Differential and Integral Equations, Volume 8, Number 3, March 1995, pp. 689 – 698.

INTEGRAL REPRESENTATIONS AND L^∞ BOUNDS FOR SOLUTIONS OF THE HELMHOLTZ EQUATION ON ARBITRARY OPEN SETS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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(Submitted by: David Sattinger)

Abstract. We establish sharp L^∞ bounds for functions defined on arbitrary open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, which vanish on the boundary and have L^2 Laplacians. All functions corresponding to the best possible constants are explicitly given. The proof is based on integral representations using the Green's function for the Helmholtz equation in arbitrary domains.

1. Introduction and main results. The main results of this paper are given in the following theorems. Throughout this paper, Ω denotes an open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\|\cdot\|$ denotes the $L^2(\Omega)$ norm, ∇ denotes the gradient and Δ denotes the Laplacian in the distributional sense. $H_0^1(\Omega)$ denotes the usual Sobolev space, which consists of functions that vanish on the boundary. When $\Omega \subset \mathbb{R}^3$, we also use the homogeneous Sobolev space $\hat{H}_0^1(\Omega)$, which is defined as the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\nabla \cdot\|$. We have $\hat{H}_0^1(\Omega) \subset L^6(\Omega)$ due to the Sobolev inequality (see e.g., [5, p. 10]):

$$\|v\|_{L^6(\Omega)} \le c\|\nabla v\|, \quad \forall\, v \in C_0^\infty(\Omega). \tag{1}$$

Theorem 1. *Let Ω be any open set in $\mathbb{R}^3$. For all $u \in \hat{H}_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, we have*

$$\sup_\Omega |u|^2 \le \frac{1}{2\pi}\|\Delta u\|\,\|\nabla u\|. \tag{2}$$

The constant $1/(2\pi)$ is the best possible for any given Ω. The equality is attained when and only when $u \equiv 0$ or

$$\Omega = \mathbb{R}^3 \quad \text{and} \quad u(x) = \alpha \frac{1 - e^{-\beta|x-x_0|}}{|x-x_0|}, \tag{3}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$ and $x_0 \in \mathbb{R}^3$ are arbitrary.

Theorem 2. *Let Ω be any open set in $\mathbb{R}^2$. For all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, we have*

$$\sup_\Omega |u|^2 \le \frac{1}{2\pi}\left(\|\Delta u\|\,\|u\| + \|\nabla u\|^2\right). \tag{4}$$

The constant $1/(2\pi)$ is the best possible for any given Ω. The equality is attained when and only when $u \equiv 0$ or

$$\Omega = \mathbb{R}^2 \quad \text{and} \quad u(x) = \alpha|x-x_0|K_1(\beta|x-x_0|), \tag{5}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$ and $x_0 \in \mathbb{R}^2$ are arbitrary; K_1 is a MacDonald function.

Received February 1993.
AMS Subject Classifications: 26D10, 35J05.

Theorem 3. *Let Ω be any open set in $\mathbb{R}^m$, $m = 2$ or 3. For all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, we have*

$$\sup_{\Omega} |u| \le C(m) \|\Delta u\|^{m/4} \|u\|^{1-m/4}, \tag{6}$$

where

$$C(2) = \frac{1}{\sqrt{\pi}}, \quad C(3) = \frac{1}{3^{3/4}} \sqrt{2/\pi}.$$

In particular, if $u \in H_0^1(\Omega)$ and $-\Delta u = \lambda u$, then

$$\sup_{\Omega} |u| \le C(m) \lambda^{m/4} \|u\|. \tag{7}$$

Strict inequalities hold unless $u \equiv 0$.

The proofs of the above theorems are based on the following representation formulas.

