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# NEW NONEXISTENCE RESULTS FOR ELLIPTIC EQUATIONS WITH SUPERCRITICAL NONLINEARITY

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(Submitted by: H. Brezis)

**1. Introduction.** In this paper we consider the problem

$$(*)\qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega;\ u \not\equiv 0 & \text{in } \Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $f$ is a continuous function such that $f(0) = 0$ and

$$f(t) \sim t|t|^{p-2} \text{ for } t \to \pm\infty, \text{ with } p \geq \frac{2n}{n-2}$$

($\frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $H^{1,2}(\Omega) \subset L^p(\Omega)$). It is well known (see Pohozaev [27]) that Problem $(*)$ has no solution if $\Omega$ is starshaped and, for instance, $f(t) = t|t|^{p-2}$ with $p \geq \frac{2n}{n-2}$. On the other hand, a remarkable result of Bahri and Coron (see [1]) guarantees that, if $\Omega$ has nontrivial topology (in a suitable sense), then Problem $(*)$ with $f(t) = t|t|^{p-2}$ and $p = \frac{2n}{n-2}$ has at least one positive solution.

Other existence, nonexistence and multiplicity results of positive solutions for Problem $(*)$ with $f(t) = t|t|^{p-2}$ and $p = \frac{2n}{n-2}$, under suitable assumptions on the shape of $\Omega$, have been obtained by several authors (see [9], [11], [10], [8], [20], [21], [24], [30]). Hence, it is natural to ask whether the result of Bahri-Coron [1] can be extended to the case $p > \frac{2n}{n-2}$ (for other problems concerning the case $p > \frac{2n}{n-2}$ see also [6], [28], [29], [16], [17], [18], [19]).

A result proved in [26] shows that the nontriviality of the topology of $\Omega$ in the sense of Bahri-Coron [1] does not guarantee the existence of solutions for Problem $(*)$ with $f(t) = t|t|^{p-2}$ for every $p > \frac{2n}{n-2}$ (this question has been posed by Rabinowitz, as Brezis reports in [4]). More precisely, in [26] it is proved that, if $p > \frac{2(n-k)}{n-k-2}$ (with $1 \leq k \leq n-3$), then there exists a bounded domain $\Omega \subset \mathbb{R}^n$, homotopically equivalent to the $k$-dimensional sphere $S_k$ (hence topologically nontrivial in the sense of [1]), such that $(*)$ has no solution when $f(t) = t|t|^{p-2}$.

In this paper we obtain more general and complete results, which enable us to state, in particular, the following proposition (see also Theorem 3.1 and Applications 3.2 for more precise and general statements).

**Proposition 1.1.** *For every positive integer $k$ such that $1 \leq k \leq n-3$ there exists a bounded domain $\Omega$ homotopically equivalent to the $k$-dimensional sphere $S_k$ (hence topologically nontrivial) such that Problem $(*)$ with $f(t) = t|t|^{p-2}$ has no solutions*

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*for* $p \geq \frac{2(n-k)}{n-k-2}$*, while for* $2 < p < \frac{2(n-k)}{n-k-2}$ *it has infinitely many solutions and at least one of them is positive.*

The techniques developed in this paper allow us to study Problem $(*)$ also for functions $f$ more general than in [26] (see Examples 3.3 and Remark 3.6); in particular, they can be applied to functions $f$ like $f(t) = t|t|^{p-2} + \lambda t$ with $\lambda \in \mathbb{R}$; so we get information on the bifurcation problem related to $(*)$. Moreover the techniques of this paper enable us to extend the class of the bounded domains $\Omega$ for which nonexistence results for $(*)$ hold (see Remark 3.6 for a comparison with nonexistence results obtained in [26]). Notice that neither the results of [26] nor those of this paper concern the cases $n = 3$ with $p > 6$ or $n \geq 4$ with $\frac{2n}{n-2} < p < \frac{2(n-1)}{n-3}$. Therefore, in these cases it is not left out that the topological nontriviality of $\Omega$ (in the sense of [1]) may guarantee the existence of (positive) solutions of $(*)$. Finally, let us point out that in [25] one can find some examples of contractible bounded domains $\Omega$ in $\mathbb{R}^n$, with $n \geq 3$, such that Problem $(*)$ with $f(t) = t|t|^{p-2}$ has at least one positive solution for every $p > \frac{2n}{n-2}$; more precisely, in [25] it is proved that for every positive integer $h$ there exists a contractible bounded domain $\Omega$ such that $(*)$ has at least $h$ positive solutions for every $p > \frac{2n}{n-2}$.

