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ON WELL-POSEDNESS OF INTEGRO-DIFFERENTIAL EQUATIONS IN WEIGHTED L^2–SPACES

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Abstract. In this paper we consider the problem of constructing a well-posed state space model for a class of singular integro-differential equations of neutral type. The work is motivated by the need to develop a framework for the analysis of numerical methods for designing control laws for aeroelastic systems. Semigroup theory is used to establish existence and well-posedness results for initial data in weighted L^2–spaces. It is shown that these spaces lead naturally to the dissipativeness of the basic dynamic operator. The dissipativeness of the solution generator combined with the Hilbert space structure of these weighted spaces make this choice of a state space more suitable for use in the design of computational methods for control than previously used product spaces.

1. Introduction. The development of an appropriate state space model for the analysis and design of control laws is often a crucial step in the overall process of constructing computational methods for control. In particular, the form of the feedback control law that results from typical linear quadratic regulator design is completely determined by the choice of state space. During the past ten years considerable effort has been devoted to the problem of constructing well-posed state space models for neutral functional differential equations with non-atomic D operator. This work was primarily motivated by the problem of developing computational algorithms for active flutter control (see [1], [3]). General well-posedness results were obtained in [2], [3] and [5] using product spaces of the form $X^p = \mathbb{R}^n \times L^p(-r, 0; \mathbb{R}^n)$, $1 < p < \infty$. However, when the results in [3] are applied to the flutter problem, the parameter p must be chosen so that $p > 2$, i.e., X^p is not a Hilbert space. Given that the ultimate goal of the work was to develop practical computation methods for control, the fact that the space X is not a Hilbert space complicates the problem of constructing and analyzing numerical algorithms. Therefore, one is lead naturally to the problem of finding a state space model that has the following three properties:
(i) The state space is a Hilbert space Z.
(ii) The problem of developing finite dimensional approximating spaces Z^N is reasonable.

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(iii) The relationship between the states $z \in Z$ and the solutions to the governing functional differential equation is as simple as possible.

In this paper we show that certain weighted L^2-spaces can be used as state spaces for such systems. We use linear semigroup theory [6] to establish well-posedness. Numerical schemes based on this framework can be found in [4].

We consider several questions concerning the well-posedness of the initial value problem

$$\frac{d}{dt}\left[\int_{-r}^{0} g(s)x(t+s)\,ds\right] = \int_{-r}^{0} d\mu(s)x(t+s) + f(t), \quad t > 0 \tag{1.1}$$

with initial data

$$x(s) = \phi(s), \quad -r < s < 0. \tag{1.2}$$

We are interested in the case where $0 < r \leq +\infty$ and g is non-negative, non-decreasing and weakly singular.

We assume that $g(\cdot) \in L^1(-r, 0)$ and is weakly singular at $s = 0$. The measure $\mu(\cdot)$ is assumed to be of the form

$$\mu(s) = \mu_0(s) - \int_{s}^{0} a(\tau)\,d\tau,$$

where $\mu_0(\cdot)$ is of bounded variation on $(-r, 0)$ and $a(\cdot)$ is locally absolutely continuous on $(-r, 0)$. Additional restrictions will be placed on $\mu_0(\cdot)$ and $a(\cdot)$ in order to ensure the well-posedness of (1.1) in an appropriate space of initial data.

Although the results in this paper apply to very general systems, it is helpful to think of $g(\cdot)$ as an Abel kernel, i.e., $g(s) = |s|^{-\alpha}$ for $0 < \alpha < 1$ on a finite interval $[-1, 0]$. Moreover, equations of the form (1.1) with Abel kernels $g(\cdot)$ are found to have applications to several problems in aerodynamics and, in particular, are useful in modeling certain aeroelastic control systems (see [1]).

The initial value problem (1.1) – (1.2) can be written as

$$\int_{-r}^{0} g(s)x(t+s)\,ds = \int_{-r}^{0} g(s)\phi(s)\,ds + \int_{0}^{t}\left\{\int_{-r}^{0} d\mu(s)\,x(\tau+s)\right\}d\tau + \int_{0}^{t} f(\tau)\,d\tau \tag{1.3}$$

provided that the functions $t \to f(t)$, $t \to \int_{-r}^{0} d\mu(s)\,x(t+s)$ are integrable, $g(\cdot)\phi(\cdot)$ belongs to $L^1(-r, 0)$ and the function $t \to \int_{-r}^{0} g(s)x(t+s)\,ds$ is absolutely continuous. There are cases of practical interest when the solution of the problem

$$\int_{-r}^{0} g(s)x(t+s)\,ds = \eta + \int_{0}^{t}\left\{\int_{-r}^{0} d\mu(s)x(\tau+s)\right\}d\tau + \int_{0}^{t} f(\tau)\,d\tau \tag{1.4}$$

with $\eta \neq \int_{-r}^{0} g(s)\phi(s)\,ds$ is also important. Thus, we shall consider both problems. It is helpful to introduce the operators D and L by setting

$$D\phi = \int_{-r}^{0} g(s)\phi(s)\,ds, \quad L\phi = \int_{-r}^{0} d\mu(s)\phi(s),$$

respectively. Using rather standard notation (1.4) can be written as

$$Dx_t = \eta + \int_{0}^{t}\{Lx_\tau + f(\tau)\}\,d\tau, \tag{1.5}$$

where $x_t : (-r, 0) \to \mathbb{R}$ is defined by $x_t(s) = x(t+s)$. Note that (1.5) is equivalent to (1.3) if $D\phi = \eta$. Moreover, since $g(\cdot) \in L^1(-r, 0)$, it follows that $D : L^2_g \to \mathbb{R}$ is a bounded linear functional satisfying

$$|D\phi| \leq \Big[\int_{-r}^{0} |g(s)|\, ds \Big]^{1/2} \Big[\int_{-r}^{0} |\phi(s)|^2 g(s)\, ds \Big]^{1/2}.$$

The well-posedness of (1.5) has been studied by several authors. In [3] it was shown that under rather general conditions on D and L, (1.5) is well-posed in the sense that the solution exists and depends continuously on the initial data $(\eta, \phi(\cdot))$ in $\mathbb{R} \times L^p(-r, 0)$ and forcing function f in $L^p(0, \tau)$ for $\tau > 0$. Most of these studies have been based on an explicit representation of the solution to Abel's equation. Moreover, as noted above when these results were applied to certain aeroelastic control problems it was shown that (1.5) is well-posed in $\mathbb{R} \times L^p(-r, 0)$ only if $p > 2$ (see [2, 3, 5] for details). We shall investigate the well-posedness of (1.5) in the (weighted) Hilbert space $L^2_g = L^2(-r, 0; g\, ds)$.

In order to introduce the basic idea, we indicate how (1.1) can be (formally) written as an evolution equation on the state space L^2_g. Given $x : (-r, +\infty) \to \mathbb{R}$, we define $\phi : [0, +\infty) \times (-r, 0) \to \mathbb{R}$ by $\phi(t, s) = x(t+s)$. If x is locally absolutely continuous and the function $t \to \int_{-r}^{0} g(s) x(t+s)\, ds$ is (locally) absolutely continuous, then (1.1) can be written as the evolution equation in L^2_g

$$\frac{\partial}{\partial t}\phi(t, s) = \frac{\partial}{\partial s}\phi(t, s) \tag{1.6}$$

with the constraint that

$$\int_{-r}^{0} g(s) \frac{\partial}{\partial s}\phi(t, s)\, ds = \int_{-r}^{0} d\mu(s)\phi(t, s) + f(t). \tag{1.7}$$

Equation (1.7) is obtained by formally passing the time derivative inside the integral and then using (1.6). The system (1.6)–(1.7) can be re-written in terms of a dynamical system on L^2_g by defining an appropriate system operator. We use $LAC(a, b)$ to denote those functions $\psi : (a, b) \to \mathbb{R}$ that are locally absolutely continuous on (a, b). If $(a, b) = (-r, 0)$, then we abbreviate and write LAC. Let A denote the operator with domain

$$\mathrm{dom}(A) = \{\phi \in L^2_g : \phi \in LAC,\ \phi' \in L^2_g,\ \ D\phi' - L\phi = 0\} \tag{1.8}$$

and for $\phi \in \mathrm{dom}(A)$

$$A\phi = \phi' = \frac{d}{ds}\phi. \tag{1.9}$$

Assume that $L(1(\cdot)) \neq 0$, where $1(s) \equiv 1$ and define the operator $B : \mathbb{R} \to L^2_g$ by

$$[Bf](s) = b1(s)f \equiv bf, \tag{1.10}$$

where $b = 1/L(1(\cdot))$. We are interested in the Cauchy problem in L^2_g defined by

$$\dot{z}(t) = A(z(t) + Bf(t)), \quad t > 0, \tag{1.11}$$

with initial data

$$z(0) = \phi. \tag{1.12}$$

Although we have assumed $L(1(\cdot)) \neq 0$, we shall be able to relax this condition later (see Theorem 3.2). As it will be shown in Corollary 3.3., $\langle ABf, \phi\rangle_{L^2_g} = f\,\phi(0)$ for all $\phi \in \mathrm{dom}(A^*)$, where A^* is the adjoint operator of A (see, Lemma 3.4). Hence, in this sense ABf is the delta distribution at $s = 0$. The goal of this paper is to show that, under mild assumptions on g and μ, A generates a $\mathcal{C}_0$ -semigroup on L^2_g and solutions of (1.1) −(1.2) are equivalent to solutions at (1.11)–(1.12). We shall also consider the case described by (1.4) of inconsistent initial data. The paper is organized as follows. We first consider the simple case where $\mu_0 = 0$ and $L = 0$. Here we can establish the dissipativeness of A. This argument is then extended to certain problems for which $L \neq 0$. Consideration of the non-homogeneous problem follows and more general measure μ are allowed. The case of inconsistent initial data is discussed in the last theorem.