Theorem 4. *Let $\mu > 0$ and $x_0 \in \Omega$. Under the hypotheses of Theorem 1, we have*

$$u(x_0) = -\int_{\Omega} (G_\mu \Delta u - \nabla(G_0 - G_\mu) \cdot \nabla u)\, dx. \tag{8}$$

Under the hypotheses of Theorems 2 or 3, we have

$$u(x_0) = -\int_{\Omega} G_\mu (\Delta - \mu) u \, dx. \tag{9}$$

Here, G_μ, $\mu > 0$ or $\mu = 0$, denotes the Green function for the Helmholtz equation:

$$(\Delta - \mu) G_\mu(x; x_0) = -\delta(x - x_0) \quad \text{in } \Omega, \tag{10}$$

$$G_\mu(x; x_0)|_{\partial\Omega} = 0, \tag{11}$$

where δ is the Dirac distribution. The arbitrary point $x_0 \in \Omega$ is always fixed in our discussions.

When Ω is a bounded and smooth domain, it is well known that the Green function is well-defined and that the integral formulas hold. ((8) is reduced to the Poisson formula

$$u(x_0) = -\int_{\Omega} G_0 \Delta u \, dx \tag{12}$$

upon integration by parts.) By using the Schwarz inequality and estimating the Green functions, we prove Theorems 1, 2 and 3 in Sections 2, 3 and 4 respectively.

When Ω is unbounded or nonsmooth, the meaning of the boundary condition (11) becomes vague, since the boundary $\partial\Omega$ may be very irregular or even empty. In Section 6, we construct the Green functions for arbitrary domains. The integral formulas are then proved for arbitrary domains with the aid of the lemmas in Section 5, which concern the domain dependence of solutions of the Helmholtz equation.

That an inequality in the form of (2) should be valid with a constant independent of the domain was suggested to the author by Professor J.G. Heywood. He conjectured that an analogous inequality also holds for the Stokes operator, and can be combined with the methods of [2] and [3] to obtain regularity results for the

Navier–Stokes equations in nonsmooth domains. The inequality (2) is applied to the Burgers' equation in [4] to demonstrate this theory.

Inequality (2) was first established in [7] by a technique of finite eigenfunction expansions. The same technique was then applied to the Stokes equations in [8], which reduce the problem of proving an analogous inequality for the Stokes operator to one of estimating the L^2 norm of the Green tensor. The new method in this paper sharpens the result of [7] by giving all functions corresponding to the best possible constant. The application of the new method to the Stokes equations will again require some estimates on the Green tensors.

Inequalities analogous to (2) and (6), but with constants depending on the regularity of the domain, were proved for the Stokes operator, see e.g., [3, p. 299], [6, p. 12] and [1, p. 39]. The inequality (4) is new.

It is easy to verify that the above theorems are also valid for complex- or vector-valued functions (but the α in (3) and (5) should accordingly be redefined as an arbitrary complex number or arbitrary constant vector). The results may also be extended to other partial differential equations, if appropriate estimates on the Green functions can be obtained.

2. Proof of Theorem 1 in bounded smooth domains. In this section we assume that Ω is a bounded smooth open set in $\mathbb{R}^3$. By the maximum principle (see e.g., [7]), it is easy to prove that the Green function G_μ, $\mu \geq 0$, satisfies

$$0 \leq G_\mu(x; x_0) \leq g_\mu(x; x_0), \forall\, x \in \Omega\backslash\{x_0\}, \tag{13}$$

where

$$g_\mu(x; x_0) = \frac{e^{-\sqrt{\mu}|x-x_0|}}{4\pi|x-x_0|} \tag{14}$$

is the fundamental solution in $\mathbb{R}^3$.