The main results of this paper are Theorem 3.1 and Proposition 3.4; their proof is based on Lemma 2.5, which is inspired by Pohozaev [27]. In 3.2 we show some applications of Theorem 3.1, which in particular proves Proposition 1.1.

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## 2. Preliminary results.

**2.1. Notations.** For every $x = (x_1 \ldots x_n) \in \mathbb{R}^n$ and for every positive integer $k < n$, we set

$$P_1^k(x) = (x_1 \ldots x_k\, ,\, x_{k+1}, 0 \ldots 0) \in \mathbb{R}^n, \quad P_2^k(x) = x - P_1^k(x).$$

Let $B_k = \{x \in \mathbb{R}^n : P_1^k(x) = 0\}$; for every $x \in \mathbb{R}^n \backslash B_k$, we set

$$N_k(x) = \{\lambda P_1^k(x) : \lambda \in \mathbb{R}\},$$

$$T_k(x) = \{y \in \mathbb{R}^n : (x \cdot y) = 0, \quad P_2^k(y) = 0\}$$

(notice that $T_k(x)$ has dimension $k$). For every $\rho_0 > 0$, let $v_{\rho_0}^{(k)}$ be the vector field in $\mathbb{R}^n \backslash B_k$ such that

$$v_{\rho_0}^{(k)}(x) = \varphi(|P_1^k(x)|)\; P_1^k(x) + P_2^k(x), \quad \forall x \in \mathbb{R}^n \backslash B_k,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is the function

$$\varphi(\rho) = \frac{\rho^{k+1} - \rho_0^{k+1}}{(k+1)\rho^{k+1}}, \quad \forall \rho > 0.$$

Notice that $\varphi$ is a $C^\infty$ function, hence $v_{\rho_0}^{(k)} \in C^\infty(\mathbb{R}^n \backslash B_k, \mathbb{R}^n)$. For every vector field $v = (v_1 \ldots v_n)$ of class $C^1$, we set

$$\operatorname{div} v = \sum_{i=1}^n D_i v_i;$$

$$dv(x)[\xi] = \frac{\partial}{\partial t} v(x + t\xi)_{|t=0}\ , \quad \text{i.e.,} \quad dv[\xi] = \sum_{i=1}^n D_i v \xi_i.$$

We recall the following Lemma, which is a generalization of the Pohozaev identity (see [27]) and whose proof is contained in [26].

**Lemma 2.2.** (See [26]). *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. Let $u$ be a solution of the problem*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*where $f$ is a continuous function in $\mathbb{R}$. Then, for every $v \in C^1(\bar{\Omega}, \mathbb{R}^n)$ we have*

$$\frac{1}{2}\int_{\partial\Omega} |Du|^2 (v \cdot \nu) d\sigma = \int_\Omega (dv[Du] \cdot Du)\, dx + \int_\Omega (\operatorname{div} v)(F(u) - \frac{1}{2}|Du|^2)\, dx,$$

*where $F(t) = \int_0^t f(\tau)d\tau$ and $\nu$ denotes the outward normal on the boundary of $\Omega$.*

(Notice that Pohozaev identity is obtained for $v(x) = x - x_0$).