We shall use the following definition of solutions to (1.1)–(1.2). This definition is analogous to the definition of a generalized solution given in [3].

Definition 1.1. A solution to (1.1)–(1.2) is a measurable function $x : (-r, \infty) \to \mathbb{R}$ satisfying

(a) $x(s) = \phi(s)$ almost everywhere on $(-r, 0)$,
(b) $x_t(\cdot) \in L^2_g$ for $t \geq 0$,
(c) x satisfies (1.3) on $0 \leq t < +\infty$.

2. Special cases. Throughout the paper we assume that $0 < r \leq +\infty$ and the functions $a(s)$, $g(s)$ satisfy

(H1) $g(s) > 0$ almost everywhere on $(-r, 0)$,
(H2) $g(\cdot) \in L^1(-r, 0)$ and $g(s) \uparrow \infty$ as $s \to 0^-$,
(H3) $g(\cdot) \in H^1_{\ell oc}(-r, 0)$ with $g'(s) \geq 0$ almost everywhere on $(-r, 0)$,
(H4) $\left[\frac{a(\cdot)}{g(\cdot)}\right] \in L^2_g$.

Note that we require that $g(\cdot)$ be $L^1(-r, 0)$, but $g(\cdot)$ can have a weak singularity at $s = 0$. Conditions (H1)–(H3) can be relaxed for certain special problems.

Remark 2.1. Conditions (H1)–(H3) imply that there exist constants r_0, $0 < r_0 < r$, and $\gamma_0 > 0$ such that

$$g(s) \geq \gamma_0 > 0 \quad \text{a.e. on } [-r_0, 0). \tag{2.1}$$

Therefore, it is straightforward to show the existence of a constant $K > 0$ such that

$$\int_0^T |x(s)|^2\, ds \leq K\Big\{\sup_{0\leq t\leq T} \int_{-r}^0 |x_t(s)|^2 g(s)\, ds\Big\} \tag{2.2}$$

for all functions $x : (-r, T) \to \mathbb{R}$ satisfying $x \in L^2(0, T)$ and $x_t \in L^2_g$ for $0 \leq t \leq T$.

Remark 2.2. If $r < +\infty$ is finite, then condition (H1) can be relaxed provided that $g(s)$ is bounded from below. In particular, if $g(s) \geq -c$, then define $\tilde{g}(s) = g(s) + c \geq 0$ and write (1.1) as

$$\frac{d}{dt}\int_{-r}^0 \tilde{g}(s)\, x(t+s)\, ds = \int_{-r}^0 d\tilde{\mu}(s)\, x(t+s) + f(t), \tag{2.3}$$

where $\tilde{\mu}(s) = [c\tilde{\mu}_a(s) - c\tilde{\mu}_b(s)] + \mu(s)$ and

$$\tilde{\mu}_a(s) = \begin{cases} 1 & s = 0 \\ 0 & s < 0 \end{cases}, \quad \tilde{\mu}_b(s) = \begin{cases} 0 & -r < s \\ -1 & -r = s. \end{cases}$$

In this case $\tilde{g}(\cdot)$ will satisfy (H1)–(H4) and the results presented below will apply.

Remark 2.3. If one defines $\hat{g}(s) = e^{\omega s} g(s)$ for $\omega > 0$, then x will satisfy (1.1) if $y(t) = e^{-\omega t} x(t)$ satisfies

$$\frac{d}{dt}\Big\{ \int_{-r}^{0} \hat{g}(s)\, y(t+s)\, ds \Big\} = \int_{-r}^{0} [-\omega \hat{g}(s)\, ds + e^{\omega s} d\mu(s)] y(t+s) + e^{-\omega t} f(t) \tag{2.4}$$

with $y(s) = e^{-\omega s} x(s)$ for $-r < s < 0$. Note that $\hat{g}'(s) = e^{\omega s}(\omega g + g')$. Thus, if $\frac{g'}{g}$ is bounded below, then one can choose an $\omega > 0$ such that $\hat{g}' \geq 0$ on $(-r, 0)$; i.e., (H3) holds for $\hat{g}$.

Remark 2.4. If $g \notin L^1(-\infty, 0)$, then we may proceed as follows. Suppose $x(t)$, $t > -r$, is the solution to

$$\int_{-r}^{0} g(s)\, x(t+s)\, ds = \int_{-r}^{0} g(s)\phi(s)\, ds + \int_{0}^{t} \left(\int_{-r}^{0} d\mu(s) x(\sigma + s) + f(\sigma) \right) d\sigma.$$

Define the function $y(t) = e^{-\omega t} x(t)$, $t \geq -r$, for $\omega \in \mathbb{R}$. Then y satisfies

$$\begin{aligned} &e^{\omega t} \int_{-r}^{0} g(s) e^{\omega s} y(t+s) ds \\ &= \int_{-r}^{0} g(s) e^{\omega s} \psi(s) ds + \int_{0}^{t} e^{\omega \sigma} \Big(\int_{-r}^{0} e^{\omega s} d\mu(s) y(\sigma + s) + e^{-\omega \sigma} f(\sigma) \Big) d\sigma, \end{aligned}$$

where $\psi(s) = e^{-\omega s} \phi(s)$ for $-r < s < 0$. If

$$\eta(t) = \int_{-r}^{0} g(s)\, y(t+s)\, ds \quad \text{and} \quad \eta^0 = \int_{-r}^{0} g(s) \phi(s)\, ds,$$

then we obtain

$$\eta(t) = e^{-\omega t} \eta^0 + \int_{0}^{t} e^{-\omega(t-\sigma)} \left(\int_{-r}^{0} e^{\omega s} d\mu(s) y(\sigma + s) + e^{-\omega \sigma} f(\sigma) \right) d\sigma.$$

Thus, y satisfies

$$\frac{d}{dt} \int_{-r}^{0} \hat{g}(s) y(t+s) ds = -\omega \int_{-r}^{0} \hat{g}(s) y(t+s) ds + \int_{-r}^{0} e^{\omega s} d\mu(s) y(t+s) + e^{-\omega t} f(t), \tag{2.5}$$

where $\hat{g}(s) = e^{\omega s} g(s)$. Assume that there exists a constant $\omega > 0$ such that $\hat{g} \in L^1(-\infty, 0)$. Then, Theorem 3.2 shows that (2.5) has the unique solution $y(t + \cdot) \in C([0,T]; L^2_{\hat{g}})$ provided that the initial condition $\psi = e^{-\omega s} \phi$ satisfies

$$\int_{-\infty}^{0} \hat{g}\, |\psi|^2\, ds = \int_{-\infty}^{0} g e^{-\omega s} |\phi|^2\, ds \tag{2.6}$$

and that $f \in L^2(0,T)$. Suppose y is the said solution and define $x(t) = e^{\omega t} y(t)$, $t > -\infty$. Then, $x(t)$, $t > -\infty$, satisfies (1.1) using exactly the same arguments as above. This implies that (1.1) has the solution $x(t)$, $t > -\infty$ provided (2.6) and $f \in L^2(0,T)$ and $x(t + \cdot) \in C([0,T]; L^2_{\bar{g}})$, where $\bar{g} = g e^{-\omega s}$.

We begin by considering special cases for the form of A as defined by (1.3)–(1.9). Note that if $\phi \in \text{dom}(A)$, then $\phi(0^-)$ and $\phi(-r^+)$ exist. Throughout the paper, we will denote these limits by $\phi(0)$ and $\phi(-r)$, respectively.