When $\mu > 0$, we have

$$\int_{\mathbb{R}^3} g_\mu^2\, dx = \int_0^\infty \big(\frac{e^{-\sqrt{\mu}r}}{4\pi r}\big)^2 4\pi r^2\, dr = \frac{1}{8\pi\sqrt{\mu}}\,.$$

Hence

$$\|G_\mu\|^2 \leq \int_\Omega g_\mu^2\, dx \leq \frac{1}{8\pi\sqrt{\mu}}. \tag{15}$$

Next, we claim that

$$\|\nabla(G_0 - G_\mu)\|^2 \leq \int_\Omega \mu g_\mu (g_0 - g_\mu) dx \leq \frac{\sqrt{\mu}}{8\pi}. \tag{16}$$

In fact, from (10) and (11), we have

$$\Delta(G_0 - G_\mu) = -\mu G_\mu, \quad (G_0 - G_\mu)|_{\partial\Omega} = 0.$$

Hence, by the Poisson formula (12), we have

$$G_0(x; x_0) - G_\mu(x; x_0) = \mu \int_\Omega G_0(y; x) G_\mu(y; x_0)\, dy.$$

Similarly, in the whole space we have

$$g_0(x; x_0) - g_\mu(x; x_0) = \mu \int_{\mathbb{R}^3} g_0(y; x) g_\mu(y; x_0)\, dy.$$

Therefore, by (13) we obtain

$$0 \le G_0(x;x_0) - G_\mu(x;x_0) \le g_0(x;x_0) - g_\mu(x;x_0),$$

and hence

$$\begin{aligned}\|\nabla(G_0-G_\mu)\|^2 &= \int_\Omega \mu G_\mu(G_0-G_\mu)dx \le \int_\Omega \mu g_\mu(g_0-g_\mu)dx \\ &\le \int_{\mathbb{R}^3} \mu g_\mu(g_0-g_\mu)dx = \int_0^\infty \mu\frac{e^{-\sqrt{\mu}r}}{4\pi r}\frac{1-e^{-\sqrt{\mu}r}}{4\pi r}4\pi r^2\,dr = \frac{\sqrt{\mu}}{8\pi}.\end{aligned}$$

Now, as in Theorem 1, assume that $u \in \hat{H}_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Since Ω is smooth and bounded, u can be approximated by smooth functions u_n that vanish on the boundary (e.g., the partial sums of its eigenfunction expansion), in the $H^2(\Omega)$ norm and pointwise. By the Poisson formula, we have

$$u_n(x_0) = -\int_\Omega G_0\Delta u_n\,dx = -\int_\Omega G_\mu \Delta u_n\,dx - \int_\Omega (G_0-G_\mu)\Delta u_n\,dx.$$

Integrating by parts on the last integral, we obtain

$$u_n(x_0) = -\int_\Omega (G_\mu\Delta u_n - \nabla(G_0-G_\mu)\cdot\nabla u_n)\,dx.$$

Noticing (15) and (16), we can pass to the limit and obtain

$$u(x_0) = -\int_\Omega \left(G_\mu \Delta u\,dx - \nabla(G_0-G_\mu)\cdot\nabla u\right)\,dx. \tag{17}$$

Now, by the Schwarz inequality and the estimates (15) and (16), we have

$$\begin{aligned}|u(x_0)| &\le \|G_\mu\|\|\Delta u\| + \|\nabla(G_0-G_\mu)\|\|\nabla u\| && (18)\\ &\le \sqrt{1/(8\pi\sqrt{\mu})}\|\Delta u\| + \sqrt{\sqrt{\mu}/(8\pi)}\|\nabla u\|. && (19)\end{aligned}$$

Except for the trivial case $u \equiv 0$, the bound is minimizing by letting

$$\mu = \|\Delta u\|^2/\|\nabla u\|^2.$$

We obtain

$$|u(x_0)| \le \frac{1}{\sqrt{2\pi}}\|\Delta u\|^{1/2}\|\nabla u\|^{1/2}. \tag{20}$$

This implies (2), since x_0 is arbitrary.

From the above estimates, it is easy to see that, except for the trivial case $u \equiv 0$, for the equality in (20) to hold, we must have $\Omega = \mathbb{R}^3$ and $u = c(g_0(x;x_0) - g_\mu(x;x_0))$ for some $c \in \mathbb{R}$ and some $\mu > 0$, i.e., (3). It is readily verified that the equality does hold for such functions.