**Lemma 2.3.** *Let $v_{\rho_0}^{(k)}$ be the vector field above defined* (see Notations 2.1). *Then, for every $x \in \mathbb{R}^n \backslash B_k$ we have*

a) $\operatorname{div} v_{\rho_0}^{(k)}(x) = n - k, \quad \forall x \in \mathbb{R}^n \backslash B_k$;
b) $dv_{\rho_0}^{(k)}(x)[\xi_{(\beta)}] = \xi_{(\beta)}, \quad \forall \xi_{(\beta)} \in B_k$;
c) $dv_{\rho_0}^{k}(x)[\xi_{(\nu)}] = [1 - k\varphi(|P_1^k(x)|)]\xi_{(\nu)}, \quad \forall \xi_{(\nu)} \in N_k(x)$;
d) $dv_{\rho_0}^{k}(x)[\xi_{(\tau)}] = \varphi(|P_1^k(x)|)\xi_{(\tau)}, \quad \forall \xi_{(\tau)} \in T_k(x)$.

**Proof.**

a)
$$\operatorname{div} v_{\rho_0}^{(k)}(x) = \left(\frac{d\varphi}{d\rho}(|P_1^k(x)|)\frac{P_1^k(x)}{|P_1^k(x)|} \cdot P_1^k(x)\right) + \varphi(|P_1^k(x)|) \operatorname{div} P_1^k(x) + \operatorname{div} P_2^k(x)$$
$$= \frac{d\varphi}{d\rho}(|P_1^k(x)|)|P_1^k(x)| + \varphi(|P_1^k(x)|)(k+1) + n - k - 1.$$

Then, in order to obtain a), it is enough to note that $\varphi$ is a solution of the differential equation

$$\frac{d\varphi}{d\rho}\rho + (k+1)\varphi = 1.$$

b) Let $\xi_{(\beta)} \in B_k$ (i.e., $P_1^k(\xi_{(\beta)}) = 0$). Then

$$v_{\rho_0}^{(k)}(x + t\xi_{(\beta)}) = \varphi(|P_1^k(x)|)P_1^k(x) + P_2^k(x) + t\xi_{(\beta)},$$

from which b) easily follows.
c) Let $\xi_{(\nu)} \in N_k(x)$, i.e., $\xi_{(\nu)} = \lambda P_1^k(x)$ with $\lambda \in \mathbb{R}$. In this case we have

$$v_{\rho_0}^{(k)}(x + t\xi_{(\nu)}) = \varphi(|P_1^k(x) + t\lambda P_1^k(x)|)(P_1^k + t\lambda P_1^k(x)) + P_2^k(x)$$

and, if $t$ is small enough,

$$v_{\rho_0}^{(k)}(x + t\xi_{(\nu)}) = \varphi((1 + t\lambda)|P_1^k(x)|)(1 + t\lambda)P_1^k(x) + P_2^k(x).$$

Hence

$$\frac{\partial}{\partial t}v_{\rho_0}^{(k)}(x + t\xi_{(\nu)})_{|t=0} = \frac{d\varphi}{d\rho}(|P_1^k(x)|)\lambda|P_1^k(x)|P_1^k(x)$$
$$+ \varphi(|P_1^k(x)|)\lambda P_1^k(x) = \left[\frac{d\varphi}{d\rho}(|P_1^k(x)|)|P_1^k(x)| + \varphi(|P_1^k(x)|)\right]\xi_{(\nu)}.$$

Since $\varphi$ is a solution of the differential equation $\frac{d\varphi}{d\rho}\rho + (k+1)\varphi = 1$, we have

$$\frac{d\varphi}{d\rho}(|P_1^k(x)|)|P_1^k(x)| + \varphi(|P_1^k(x)|) = 1 - k\varphi(|P_1^k(x)|)$$

from which we get c).
d) Let $\xi_{(\tau)} \in T_k(x)$; then $P_2^k(\xi_{(\tau)}) = 0$ and $(\xi_{(\tau)} \cdot P_1^k(x)) = 0$. Since $P_2^k(\xi_{(\tau)}) = 0$,

$$v_{\rho_0}^{(k)}(x + t\xi_{(\tau)}) = \varphi(|P_1^k(x) + t\xi_{(\tau)}|)(P_1^k(x) + t\xi_{(\tau)}) + P_2^k(x).$$

So, we have

$$\frac{\partial}{\partial t} v_{\rho_0}^{(k)}(x + t\xi_{(\tau)})_{|t=0} = \frac{d\varphi}{d\rho}(|P_1^k(x)|)\frac{(P_1^k(x) \cdot \xi_{(\tau)})}{|P_1^k(x)|}P_1^k(x) + \varphi(|P_1^k(x)|)\xi_{(\tau)}.$$

Hence d) is obtained recalling that $(P_1^k(x) \cdot \xi_{(\tau)}) = 0$.