Theorem 2.1. *Assume that* (H1)–(H3) *hold. If* $\mu = 0$*, then* A *is dissipative and generates a strongly continuous contraction semigroup on* L^2_g*.*

Proof. If $\phi \in D(A)$, then

$$\begin{aligned}\langle A\phi, \phi\rangle &= \int_{-r}^0 \phi'(s)\phi(s)g(s)ds = \int_{-r}^0 \phi'(s)(\phi(s) - \phi(0))g(s)ds + \int_{-r}^0 \phi'(s)\phi(0)g(s)ds \\ &= \int_{-r}^0 \psi'(s)\psi(s)g(s)\, ds,\end{aligned}$$

where $\psi(s) = \phi(s) - \phi(0)$, $-r < s < 0$. For each k and ϵ satisfying $-r < -k \le -r_0 \le -\epsilon < 0$ define the integral

$$I_{\epsilon,k} = \int_{-k}^{-\epsilon} \psi'(s)\psi(s)g(s)\, ds.$$

Since $g \in H^1[-k, -\epsilon]$, we have

$$\begin{aligned}I_{\epsilon,k} &= \frac{1}{2}\int_{-k}^{-\epsilon} \frac{d}{ds}|\psi(s)|^2 g(s)ds \\ &= \frac{1}{2}\left\{|\psi(-\epsilon)|^2 g(-\epsilon) - |\psi(-k)|^2 g(-k)\right\} - \frac{1}{2}\int_{-k}^{-\epsilon} g'(s)|\psi(s)|^2 ds.\end{aligned} \tag{2.7}$$

Also,

$$\psi(-\epsilon) = \psi(0) - \int_{-\epsilon}^0 \psi'(s)\, ds = -\int_{-\epsilon}^0 \psi'(s)\, ds$$

so it follows that

$$\begin{aligned}g(-\epsilon)|\psi(-\epsilon)|^2 &= g(-\epsilon)\Big|\int_{-\epsilon}^0 \psi'(s)\, ds\Big|^2 \le g(-\epsilon)\Big[\int_{-\epsilon}^0 |\psi'(s)|\sqrt{g(s)}\frac{ds}{\sqrt{g(s)}}\Big]^2 \\ &\le g(-\epsilon)\Big(\int_{-\epsilon}^0 \frac{ds}{g(s)}\Big)\Big(\int_{-\epsilon}^0 |\psi'(s)|^2 g(s)\, ds\Big) \\ &\le \Big(\int_{-\epsilon}^0 \frac{g(-\epsilon)}{g(s)}ds\Big)\int_{-r}^0 |\psi'(s)|^2 g(s)ds \le \epsilon \int_{-r}^0 |\psi'(s)|^2 g(s)\, ds \to 0\end{aligned}$$

as $\epsilon \to 0^+$. Note that we used the fact that g is non-decreasing and $g(s) \ge \gamma_0 > 0$. Since,

$$\langle A\phi, \phi\rangle = \lim_{\substack{\epsilon\to 0^+ \\ k\to r^-}} I_{\epsilon,k}$$

and

$$I_{\epsilon,k} \le \frac{\epsilon}{2}\int_{-r}^0 |\phi'(s)|^2 g(s)\, ds - \frac{1}{2}\left(|\psi(-k)|^2 g(-k) + \int_{-k}^{\epsilon} |\psi(s)|^2 g'(s)\, ds\right)$$

it follows that $\langle A\phi, \phi\rangle \le 0$ and A is dissipative. Moreover, if r is finite, then

$$\langle A\phi, \phi\rangle = -\frac{1}{2}g(-r)|\psi(-r)|^2 - \frac{1}{2}\int_{-r}^0 g'(s)|\psi(s)|^2\, ds. \tag{2.8}$$

Next consider the resolvent equation for A

$$\lambda\phi(s) - \phi'(s) = h(s) \in L^2_g \tag{2.9}$$

and

$$0 = D\phi' = \int_{-r}^0 g(s)\phi'(s)\, ds. \tag{2.10}$$

Solving (2.9) we obtain the representation

$$\phi(s) = e^{\lambda s}\phi(0) + \int_s^0 e^{\lambda(s-\xi)} h(\xi)\, d\xi, \qquad -r < s \le 0. \tag{2.11}$$

For $\mathrm{Re}\,\lambda > 0$

$$\begin{aligned}
&\int_{-r}^0 g(s)|\int_s^0 e^{\lambda(s-\xi)}h(\xi)d\xi|^2\, ds \le \int_{-r}^0 |\int_s^0 e^{(s-\xi)\mathrm{Re}\lambda}\sqrt{g(\xi)}|h(\xi)|\, d\xi|^2\, ds \\
&\le \int_{-r}^0 (\int_s^0 e^{(s-\xi)\mathrm{Re}\lambda}\, d\xi) \int_s^0 e^{(s-\xi)\mathrm{Re}\lambda} g(\xi)|h(\xi)|^2\, d\xi ds \\
&\le \frac{1}{\mathrm{Re}\lambda}\int_{-r}^0 (\int_{-r}^\xi e^{(s-\xi)\mathrm{Re}\lambda}\, ds) g(\xi)|h(\xi)|^2\, d\xi \le \frac{1}{(\mathrm{Re}\lambda)^2}\int_{-r}^0 g(\xi)|h(\xi)|^2\, d\xi.
\end{aligned}$$

Thus, the boundary condition (2.10) becomes

$$\Big[\int_{-r}^0 \lambda e^{\lambda s} g(s)\, ds\Big]\phi(0) = \int_{-r}^0 g(s)\Big[h(s) - \int_s^0 \lambda\, e^{\lambda(s-\xi)}h(\xi)d\xi\Big]ds. \tag{2.12}$$

Consequently, it follows that $\sigma(A) \cap \{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ contains only point spectra and $\lambda \in \sigma(A)$ with $\mathrm{Re}\lambda \ge 0$ if and only if $\int_{-r}^0 \lambda e^{\lambda s} g(s)\, ds = 0$. Since $\lambda \to \int_{-r}^0 e^{\lambda s} g(s)\, ds$ is continuous on $[0, \infty)$, there exists a $\lambda_0 > 0$ such that $\lambda_0 \int_{-r}^0 e^{\lambda_0 s} g(s)\, ds > 0$. Moreover, from (2.10) and (2.11) there exists $M > 0$ such that

$$|(\lambda_0 I - A)^{-1}h|_{L^2_g} \le M|h|_{L^2_g} \quad \text{for all} \quad h \in L^2_g.$$

Hence, one can conclude by Lumer-Phillips Theorem (e.g., Theorems 4.3 and 4.6 in [6, Chap. 1]) that if $(H1)$-$(H3)$ are satisfied, then A generates a C_0-semigroup on L^2_g.

The dissipativity argument in the proof of Theorem 2.1 can now be applied to a certain case where the right hand side is not zero.

Corollary 2.2. *Assume that r is finite, $\mu_0(s) = a_0\tilde{\mu}_a(s) + a_1\tilde{\mu}_b(s)$ and g satisfies:*

$$g' \ge c\, g \quad \text{on} \quad (-r, 0) \quad \text{for some} \quad c > 0. \tag{H5}$$

If

$$a_0 + a_1 + \frac{a_1^2}{2g(-r)} - \frac{c}{2}\Big(\int_{-r}^0 g(s)\, ds\Big) < 0, \tag{2.13}$$

then there exists $\omega \ge 0$ such that $A - \omega I$ is dissipative.

Proof. Note that for $\phi \in \mathrm{dom}(A)$

$$\langle A\phi, \phi\rangle = \int_{-r}^0 \langle\psi'(s), \psi(s)\rangle g(s)\, ds + \phi(0)\Big(a_0\phi(0) + a_1\phi(-r) + \int_{-r}^0 a(s)\phi(s)\, ds\Big).$$

Thus, from (2.8)

$$\langle A\phi,\phi\rangle = -\frac{1}{2}g(-r)|\psi(-r)|^2 - \frac{1}{2}\int_{-r}^0 g'(s)|\psi(s)|^2\,ds$$
$$+\,\phi(0)\Big((a_0+a_1)\phi(0)+a_1\psi(-r)+\int_{-r}^0 a(s)\phi(s)\,ds\Big).$$

Here for $\epsilon > 0$

$$\Big(\int_{-r}^0 g(s)\,ds\Big)|\phi(0)|^2 = \int_{-r}^0 g(s)|\phi(s)-\psi(s)|^2\,ds$$
$$\le (1+\frac{1}{\epsilon})\int_{-r}^0 g(s)|\phi(s)|^2\,ds + (1+\epsilon)\int_{-r}^0 g(s)|\psi(s)|^2\,ds$$
$$\le \frac{1+\epsilon}{\epsilon}\int_{-r}^0 g(s)|\phi(s)|^2\,ds + \frac{1+\epsilon}{c}\int_{-r}^0 g'(s)|\psi(s)|^2\,ds.$$

Hence,

$$\langle A\phi,\phi\rangle \le \Big(a_0+a_1+\frac{a_1^2}{2g(-r)} - \frac{c}{2(1+\epsilon)}\int_{-r}^0 g(s)\,ds\Big)|\phi(0)|^2$$
$$+\Big(\int_{-r}^0 (\frac{a^2}{g})\,ds\Big)^{1/2}|\phi(0)|\,|\phi|_{L_g^2} + \frac{c}{2\epsilon}|\phi|^2_{L_g^2}.$$

If (2.13) is satisfied, then there exists $\omega \ge 0$ such that $\langle A\phi,\phi\rangle \le \omega|\phi|^2_{L_g^2}$ for all $\phi \in \mathrm{dom}(A)$.