The sharpness of the constant in (20) for any given Ω was proved in [7].

3. Proof of Theorem 2 in bounded smooth domains. In this section we assume that Ω is a bounded smooth open set in $\mathbb{R}^2$. Let $\mu > 0$. In $\mathbb{R}^2$, the fundamental solution satisfying

$$(\Delta - \mu)g_\mu = -\delta(x - x_0) \tag{21}$$

is

$$g_\mu(x;x_0) = \frac{1}{2\pi} K_0(\sqrt{\mu}|x-x_0|), \tag{22}$$

where K_0 is a MacDonald function. We claim that

$$\int_{\mathbb{R}^2} g_\mu^2\,dx = \frac{1}{4\pi\mu}\,. \tag{23}$$

To prove this, differentiating (21) with respect to μ, we obtain

$$(\Delta - \mu)\frac{\partial g_\mu}{\partial \mu} = g_\mu. \tag{24}$$

Hence

$$\frac{\partial g_\mu}{\partial \mu}(x;x_0) = -\int_{\mathbb{R}^2} g_\mu(y;x)g_\mu(y;x_0)\,dy.$$

Differentiating (22) and noticing

$$\frac{d}{dt}K_0(t) = -K_1(t),$$

where K_1 is another MacDonald function, we obtain

$$\frac{\partial g_\mu}{\partial \mu}(x;x_0) = -\frac{1}{4\pi\sqrt{\mu}}|x-x_0|K_1(\sqrt{\mu}|x-x_0|). \tag{25}$$

Letting $x \to x_0$ and using the fact that $\lim_{t\to 0} tK_1(t) = 1$, we obtain (23).

Similar to the three-dimensional case in the previous section, we have

$$\|G_\mu\|^2 \le \int_\Omega g_\mu^2\,dx \le \frac{1}{4\pi\mu}, \tag{26}$$

and we can also prove the well-known representation formula

$$u(x_0) = -\int_\Omega G_\mu(\Delta u - \mu u)\,dx, \tag{27}$$

for all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. Then, by the Schwarz inequality, we have

$$|u(x_0)| \le \|G_\mu\|\,\|\Delta u - \mu u\| \tag{28}$$

$$\le \sqrt{\tfrac{1}{4\pi\mu}}\|\Delta u - \mu u\|. \tag{29}$$

Using

$$-\int_\Omega u\Delta u\,dx = \|\nabla u\|^2,$$

we obtain

$$|u(x_0)|^2 \le \frac{1}{4\pi\mu}\left(\|\Delta u\|^2 + 2\mu\|\nabla u\|^2 + \mu^2\|u\|^2\right).$$

Except for the trivial case $u \equiv 0$, the bound is minimized by letting

$$\mu = \frac{\|\Delta u\|}{\|u\|}\,. \tag{30}$$

We obtain

$$|u(x_0)|^2 \le \frac{1}{2\pi}\left(\|\Delta u\|\,\|u\| + \|\nabla u\|^2\right). \tag{31}$$

Except for the trivial case $u \equiv 0$, it is clear that the equality in (31) is possible to hold only when $\Omega = \mathbb{R}^2$ and

$$(\Delta - \mu)u = cg_\mu, \tag{32}$$

for some $\mu > 0$ and $c \in \mathbb{R}$. Comparing with (24), we obtain

$$u = c\frac{\partial g_\mu}{\partial \mu}. \tag{33}$$

Noticing (25), u is of the form given in (5).

