9), \( f'(u(r)) = a(r) \). Substituting this and \( w = u_r \) in (A.1), we obtain, after integrating from \( r \) to 1, that the left-hand side of (A.10) is equal to \( u_{rr}(1) + (N - 1)u_r(1) + f(u(1)) \), which equals 0 by (A.9), (A.7). Thus \( u = u(|x|) \) is a positive equilibrium of (1.1), (1.2).

Finally, since \( a(r) = f'(u(r)) \), the eigenvalues of (A.2), (A.3) are the eigenvalues of the linearization of (1.1), (1.2) at \( u \). Thus, by property (b), the Morse index of \( u \) is at least \( n \). This completes the proof. \( \square \)

**Remark A.2.** Note that for the function \( a \) constructed in Step 1, the problem (3.5), (3.6) has all eigenvalues positive. This follows from the comparison argument used above. By (b), if \( n \geq 2 \), we have \( \nu_1(a) > \mu_2(a) \) as needed in the proof of Proposition 3.2.

**REFERENCES**