Next, we consider the case $RHS \ne 0$. To this end, first we study the equation of the form

$$\frac{d}{dt}\Big(\int_{-r}^0 g(s)\,x(t+s)\,ds\Big) = -x(t)+f(t). \tag{2.14}$$

In this case, $\langle A\phi,\phi\rangle \le -|\phi(0)|^2$ for $\phi \in \mathrm{dom}(A)$. Using exactly the same arguments as in the proof of Theorem 2.1, one thus can show that A associated with (2.14) generates a C_0-semigroup $S(t), t \ge 0$ on L_g^2.

Theorem 2.3. *For $\phi \in L_g^2$ and $f = 0$, there exists a unique solution $x(t)$, $t \ge 0$ to* (2.14) *and we have*

$$(S(t)\phi)\,(s) = x(t+s) \quad on \quad (-r,0) \quad for \quad t \ge 0.$$

Proof. If $\phi \in \mathrm{dom}(A^2)$, then $z(t) = \phi(t,\cdot) = S(t)\phi \in \mathrm{dom}(A^2)$, $t \ge 0$ and satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t,s) = \dfrac{\partial}{\partial s}\phi(t,s) \quad \text{in} \quad (-r,0) \\ \displaystyle\int_{-r}^0 g(s)\frac{\partial}{\partial s}\phi(t,s)\,ds = -\phi(t,0). \end{cases}$$

Since $z(t) \in C^2(0,T;L_g^2)$ for any $T \ge 0$,

$$\phi(t,0) = -\frac{d}{dt}\int_{-r}^0 g(s)\phi(t,s)\,ds \in H^1(0,T). \tag{2.15}$$

Thus, $\phi(t,s)$, $t \geq 0$ satisfies the initial-boundary value problem $\frac{\partial}{\partial t}\phi = \frac{\partial}{\partial s}\phi$ in $(-r,0)$ and $\phi(t,0) =$ "given H^1-function"; so that there exists a $x \in H^1[-r,T]$ such that

$$\phi(t,s) = x(t+s),\ t \geq 0 \quad \text{and} \quad -r \leq s \leq 0.$$

It follows from (2.15) that,

$$\frac{d}{dt}\Big(\int_{-r}^{0} g(s)\, x(t+s)\, ds\Big) = -x(t), \quad t \geq 0. \tag{2.16}$$

Since $\text{dom}(A^2)$ is dense in L^2_g and $\phi \in L^2_g$, there exists a sequence $\{\phi^n\}$ in $\text{dom}\,(A^2)$ such that $|\phi^n - \phi|_{L^2_g} \to 0$ as $n \to \infty$. Suppose $x^n(t+s) = \phi^n(t,s) = (S(t)\phi^n)\,(s)$, $t \geq 0$, then from (2.16)

$$\int_{-r}^{0} g(s)\, x^n(t+s)\ ds = \int_{-r}^{0} g(s)\phi^n(s)\, ds - \int_0^t x^n(s)\, ds.$$

Since $|S(t)\phi^n - S(t)\phi|_{L^2_g} \to 0$ as $n \to \infty$, uniformly on $[0,T]$, there exists a unique limit $x(t+\cdot) \in C(0,T;L^2_g)$ that satisfies

$$(S(t)\phi)(s) = x(t+s), \quad -r < s < 0, \quad t \geq 0$$

and

$$\int_{-r}^{0} g(s)x(t+s)\, ds = \int_{-r}^{0} g(s)\phi(s)\, ds - \int_0^t x(s)\, ds.$$

Conversely, suppose $x(t)$ is a solution to (2.14). Define $z(t) = x(t+\cdot) \in L^2_g$. Then $z(t)$, $t \geq 0$ satisfies

$$z(t) = \phi + A\int_0^t z(s)\, ds; \tag{2.17}$$

i.e., $z(t)$ is the weak-solution to (1.11). In fact, if $\xi = \int_0^t z(\sigma)\, d\sigma$; i.e.,

$$\xi(s) = \int_0^t x(\sigma+s)\, d\sigma, \qquad s \in [-r,0],$$

then ξ is locally absolutely continuous with

$$\frac{d}{ds}\xi = x(t+s) - x(s) \in L^2_g.$$

Moreover, since $x(t)$ satisfies (2.14), we have

$$\int_{-r}^{0} g(s)\frac{d}{ds}\xi\, ds = -\int_0^t x(\sigma)\ d\sigma = -\xi(0).$$

Hence, $\xi \in \text{dom}(A)$ and $A\xi = z(t) - \phi$ which implies z satisfies (2.17). Since the weak solution to (1.11) is unique, (2.14) has a unique solution.

Now, we consider the case when $\phi \in L^2_g$ and $f \in L^2_{\ell oc}(0,\infty)$.

Theorem 2.4. *For any $\phi \in L_g^2$ and $f \in L_{\ell oc}^2(0,\infty)$, there exists a unique solution $x(t)$, $t \geq 0$ to (2.14) and it satisfies*

$$|x(t+\cdot)|_{L_g^2}^2 \leq \left(|\phi|_{L_g^2}^2 + \int_0^t |f(s)|^2\, ds\right), \quad t \geq 0.$$

If we define the function $z(t) = x(t+s) \in L_g^2$, then

$$z(t) = S(t)\phi + A\int_0^t S(t-s)\psi f(s)\, ds,$$

and moreover, if $\phi + f(0)\psi \in dom(A)$ and $f \in H_{\ell oc}^1(0,\infty)$, then $z(t)$ is the strong solution to

$$\frac{d}{dt}z(t) = A\left(z(t) + \psi f(t)\right), \quad t > 0,$$

where $\psi = -1(\text{constant}) \in L_g^2$.

Proof. We first consider the case where $\phi = 0$ and $f(t) \equiv f$ (constant). Note that

$$A\int_0^t S(t-s)\,\hat{\phi}\, ds = S(t)\hat{\phi} - \hat{\phi} \quad \text{for } \hat{\phi} \in L_g^2. \tag{2.18}$$

Let $\xi \in L_g^2$ be the solution to the boundary-value problem:

$$A\xi = \xi' = 0 \quad \text{with} \quad \int_{-r}^0 g(s)\xi'(s)\, ds + \xi(s) = -f;$$

i.e., $\xi(s) = -f$ (constant). Define the function

$$\hat{\phi}(t,\cdot) = S(t)\xi - \xi \in L_g^2, \quad t \geq 0$$

and for $n > 0$

$$\hat{\phi}_n(t,\cdot) = S(t)J_n\xi - \xi \in L_G^2, \quad t \geq 0,$$

where $J_n = n(nI - A)^{-1}$. Then $\hat{\phi}_n(t,\cdot) \in L_g^2$ satisfies

$$\begin{cases} \dfrac{\partial\hat{\phi}_n}{\partial t} - \dfrac{\partial\hat{\phi}_n}{\partial s} = 0, \\ \displaystyle\int_{-r}^0 g(s)\frac{\partial}{\partial s}\hat{\phi}_n(t,s)\, ds = -\hat{\phi}_n(t,0) + f. \end{cases}$$