To verify that the equality in (31) does hold for such u, it suffices to verify (30). From (32) and (33), we have

$$\|\Delta u\|^2 = \mu^2\|u\|^2 + 2\mu c\int_{\mathbb{R}^2} ug_\mu\,dx + c^2\int_{\mathbb{R}^2} g_\mu^2\,dx = \mu^2\|u\|^2 + c^2\frac{\partial}{\partial\mu}\Big(\mu\int_{\mathbb{R}^2} g_\mu^2\,dx\Big).$$

Using (23), (30) is obtained.

Finally, given any open set $\Omega \in \mathbb{R}^2$, and any $x_0 \in \Omega$, we prove that the constant $1/(2\pi)$ in (31) cannot be reduced if (31) is to hold for all $C_0^\infty(\Omega)$ functions. Clearly, the function $u(x) = |x-x_0|K_1(|x-x_0|)$ belongs to the Sobolev space $H^2(\mathbb{R}^2)$, hence there exist $u_n \in C_0^\infty(\mathbb{R}^2)$ such that $u_n \to u$ in $H^2(\mathbb{R}^2)$. Therefore

$$\frac{|u_n(x_0)|^2}{\|\Delta u_n\|_{L^2(\mathbb{R}^2)}\|u_n\|_{L^2(\mathbb{R}^2)} + \|\nabla u_n\|^2_{L^2(\mathbb{R}^2)}} \to \frac{1}{2\pi} \quad \text{as} \quad n \to \infty.$$

The values of the above quotients do not change if we substitute $u_n(x)$ by $v_n(x) \equiv u_n(x_0 + c_n(x - x_0))$. And each c_n can be chosen large enough so that $v_n \in C_0^\infty(\Omega)$. Therefore the constant is the best possible.

4. Proof of Theorem 3 in bounded smooth domains. When $\Omega \subset \mathbb{R}^3$, similar to (29), we obtain

$$|u(x_0)| \le \sqrt{1/(8\pi\sqrt{\mu})}\,\|\Delta u - \mu u\|. \tag{34}$$

But the inequality obtained by minimizing the bound is not neat. Therefore we sacrifice sharpness and use the triangle inequality

$$\|(\Delta - \mu)u\| \le \|\Delta u\| + \mu\|u\|. \tag{35}$$

Minimizing $\sqrt{1/(8\pi\sqrt{\mu})}(\|\Delta u\| + \mu\|u\|)$ by letting $\mu = \frac{\|\Delta u\|}{3\|u\|}$, (again, there is a trivial exception $u \equiv 0$,) we obtain

$$|u(x_0)| \le \frac{1}{3^{3/4}}\sqrt{2/\pi}\,\|\Delta u\|^{3/4}\|u\|^{1/4}. \tag{36}$$

The equality in (34) holds only when $(\Delta - \mu)u = cg_\mu$ for some $\mu > 0$ and $c \in \mathbb{R}$. But, for such functions, the equality in (35) does not hold unless $u \equiv 0$. Therefore, strict inequality in (36) holds unless $u \equiv 0$.

Using the above method in the case $\Omega \subset \mathbb{R}^2$, we obtain

$$|u(x_0)| \leq \frac{1}{\sqrt{\pi}} \|\Delta u\|^{1/2} \|u\|^{1/2}. \tag{37}$$

This completes the proof of the inequalities (6). The inequalities (7) for the eigenfunctions for the negative Laplacian obviously follow from (6).

Remark. The inequality (37) may also be obtained from (31) by using

$$\|\nabla u\|^2 = -\int_\Omega u \Delta u \, dx \leq \|\Delta u\| \, \|u\|. \tag{38}$$

When $\Omega \subset \mathbb{R}^3$, one may also use (38) and (20) to obtain the inequality

$$|u(x_0)| \leq \frac{1}{\sqrt{2\pi}} \|\Delta u\|^{3/4} \|u\|^{1/4}.$$

But the constant $1/\sqrt{2\pi}$ is larger than the one given in (36).

5. Approximations in increasing domains. Let Ω be an arbitrary open set in $\mathbb{R}^m$, $m = 2$ or 3. Let Ω_n be a sequence of bounded smooth open subsets of Ω increasing to Ω, i.e., $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$.

Lemma 1. *Suppose $\Omega \subset \mathbb{R}^3$, $u \in \hat{H}_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Then, the problems*

$$\Delta u_n = (\Delta u)|_{\Omega_n} \quad in \;\; \Omega_n, \quad u_n \in \hat{H}_0^1(\Omega)n \tag{39}$$

have unique solutions. As $n \to \infty$, they satisfy

$$\|\nabla u_n - \nabla u\|_{L^2(\Omega_n)} \to 0, \tag{40}$$

$$|u_n(x_0) - u(x_0)| \to 0, \quad \forall \, x_0 \in \Omega. \tag{41}$$

Lemma 2. *Suppose $\Omega \subset \mathbb{R}^m$, $m = 2$ or 3. Suppose $u \in H_0^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and $\mu > 0$. Then the problems*

$$\Delta u_n - \mu u_n = (\Delta u - \mu u)|_{\Omega_n} \quad in \;\; \Omega_n, \quad u_n \in H_0^1(\Omega_n) \tag{42}$$

have unique solutions. As $n \to \infty$, they satisfy

$$\|u_n - u\|_{L^2(\Omega_n)} \to 0, \tag{43}$$

$$\|\nabla u_n - \nabla u\|_{L^2(\Omega_n)} \to 0, \tag{44}$$

$$\|\Delta u_n - \Delta u\|_{L^2(\Omega_n)} \to 0, \tag{45}$$

$$|u_n(x_0) - u(x_0)| \to 0, \; \forall x_0 \in \Omega. \tag{46}$$

Proof of Lemma 2. In the space $H_0^1(\Omega)$ we define the norm

$$\|u\|_{H_0^1(\Omega)} = \sqrt{\|\nabla u\|^2 + \mu \|u\|^2}$$

and use the corresponding inner product. We do the same for $H_0^1(\Omega_n)$. For all $v \in H_0^1(\Omega_n)$, we have

$$\Big| \int_{\Omega_n} (\nabla u \cdot \nabla v + \mu uv)\, dx \Big| \leq \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega_n)}.$$

It follows from the Riesz Theorem that there exists a unique $u_n \in H_0^1(\Omega_n)$ such that

$$\|u_n\|_{H_0^1(\Omega_n)} \leq \|u\|_{H_0^1(\Omega)}$$

and

$$\int_{\Omega_n} (\nabla u_n \cdot \nabla v + \mu u_n v)\, dx = \int_{\Omega_n} (\nabla u \cdot \nabla v + \mu uv)\, dx, \ \ \forall v \in H_0^1(\Omega_n).$$

The last equation implies (42).

We extend u_n by zero values to Ω. Then we have $u_n \in H_0^1(\Omega)$,

$$\|u_n\|_{H_0^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)}, \tag{47}$$

and

$$\lim_{n\to\infty} \int_\Omega \left(\nabla(u_n - u) \cdot \nabla v + \mu(u_n - u)v\right)\, dx = 0, \ \ \forall v \in C_0^\infty(\Omega). \tag{48}$$

The uniform boundedness of $\|u_n\|_{H_0^1(\Omega)}$ and (48) imply that $u_n \to u$ weakly in $H_0^1(\Omega)$. And then the weak convergence with the inequality (47) implies strong convergence. Therefore we have (43) and (44). (45) follows from (43) and (42).