In fact, if $\hat{x}_n(t+\cdot) = S(t)J_n\xi$, then it follows from Theorem 2.3 that $\hat{x}_n$ satisfies

$$\int_{-r}^0 g(s)\hat{x}_n(t+s)\, ds = \int_{-r}^0 g(s)\xi_n(s)\, ds - \int_0^t \hat{x}_n(s)\, ds,$$

where $\xi_n = J_n\xi$. Thus, if we define $x_n(t) = \hat{x}_n(t) + f$, $t \geq -r$, then x_n satisfies

$$\frac{d}{dt}\left(\int_{-r}^0 g(s)x_n(t+s)\, ds\right) = -x_n(t) + f,$$

where $x_n(t+\cdot)=\hat{\phi}_n(t,\cdot)\in L^2_g$. Moreover, it follows that $\hat{\phi}_n(t)\in C(0,T;\operatorname{dom}(A))\cap C^1(0,T;L^2_g)$ and we have

$$I_n=\frac{1}{2}\frac{d}{dt}|\hat{\phi}_n|^2_{L^2_g}=\int_{-r}^0 g(s)\frac{\partial}{\partial s}\hat{\phi}_n(t,s)\hat{\phi}_n(t,s)\,ds$$
$$=\int_{-r}^0 g(s)\frac{\partial}{\partial s}(\hat{\phi}_n(t,s)-\hat{\phi}_n(t,0))(\hat{\phi}_n(t,s)-\hat{\phi}_n(t,0))ds+\hat{\phi}_n(t,0)\int_{-r}^0 g(s)\frac{\partial}{\partial s}\hat{\phi}_n(t,s)ds$$

and, using the same arguments as in the proof of Theorem 2.1, we have

$$\int_{-r}^0 g(s)\frac{\partial}{\partial s}(\hat{\phi}_n(t,s)-\hat{\phi}_n(t,0))(\hat{\phi}_n(t,s)-\hat{\phi}_n(t,0))\,ds\le 0.$$

Since

$$\int_{-r}^0 g(s)\frac{\partial}{\partial s}\hat{\phi}_n(t,s)\,ds=-\hat{\phi}_n(t,0)+f$$

it follows that

$$I_n\le\hat{\phi}_n(t,0)(-\hat{\phi}_n(t,0)+f)\le\frac{1}{2}|f|^2.$$

Thus, we have

$$|\hat{\phi}_n(t,\cdot)|^2_{L^2_g}\le t|f|^2,\quad t\ge 0.$$

Since for all $\xi\in L^2_g$, $J_n\xi\to\xi$ as $n\to\infty$ [6], it follows that $|\hat{\phi}_n(t,\cdot)-\hat{\phi}(t,\cdot)|_{L^2_g}\to 0$ as $n\to\infty$ and hence

$$|\hat{\phi}(t,\cdot)|^2_{L^2_g}\le t|f|^2,\quad t\ge 0,$$

where $x(t+\cdot)=\hat{\phi}(t,\cdot)$ satisfies

$$\frac{d}{dt}\left(\int_{-r}^0 g(s)x(t+s)\,ds\right)=-x(t)+f,\quad \text{a.e.}\ \ t\in(0,T).$$

Next, we assume $f(t)$ is given by

$$f(t)=\sum_{i=1}^n f_i\chi_{[t_{i-1},t_i)}$$

with $0=t_0<t_1<\cdots<t_n=T$. Applying the above arguments, successively on the intervals $[t_{i-1},t_i)$, $i=1,2,\ldots,n$, we obtain the function x defined by

$$x(t+s)=\Big[\sum_{i=1}^m f_i\left(S(t-t_{i-1})\psi-S(t-t_i)\psi\right)\Big](s) \tag{2.19}$$

for $t\in[t_m,t_{m+1})$, satisfying

$$\frac{d}{dt}\Big(\int_{-r}^0 g(s)x(t+s)\,ds\Big)=-x(t)+f_i,\quad \text{a.e.}\ \ t\in(t_{i-1},t_i), \tag{2.20}$$

and we have

$$|x(t+\cdot)|^2_{L^2_g}\le\sum_{i=1}^m|f_i|^2|t_i-t_{i-1}|^2,\quad t\ge 0. \tag{2.21}$$

Finally, for any $f \in L^2(0,T)$, there exists a sequence of step functions $\{f^n\}$ such that $|f^n - f|_{L^2(0,T)} \to 0$ as $n \to \infty$. Let $x^n(t,\cdot)$ be the corresponding solution to f^n defined by (2.19). Then from (2.21), $t \to x(t+\cdot) \in L^2_g$ defines a Cauchy sequence in $C(0,T;L^2_g)$. Thus

$$|x^n(t+\cdot) - x^m(t+\cdot)|_{L^2_g} \to 0 \quad \text{as} \quad n,\ m \to \infty$$

uniformly in t on $[0,T]$. Hence there exists a unique limit

$$x(t+s) = \lim_{n\uparrow\infty} x^n(t+s),$$

$t \ge 0$ and $-r \le s \le 0$ and x satisfies

$$\int_{-r}^0 g(s)x(t+s)\,ds = -\int_0^t (x(s) - f(s))\,ds, \quad t \ge 0,$$

and moreover,

$$|x(t+\cdot)|^2_{L^2_g} \le \int_0^t |f(s)|^2\,ds, \quad t \ge 0,$$

when $x(t+\cdot) \in C(0,T;L^2_g)$. From (2.18) and (2.19)

$$x^n(t+\cdot) = A\int_0^t S(t-s)\psi f^n(s)\,ds.$$

Since A is closed, the above convergence implies that $\int_0^t S(t-s)\psi f(s)\,ds \in$ dom(A) and

$$x(t+\cdot) = A\int_0^t S(t-s)\psi f(s)\,ds \tag{2.22}$$

for all $f \in L^2(0,T)$. Moreover, suppose $f \in H^1(0,T)$, then

$$z(t) = x(t+\cdot) = S(t)\psi f(0) - \psi f(t) + \int_0^t S(t-s)\psi f'(s)\,ds.$$

If $\phi \ne 0$, then linearity implies that

$$z(t) = S(t)(\phi + \psi f(0)) - \psi f(t) + \int_0^t S(t-s)\psi f'(s)\,ds,$$

and finally, if $\phi + \psi f(0) \in \mathrm{dom}(A)$, then

$$z(t) + \psi f(t) \in C(0,T;\mathrm{dom}(A))$$

and

$$A(z(t) + \psi f(t)) = A\,S(t)(\phi + \psi f(0)) + A\int_0^t S(t-s)\psi f'(s)\,ds.$$

Similarly, one can show that $\frac{d}{dt}z(t)$ exists for all $t > 0$ and $\frac{d}{dt}z(t) = A(z(t) + \psi f(t))$.

3. The general case. Let us now consider the general equation

$$\frac{d}{dt}\Big(\int_{-r}^0 g(s)\,x(t+s)\,ds\Big) = \int_{-r}^0 d\mu(s)\,x(t+s) + f(t), \tag{3.1}$$

where $\mu = \mu_0 - \int_s^0 a(\xi)d\xi$, μ_0 is of bounded variation on $(-r,0)$ and we assume

$$\int_{-r}^0 \frac{|d\mu_0(s)|}{g(s)} < \infty. \tag{H6}$$

From Remark 2.2, if r is finite, then without loss of generality one can assume that $g \ge g(-r) > 0$ on $[-r,0)$. Thus, in this case, (H6) is equivalent to μ_0 being of bounded variation on $(-r,0)$.

Theorem 3.1. *Suppose* (H6) *holds. Then for any* $T > 0$, $\phi \in L^2_g$ *and* $f \in L^2(0,T)$, *equation* (3.1) *has a unique solution* $x(t)$, $t \geq 0$, *and there exists a* K, *which depends only on* T, *such that*

$$|x(t+\cdot)|^2_{L^2_g} \leq K\Big(|\phi|^2_{L^2_g} + \int_0^t |f(s)|^2\, ds\Big). \tag{3.2}$$

Proof. Let M be the set defined by

$$M = \{x(t+\cdot) \in C(0,T;L^2_g) \text{ and } x(s) = \phi(s) \text{ on } (-r,0)\}.$$