Without loss of generality assume that $x_0 \in \Omega_1$. Let ξ be a function in $C_0^\infty(\Omega_1)$ such that $\xi(x_0) = 1$. Applying the inequality (6) for the bounded smooth domain Ω_1, we have

$$|u_n(x_0) - u(x_0)| \leq \sup_{\Omega_1} |\xi(u_n - u)| \leq C(m) \|\Delta(\xi(u_n - u))\|_{\Omega_1}^{m/4} \|\xi(u_n - u)\|_{\Omega_1}^{1-m/4}$$
$$\leq C_\xi \left(\|\Delta u_n - \Delta u\|_{\Omega_1} + \|\nabla u_n - \nabla u\|_{\Omega_1} + \|u_n - u\|_{\Omega_1}\right)^{m/4} \|u_n - u\|_{\Omega_1}^{1-m/4},$$

where $\|\cdot\|_{\Omega_1}$ denotes the $L^2(\Omega_1)$ norm, and C_ξ denotes a constant depending only on ξ. Therefore (46) follows from (43), (44) and (45).

The proof of Lemma 1 is similar. The only exception is that we do not have (43). But we can use the Sobolev inequality (1) and obtain

$$\|u_n - u\|_{L^2(\Omega_1)} \leq c_1 \|u_n - u\|_{L^6(\Omega)} \leq c_1 c \|\nabla u_n - \nabla u\| \to 0.$$

Therefore we can prove (41) as proving (46).

Remark. The above lemmas may be generalized as follows. Ω_n may be arbitrary open sets increasing to Ω. The pointwise convergence (41) and (46) may be strengthened to the uniform convergence

$$\lim_{n\to\infty} \sup_{\Omega'} |u_n - u| = 0,$$

where Ω' is any bounded subset of Ω_k for any k. Excluding the pointwise limits, all other conclusions in the Lemmas also hold in higher dimensions.

6. The Green functions and integral formulas for arbitrary domains. The inequalities (2), (4) and (6), already proven for bounded smooth domains, are readily extended to arbitrary domains by using the lemmas in the previous section. It is not necessary to prove the representation formulas (8) and (9) in arbitrary domains and then derive these inequalities again. However, this latter approach will not only prove the inequalities, but also tell us that the only functions that make the equalities hold are those given in Theorems 1, 2 and 3, even when Ω is not assumed bounded and smooth. For this purpose, we proceed to prove Theorem 4 in arbitrary domains.

First we construct the Green functions in arbitrary domains. Let Ω be any open set in $\mathbb{R}^m$, $m = 2$ or 3. When $m = 2$, let $\mu > 0$; when $m = 3$, let $\mu \geq 0$. Let x_0 be an arbitrarily fixed point in Ω. Let $g_\mu(x; x_0)$ be the fundamental solution given in (22) or (14), satisfying

$$(\Delta - \mu)g_\mu = -\delta(x - x_0).$$

Let η be a function in $C_0^\infty(\Omega)$ such that $\eta = 1$ in a neighborhood of x_0. Let k_μ be the solution of the following problem:

$$\begin{aligned} &(\Delta - \mu)k_\mu = (\Delta - \mu)(g_\mu - \eta g_\mu) \quad \text{in } \Omega, \\ &k_\mu \in H_0^1(\Omega) \quad \text{in the case } \mu > 0, \\ &k_\mu \in \hat{H}_0^1(\Omega) \quad \text{in the case } \mu = 0. \end{aligned} \tag{49}$$

Denoting the right side of (49) by f_μ, clearly we have $f_\mu \in C_0^\infty(\Omega)$. The equivalent integral form of (49) is

$$\int_\Omega (\nabla k_\mu \cdot \nabla v + \mu k_\mu v)\,dx = -\int_\Omega f_\mu v\,dx, \quad \forall\, v \in C_0^\infty(\Omega).$$

In the case $\mu > 0$ ($m = 2$ or 3), we have

$$\Big| -\int_\Omega f_\mu v\,dx \Big| \leq \|f_\mu\|\,\|v\| \leq \frac{1}{\sqrt{\mu}}\|f_\mu\|\,\sqrt{\|\nabla v\|^2 + \mu\|v\|^2};$$

in the case $\mu = 0$ ($m = 3$), using Hölder's inequality and the Sobolev inequality (1), we have

$$\Big| -\int_\Omega f_\mu v\,dx \Big| \leq \|f_\mu\|_{L^{6/5}(\Omega)}\|v\|_{L^6(\Omega)} \leq c\|f_\mu\|_{L^{6/5}(\Omega)}\|\nabla v\|. \tag{50}$$