Define the solution map T on M by $y = Tx$, where y is defined as the solution to

$$\frac{d}{dt}\Big(\int_{-r}^0 g(s)\, y(t+s)\, ds\Big) = -y(t) + x(t) + \int_{-r}^0 d\mu(s)\, x(t+s) + f(t)$$

with $y(s) = \phi(s)$, $s < 0$. We will show that for $x \in M$,

$$t \to x(t) + \int_{-r}^0 d\mu(s)x(t+s) \in L^2(0,T).$$

From Remark 2.1, there exists a $r_0 > 0$ such that $g \geq 1$ on $[-r_0, 0)$. Thus, $x \in M$ implies $x \in L^2(0,T)$. From (H4), for $x \in M$,

$$\int_{-r}^0 a(s)x(t+s)\, ds \in C(0,T).$$

Suppose $r = \infty$. Then for any $R > 0$,

$$\int_{-R}^0 d\mu_0(s)x(t+s) \in L^2(0,T),$$

and we have

$$\begin{aligned}
&\int_0^T \Big|\int_{-R}^0 d\mu_0(s)x(t+s)\Big|^2\, dt \leq \int_0^T \int_{-R}^0 |d\mu_0(s)| \int_{-R}^0 |d\mu_0(s)|\, |x(t+s)|^2\, dt\\
&= \int_{-R}^0 |d\mu_0(s)| \int_{-R}^0 |d\mu_0(s)| \int_0^T |x(t+s)|^2\, dt\\
&\leq \int_{-R}^0 |d\mu_0(s)| \Big(\int_{-R}^0 \frac{|d\mu_0(s)|}{g(s)} \int_s^{\min(0,T+s)} g(\xi)|\phi(\xi)|^2\, d\xi\\
&\qquad + \int_{-R}^0 |d\mu_0(s)| \int_0^{\max(0,T+s)} |x(\xi)|^2\, d\xi\Big)\\
&\leq \int_{-R}^0 |d\mu_0(s)| \Big[\int_{-R}^0 \frac{|d\mu_0(s)|}{g(s)} |\phi|^2_{L^2_g} + \int_{-R}^0 |d\mu_0(s)|\, |x|^2_{L^2(0,T)}\Big].
\end{aligned} \tag{3.3}$$

Similarly, for $T \leq R_1 < R_2$

$$\Big|\int_{-R_2}^{-R_1} d\mu_0(s)x(t+s)\Big|^2_{L^2(0,T)} \leq \int_{-R_2}^{-R_1} |d\mu_0(s)| \int_{-R_2}^{-R_1} \frac{|d\mu_0(s)|}{g(s)} |\phi|^2_{L^2_g}.$$

Thus, $\left\{\int_{-k}^0 d\mu_0(s)x(t+s)\right\}_{k\geq 1}$ is a Cauchy sequence in $L^2(0,T)$ and hence from (3.3)

$$\Big|\int_{-\infty}^0 d\mu_0(s)x(t+s)\Big|^2_{L^2(0,T)} \leq \int_{-\infty}^0 |d\mu_0(s)|\Big[\int_{-\infty}^0 \frac{|d\mu_0|}{g}|\phi|^2_{L^2_g} + \int_{-\infty}^0 |d\mu_0|\,|x|^2_{L^2(0,T)}\Big]. \tag{3.4}$$

For r finite, we have

$$\Big|\int_{-r}^0 d\mu_0(s)x(t+s)\Big|^2_{L^2(0,T)} \leq \Big(\int_{-r}^0 |d\mu_0(s)|\Big)^2\Big[\frac{1}{g(-r)}|\phi|^2_{L^2_g} + |x|^2_{L^2(0,T)}\Big]. \tag{3.5}$$

Hence it follows from Theorem 2.4 that $Tx \in M$ for all $x \in M$. Define the sequence $\{x^k\}$ in M by $x^{k+1} = Tx^k$, where $x^0(t) = 0,\ t > 0$. Then, from Theorem 2.4

$$|x^1(t+\cdot)|^2_{L^2_g} \leq |\phi|^2_{L^2_g} + \int_0^t |f(s)|^2\,ds. \tag{3.6}$$

Now, let $y^k = x^{k+1} - x^k$ for $k \geq 1$, then y^k satisfies

$$\frac{d}{dt}\left(\int_{-r}^0 g(s)y^k(t+s)\,ds\right) = -y^k(t) + y^{k-1}(t) + \int_{-r}^0 d\mu(s)y^{k-1}(t+s) \tag{3.7}$$

with $y^k(s) = 0,\ s < 0$. For $t < r$, it follows from Theorem 2.4 and (3.4)–(3.5) that

$$\begin{aligned}
|y^k(t+\cdot)|^2_{L^2_g} &\leq 3\Big[1 + \Big(\int_{-r}^0 |d\mu_0(s)|\Big)^2\Big]\int_0^t |y^{k-1}(s)|^2 ds \\
&\quad + 3\Big(\int_{-t}^0 \frac{a^2(s)}{g(s)}ds\Big)\int_0^t\int_{-\sigma}^0 g(s)|y^{k-1}(\sigma+s)|^2 ds d\sigma \\
&\leq 3\Big[1 + \Big(\int_{-r}^0 |d\mu_0(s)|\Big)^2\Big]\frac{1}{g(-t)}|y^{k-1}(t+\cdot)|^2_{L^2_g} + 3t\Big(\int_{-t}^0 \frac{a^2}{g}ds\Big)\Big(\max_{s\in[0,t]}|y^{k-1}(s+\cdot)|^2_{L^2_g}\Big),
\end{aligned}$$

where we used $g(s) \geq g(-t),\ s \geq -t$. Since $g(-t) \uparrow \infty$ as $t \to 0^+$, for t_0 sufficiently small

$$|y^k(t+\cdot)|_{C(0,t_0;L^2_g)} \leq \frac{1}{2}|y^{k-1}(t+\cdot)|_{C(0,t_0;L^2_g)}, \quad k \geq 2.$$

This implies that $\{x^k(t+\cdot)\}_{k\geq 1}$ is a Cauchy sequence in $C(0,t_0;L^2_g)$ and hence from (3.7) its unique limit

$$x(t+\cdot) = \lim x^k(t+\cdot) \ \text{ as } \ k \to \infty \ \text{ in } \ C(0,t_0;L^2_g)$$

satisfies (3.1) on $t \in [0,t_0]$. From Theorem 2.4 and (3.4)-(3.6), there exists a constant $M_1 > 0$ (independent of $\phi \in L^2_g$ and $f \in L^2_{\ell oc}$) such that

$$|y^1(t+\cdot)|^2_{C(0,t_0;L^2_g)} \leq \Big|x^1(t) + \int_{-r}^0 d\mu(s)x^1(t+s)\Big|_{L^2(0,t_0)} \leq M_1\Big(|\phi|^2_{L^2_g} + \int_0^{t_0}|f(s)|^2\,ds\Big).$$

Since

$$|x(t+\cdot)|_{C(0,t_0;L^2_g)} \leq 2|y^1(t+\cdot)|_{C(0,t_0;L^2_g)} + |x^1(t+\cdot)|_{C(0,t_0;L^2_g)}$$

we have

$$|x(t+\cdot)|^2_{C(0,t_0;L^2_g)} \leq (2M_1+1)\Big(|\phi|^2_{L^2_g} + \int_0^{t_0}|f(s)|^2\,ds\Big).$$

Since (3.1) is linear and time-invariant in x, one can extend the interval of existence to an arbitrary bounded interval $[0,T]$, employing the above construction of solutions, successively on the intervals $[jt_0,(j+1)t_0]$, $j \geq 1$. The following theorem proves the equivalence of solutions to the equation(3.1) and the Cauchy problem (1.11).

Theorem 3.2. *Let $x(t)$, $t \ge -r$, be the solution to (3.1) with $f \equiv 0$ and define the solution semigroup $S(t)$, $t \ge 0$, on L^2_g by*

$$(S(t)\phi)(s) = x(t+s) \quad \text{on} \quad (-r, 0). \tag{3.8}$$

Then, $S(t)$, $t \ge 0$, forms a $\mathcal{C}_0$-semigroup on L^2_g whose infinitesimal generator A is given by

$$\begin{aligned} dom(A) = \Big\{\phi \in L^2_g : \phi \ \text{ is locally absolutely continuous with } \ \dot{\phi} \in L^2_g \ \text{ and} \\ \int_{-r}^0 g(s)\dot{\phi}(s)\,ds = \int_{-r}^0 d\mu(s)\phi(s)\Big\} \end{aligned} \tag{3.9}$$

and $A\phi = \phi'$ for $\phi \in dom(A)$. Moreover, $\lambda \in \rho(A)$ if and only if $\Delta(\lambda) \ne 0$, where the characteristic function $\Delta(\lambda)$ is given by

$$\Delta(\lambda) = \lambda \int_{-r}^0 g(s)e^{\lambda s}\,ds - \int_{-r}^0 d\mu(s)e^{\lambda s}. \tag{3.10}$$

If $\lambda \in \rho(A) \cap \mathbb{R}^+$ and $\psi_\lambda = \Delta(\lambda)^{-1}e^{\lambda s} \in L^2_g$, then the solution $x(t+\cdot) \in L^2_g$ to (3.1) can be represented as

$$x(t+\cdot) = S(t)\phi - (A - \lambda I)\int_0^t S(t-s)\psi_\lambda f(s)\,ds. \tag{3.11}$$

In particular, if $L(1(\cdot)) \ne 0$, then we may set $\lambda = 0$ and $\psi_\lambda(s) \equiv b$ as defined in (1.10).

Proof. Since $x(t)$ satisfies (2.14) with

$$f(t) = x(t) + \int_{-r}^0 d\mu(s)x(t+s) \in L^2(0,T),$$

it follows from Theorem 2.4, the semigroup $S(t)$ defined by (3.8) is strongly continuous. Let $\phi \in \text{dom}(A)$ and choose $\lambda \in \mathbb{R}^+$ sufficiently large. Then for some $\psi \in L^2_g$

$$\phi = (\lambda I - A)^{-1}\psi = \int_0^\infty e^{-\lambda t}S(t)\psi\,dt,$$

which implies

$$\phi(s) = \int_0^\infty e^{-\lambda t}x(t+s)\,dt = e^{\lambda s}\int_0^\infty e^{-\lambda\tau}x(\tau)d\tau + \int_s^0 e^{\lambda(s-\sigma)}\psi(\sigma)\,d\sigma,$$

where $x(t)$, $t \ge -r$ is the solution to (3.1) with initial data ψ and $f \equiv 0$. This equality shows that ϕ is locally absolutely continuous on $(-r, 0]$ and $\phi' = \lambda\phi - \psi \in L^2_g$. Thus, $A\phi = \phi'$ and

$$\begin{aligned} &\int_{-r}^0 g(s)\phi'(s)\,ds = \int_0^\infty \lambda e^{-\lambda t}\Big(\int_{-r}^0 g(s)x(t+s)\,ds - \int_{-r}^0 g(s)\psi(s)\,ds\Big)\,dt \\ &= \int_0^\infty \lambda e^{-\lambda t}\Big(\int_0^t\int_{-r}^0 d\mu(s)x(\sigma+s)\,d\sigma\Big)dt = \int_{-r}^0 d\mu(s)\int_0^\infty \Big(\int_\sigma^\infty \lambda e^{-\lambda t}dt\Big)x(\sigma+s)ds \\ &= \int_{-r}^0 d\mu(s)\int_0^\infty e^{-\lambda\sigma}x(\sigma+s)\,ds = \int_{-r}^0 d\mu(s)\phi(s). \end{aligned}$$

Hence we have proved that dom(A) is contained in the set D defined by (3.9). Conversely, suppose $\phi \in D$. For λ sufficiently large

$$\psi = \lambda\phi - \phi' \in L^2_g \quad \text{and} \quad \phi_\lambda = (\lambda I - A)^{-1}\psi \in \operatorname{dom}(A),$$

i.e., $\psi = (\lambda I - A)\phi_\lambda$. Then,

$$\lambda\phi - \phi' = \lambda\phi_\lambda - \phi'_\lambda,$$
$$\int_{-r}^0 g(s)\phi'(s)ds - \int_{-r}^0 d\mu(s)\phi(s) = \int_{-r}^0 g(s)\phi'_\lambda(s)\,ds - \int_{-r}^0 d\mu(s)\phi_\lambda(s).$$

These two equations imply

$$\xi(s) = \phi(s) - \phi_\lambda(s) = e^{\lambda s}\xi(0)$$

and

$$\Delta(\lambda)\xi(0) = \Big(\lambda \int_{-r}^0 g(s)e^{\lambda s}ds - \int_{-r}^0 d\mu(s)e^{\lambda s}\Big)\xi(0) = 0.$$

Since for $\lambda > 0$ and $\varepsilon > 0$,

$$\lambda \int_{-r}^0 g(s)e^{\lambda s}\,ds \geq g(-\varepsilon)(1 - e^{-\lambda\varepsilon}),$$

for λ sufficiently large, $\Delta(\lambda) \neq 0$ and therefore, $\phi = \phi_\lambda \in \operatorname{dom}(A)$. Exactly the same calculation as above shows that $\lambda \in \rho(A)$ if and only if $\Delta(\lambda) \neq 0$. For $\lambda \in \rho(A) \cap \mathbb{R}^+$, let $y(t,s) = e^{-\lambda t}x(t+s)$, $t \geq 0$ on $(-r,0)$, where $x(t)$, $t \geq -r$, is the solution to (3.1). Then, $y(t,\cdot)$, $t \geq 0$, formally satisfies

$$\frac{d}{dt}y(t,\cdot) = (A - \lambda I)y(t,\cdot) \quad \text{in} \;\; (-r,0)$$
$$\int_{-r}^0 g(s)\frac{\partial}{\partial s}y(t,s)\,ds = \int_{-r}^0 d\mu(s)y(t,s) + e^{-\lambda t}f(t).$$

Using the same arguments as in the proof of Theorem 2.4, one obtains

$$y(t,\cdot) = e^{-\lambda t}S(t)\phi - (A - \lambda I)\int_0^t e^{-\lambda(t-s)}S(t-s)\psi_\lambda e^{-\lambda s}f(s)\,ds,$$

which implies (3.11). Also, $(A - \lambda I)\psi_\lambda$ in (3.11) is the delta distribution at $s = 0$ in the following sense.

Corollary 3.3. *For $\phi \in$ dom(A^*)*

$$-\langle (A - \lambda I)\psi_\lambda, \phi\rangle = \phi(0), \tag{3.12}$$

where ψ_λ is defined as in Theorem 3.2.

In order to prove Corollary 3.3 we need

Lemma 3.4. *The adjoint operator A^* of A is given by*

$$A^*\psi = -\frac{1}{g}\frac{d}{ds}(g(s)(\psi(s)-\psi(0)) - \mu(s)\psi(0))$$

with domain $dom(A^*) = \{\phi \in L^2_g(-r,0) : z(s) = g(s)(\phi(s)-\phi(0)) - \mu(s)\phi(0)$ is locally absolute continuous with $\frac{z'}{g} \in L^2_g, \lim_{\theta\to 0^-} z(\theta) = -\mu(0)\psi(0)$ and $\lim_{\theta\to -r^+} z(\theta) = 0\}$.

Proof. Since A is densely defined, the adjoint operator A^* exists and it satisfies

$$\langle A\phi, \psi\rangle = \langle \phi, y\rangle \quad \text{for all} \quad \phi \in \mathrm{dom}(A),$$

where $\psi \in \mathrm{dom}(A^*)$ and $y = A^*\psi \in L^2_g$. Thus,

$$\begin{aligned}\int_{-r}^0 g(s)\phi'(s)\psi(s)\,ds &= \int_{-r}^0 g(s)\phi(s)y(s)\,ds \\ = \phi(0)\Big(\int_{-r}^0 g(\xi)y(\xi)d\xi\Big) &- \int_{-r}^0 \Big(\int_{-r}^s g(\xi)y(\xi)d\xi\Big)\phi'(s)\,ds.\end{aligned} \tag{3.13}$$

First we assume that $\mu(0) = \int_{-r}^0 d\mu(\theta) \neq 0$, where one can assume $\mu(-r) = 0$ without loss of generality. Then

$$\phi(s) = \phi(0) - \int_s^0 \phi'(\xi)d\xi$$

and therefore,

$$\begin{aligned}\int_{-r}^0 g(s)\phi'(s)\,ds &= \mu(0)\phi(0) - \int_{-r}^0 d\mu(s)\int_s^0 \phi'(\xi)d\xi \\ = \mu(0)\phi(0) - \int_{-r}^0 \Big(\int_{-r}^\xi d\mu(s)\Big)\phi'(\xi)\;d\xi &= \mu(0)\phi(0) - \int_{-r}^0 \mu(s)\phi'(s)\,ds.\end{aligned}$$

Thus, $\phi(0) = \frac{1}{\mu(0)}\int_{-r}^0 (g(s)+\mu(s))\phi'(s)\,ds$ and it follows from (3.13) that

$$\int_{-r}^0 \Big[g(s)(\psi(s) - \frac{1}{\mu(0)}\Big(\int_{-r}^0 gy\,d\xi)\Big) - \frac{\mu(s)}{\mu(0)}\Big(\int_{-r}^0 gy\,d\xi\Big) + \int_{-r}^s gy\,d\xi\Big]\phi'(s)\,ds = 0$$

for all $\phi' \in L^2_g(-r,0)$. This implies that

$$g(s)(\psi(s) - \frac{1}{\mu(0)}\Big(\int_{-r}^0 gy\,d\xi)\Big) - \frac{\mu(s)}{\mu(0)}\Big(\int_{-r}^0 gy\,d\xi\Big) + \int_{-r}^s gy\,d\xi = 0, \quad \text{a.e.}$$