Therefore, by the Riesz Theorem, in both cases the solution k_μ exists uniquely. Let $G_\mu = \eta g_\mu + k_\mu$. We have

$$(\Delta - \mu)G_\mu = (\Delta - \mu)g_\mu = -\delta(x - x_0).$$

And near the boundary, G_μ equals to k_μ. If Ω is bounded and smooth, we have

$$G_\mu|_{\partial\Omega} = k_\mu|_{\partial\Omega} = 0.$$

Therefore G_μ is the Green function classically defined by (10) and (11).

If one uses another cut-off function $\tilde{\eta}$ and obtain another function $\tilde{G}_\mu$, it is easy to see that

$$\begin{aligned} &(\Delta - \mu)(G_\mu - \tilde{G}_\mu) = 0 \quad \text{in } \Omega, \\ &(G_\mu - \tilde{G}_\mu) \in H_0^1(\Omega) \quad \text{in the case } \mu > 0, \\ &(G_\mu - \tilde{G}_\mu) \in \hat{H}_0^1(\Omega) \quad \text{in the case } \mu = 0, \end{aligned}$$

which imply that $\tilde{G}_\mu - G_\mu = 0$. Therefore the function G_μ is unique, independent of the cut-off function η used in its construction.

Remark. Clearly, the Green function G_μ in other dimensions can also be constructed in the same way. In dimensions higher than three, G_0 can be constructed with an estimate similar to (50), but these are not used in this paper.

Now, we choose Ω_n increasing to Ω as in the previous section, such that $x_0 \in \Omega_1$. For easy comparison, we use one and the same cut-off function $\eta \in C_0^\infty(\Omega_1)$ to construct all the Green functions. We use the subscripts Ω_n and Ω to distinguish them. By Lemma 2 and Lemma 1 we have

$$\|k_{\mu,\Omega_n} - k_{\mu,\Omega}\|_{L^2(\Omega_n)} \to 0,$$
$$\|\nabla k_{\mu,\Omega_n} - \nabla k_{\mu,\Omega}\|_{L^2(\Omega_n)} \to 0,$$

and

$$\|\nabla k_{0,\Omega_n} - \nabla k_{0,\Omega}\|_{L^2(\Omega_n)} \to 0.$$

Hence we obtain

$$\|G_{\mu,\Omega_n} - G_{\mu,\Omega}\|_{L^2(\Omega_n)} \to 0,$$

and

$$\|\nabla(G_{0,\Omega_n} - G_{\mu,\Omega_n}) - \nabla(G_{0,\Omega} - G_{\mu,\Omega})\|_{L^2(\Omega_n)} \to 0.$$

It follows that the inequalities (15), (16) and (26) also hold for arbitrary domains. Pointwise inequalities such as (13) for the Green functions also hold similarly.

Given Ω and u as in Theorem 1. Let u_n be as defined in Lemma 1. Then, from (17) we have

$$u_n(x_0) = -\int_{\Omega_n} \left(G_{\mu,\Omega_n}\Delta u_n - \nabla(G_{0,\Omega_n} - G_{\mu,\Omega_n}) \cdot \nabla u_n\right) dx.$$

Passing to the limit, we obtain (8).

Similarly, given Ω and u as in Theorems 2 or 3, let u_n be as defined in Lemma 2. Then, from (27), we have

$$u_n(x_0) = -\int_{\Omega_n} G_{\mu,\Omega_n}(\Delta - \mu)u_n \, dx.$$

Passing to the limit, we obtain (9).

Acknowledgments. The author wishes to thank H.F. Weinberger and E.B. Fabes for many helpful discussions.

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