Since $g(\theta) \uparrow \infty$ as $\theta \to 0^-$, it follows that $\int_{-r}^0 g(\xi)y(\xi)d\xi = \mu(0)\psi(0)$ and hence

$$z(s) = g(s)(\psi(s)-\psi(0)) - \mu(s)\psi(0) = \int_{-r}^s -g(\xi)y(\xi)\,d\xi$$

and

$$\lim_{s\to 0^-} z(s) = -\mu(0)\psi(0), \qquad \lim_{s\to -r^+} z(s) = 0.$$

Next we assume that $\mu(0) = 0$. Then,

$$\int_{-r}^{0} (g(s) + \mu(s))\phi'(s)\, ds = 0.$$

Note that $\int_{-r}^{0} d\mu(s) = \mu(0) = 0$ implies that all constant functions are in $\mathrm{dom}(A)$. Hence $\phi(0) \in \mathbb{R}$ can be chosen arbitrary. It thus follows from (3.13) that

$$\int_{-r}^{0} g(\xi)y(\xi)d\xi = 0$$

and

$$g(s)\psi(s) + \int_{-r}^{s} g(\xi)y(\xi)d\xi = c(g(s) + \mu(s)),$$

where c is a constant. Since $g(s) \uparrow \infty$ as $s \to 0^-, c = \psi(0)$ and we obtain

$$g(s)(\psi(s) - \psi(0)) - \mu(s)\psi(0) + \int_{-r}^{s} g(\xi)y(\xi)d\xi = 0.$$

Hence $\lim_{s\to 0^-} z(s) = 0 = \mu(0)\psi(0)$ and $\lim_{s\to -r^+} z(s) = 0$.

We now return to the proof of Corollary 3.3. It follows from Lemma 3.3 that for $\phi \in \mathrm{dom}(A^*)$ and $z(s) = g(s)(\phi(s) - \phi(0)) - \mu(s)\phi(0)$,

$$\begin{aligned} -\langle \psi_\lambda, (A^* - \lambda I)\phi\rangle &= \frac{1}{\Delta(\lambda)} \int_{-r}^{0} e^{\lambda s}(\lambda g(s)\phi(s) + \frac{d}{ds} z(s))\, ds \\ &= \frac{1}{\Delta(\lambda)} \left(\int_{-r}^{0} \lambda e^{\lambda s}(g(s) + \mu(s))\phi(0)\, ds - \mu(0)\phi(0) \right) = \phi(0), \end{aligned}$$

which implies (3.12).

Finally, we discuss the inconsistent initial data case. Given $\eta = 1$, consider the equation

$$\int_{-r}^{0} g(s)x(t+s)\, ds = 1 + \int_{0}^{t}\int_{-r}^{0} d\mu(s)x(\sigma+s)d\sigma, \quad x(s) = 0, \quad s < 0. \tag{3.14}$$

Theorem 3.5. *Assume that there exists a $\lambda \in \rho(A) \cap \mathbb{R}^+$ such that*

$$|\Delta^{-1}(i\omega + \lambda)| \in L^2(0, \infty),$$

where Δ is the characteristic function defined by (3.10). *Then, there exists a locally measurable function x which satisfies* (3.14) *almost everywhere, with $x(t+\cdot) \in L^2_{\ell oc}(0,\infty; L^2_g)$.*

Proof. Consider the family of equations

$$\int_{-r}^{0} g(s)x(t+s)\, ds = \int_{0}^{t} \left[\int_{-r}^{0} d\mu(s)\, x(\sigma+s) + f_\gamma(\sigma) \right] \, d\sigma \tag{3.15}$$

with $f_\gamma(s) = \gamma e^{-\gamma s}$, $s > 0$, $\gamma \geq 0$ and $x(s) = 0$, $s < 0$. By Theorems 3.1 and 3.2, equation (3.15) possesses a unique solution $x_\gamma(t)$, $t \geq -r$, and

$$z_\gamma(t) = x_\gamma(t+\cdot) = -(A - \lambda I)\int_{0}^{t} S(t-s)\psi_\lambda f_\gamma(s)\, ds \in L^2_g. \tag{3.16}$$

Let $y_\gamma(t) = z_\gamma(t)e^{-\lambda t} \in L^2_g,\ t \geq 0$. Then $y_\gamma \in L^2(0,\infty; L^2_g)$ and its (one-sided) Laplace transformation is given by

$$\begin{aligned}\hat{y}_\gamma(z) &= \int_0^\infty e^{-zt} y_\gamma(t)\, dt = -\,(A-\lambda I)((z+\lambda)\,I - A)^{-1} \frac{\gamma}{\gamma+\lambda+z}\psi_\lambda \\ &= \Delta^{-1}(z+\lambda)\frac{\gamma}{\gamma+\lambda+z} e^{(z+\lambda)s} \in L^2_g, \quad \operatorname{Re} z > 0.\end{aligned}$$

In fact, if we let $\psi = ((z+\lambda)\,I - A)^{-1} e^{\lambda\cdot}$, then $\psi' = (z+\lambda)\,\psi - e^{\lambda s},\ s \in [-r,0]$, and consequently

$$\psi(s) = e^{(z+\lambda)s}\psi(0) + \frac{1}{z}\,(e^{\lambda s} - e^{(z+\lambda)s}), \quad -r \leq s \leq 0.$$

The condition $D\psi' = L\psi$ implies that $\Delta(z+\lambda)\psi(0) = z^{-1}(\Delta(z+\lambda) - \Delta(\lambda))$, so that

$$\psi(s) = \frac{1}{z}\,e^{(z+\lambda)s}(1 - \frac{\Delta(\lambda)}{\Delta(z+\lambda)}) + \frac{1}{z}\,(e^{\lambda s} - e^{(z+\lambda)s}), \quad -r \leq s \leq 0.$$

A simple calculation yields

$$((A-\lambda I)\psi)\,(s) = -\,e^{(z+\lambda)s}\,\frac{\Delta(\lambda)}{\Delta(z+\lambda)}, \quad -r \leq s \leq 0,$$

which implies the desired identity. Note that if we let $\hat{y}(z) = \Delta^{-1}(z+\lambda)e^{(z+\lambda)x}$, $\operatorname{Re} z > 0$, then $\hat{y}$ belongs to the Hardy-Lebesgue class $H^2(0)$ and

$$\int_{-\infty}^\infty |\hat{y}_\gamma(i\omega) - \hat{y}(i\omega)|^2_{L^2_g}\, d\omega \to 0 \quad \text{as} \quad \gamma \to \infty.$$

Let $y \in L^2(0,\infty; L^2_g)$ be the inverse Fourier transform of $\hat{y}(i\omega)$:

$$y(t) = \lim \int_{-N}^N e^{i\omega t}\hat{y}(i\omega)\, d\omega \in L^2_g \quad \text{as} \quad N \to \infty.$$

Then by Fourier-Plancherel's theorem

$$\int_0^\infty |y_\gamma(t) - y(t)|^2_{L^2_g}\, dt \to 0 \quad \text{as} \quad \gamma \to \infty.$$

Suppose $z(t) = e^{\lambda t} y(t)$. Then for any $T > 0$

$$\int_0^T |z_\gamma(t) - z(t)|^2_{L^2_g}\, dt \to 0 \quad \text{as} \quad \gamma \to \infty.$$

Since

$$\begin{aligned}\int_0^t S(t-s)\psi_\lambda f_\gamma(s)\, ds &= S(t)\psi_\lambda - e^{-\gamma t}\psi_\lambda - A\int_0^t S(t-s)\psi_\lambda e^{-\gamma s}\, ds \\ &\to S(t)\psi_\lambda \quad \text{in} \quad L^2(0,T; L^2_g) \quad \text{as} \quad \gamma \to \infty,\end{aligned}$$

and A is closed on $L^2(0,T;L^2_g)$, from (3.16) we have

$$z(t) = -(A-\lambda I)S(t)\psi_\lambda \in L^2(0,T;L^2_g).$$

This implies that there exists a locally measurable function x such that $z(t) = x(t+\cdot) \in L^2_g$ almost everywhere. Now, since

$$\int_0^t f_\gamma(s)\,ds = 1 - e^{-\gamma t}, \quad t \geq 0,$$

it follows from (3.15) that $x(t+\cdot) \in L^2(0,T;L^2_g)$ satisfies (3.14) almost everywhere in $(0,T)$ and this completes the proof.

In conclusion, we note that the Hilbert space $L^2_g(-r,0)$ provides a state space in which (3.1) is well-posed. Theorem 3.2 establishes the basic equivalence between the solutions to (3.1) and the semigroup generated by A. Moreover, the space $L^2_g(-r,0)$ is simple enough to be used in the development of approximating systems and hence we have shown that $L^2_g(-r,0)$ is a state space that meets the basic requirements (i)-(iii) as given in the introduction. Numerical results based on this framework can be found in [4].

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