**Corollary 2.4.** *The vector field* $v_{\rho_0}^{(k)}$ (*see Notations* 2.1) *satisfies the following properties:*

a) *for every* $x \in \mathbb{R}^n \backslash B_k$ *and for every* $\xi \in \mathbb{R}^n$, *we have*

$$(dv_{\rho_0}^{(k)}(x)[\xi] \cdot \xi) = |\xi_{(\beta)}|^2 + [1 - k\varphi(|P_1^k(x)|)]|\xi_{(\nu)}|^2 + \varphi(|P_1^k(x)|)|\xi_{(\tau)}|^2$$

*where* $\xi = \xi_{(\beta)} + \xi_{(\nu)} + \xi_{(\tau)}$ *with* $\xi_{(\beta)} \in B_k$, $\xi_{(\nu)} \in N_k(x)$, $\xi_{(\tau)} \in T_k(x)$.

b) *For every* $x \in \mathbb{R}^n$ *such that* $|P_1^k(x)| \geq \rho_0$ *and for every* $\xi \in \mathbb{R}^n$,

$$(dv_{\rho_0}^{(k)}(x)[\xi] \cdot \xi) \leq |\xi|^2.$$

c) *If* $k \geq 1$, *for every* $x \in \mathbb{R}^n$ *such that* $|P_1^k(x)| > \rho_0$ *and for every* $\xi \in \mathbb{R}^n$ *such that* $P_1^k(\xi) \neq 0$,

$$(dv_{\rho_0}^{(k)}(x)[\xi] \cdot \xi) < |\xi|^2$$

(*notice that for* $k = 0$ *we should have* $(dv_{\rho_0}^{(k)}(x)[\xi] \cdot \xi) = |\xi|^2, \forall \xi \in \mathbb{R}^n$).

**Proof.** a) easily follows from b), c) and d) of Lemma 2.3. For b) it is enough to observe that $\varphi(\rho) < \frac{1}{k+1} \leq 1$, $\forall \rho > 0$ and

$$1 - k\varphi(\rho) \leq 1 \quad \text{for } \varphi(\rho) \geq 0, \text{ i.e., for } \rho \geq \rho_0.$$

In order to prove c), notice that

$$|P_1^k(\xi)|^2 = |\xi_{(\tau)}|^2 + |\xi_{(\nu)}|^2,$$

so, if $P_1^k(\xi) \neq 0$, it must be that $|\xi_{(\tau)}|^2 > 0$ or $|\xi_{(\nu)}|^2 > 0$. If $|\xi_{(\tau)}|^2 > 0$, c) follows from the fact that

$$\varphi(\rho) < \frac{1}{k+1} < 1, \quad \forall \rho > 0;$$

if $|\xi_{(\nu)}|^2 > 0$, c) follows because $\varphi(\rho) > 0$ for $\rho > \rho_0$.

**Lemma 2.5.** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and assume that there exists $\rho_0 > 0$ such that $|P_1^k(x)| > \rho_0$ for every $x \in \Omega$, with $k$ a positive integer and $k < n$. Let $u \not\equiv 0$ be a solution of the problem*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*where $f$ is a continuous real function. Then*

$$\frac{1}{2}\int_{\partial\Omega} |Du|^2 \cdot (v_{\rho_0}^{(k)} \cdot \nu) d\sigma < \int_\Omega |Du|^2 dx + (n-k)\int_\Omega (F(u) - \frac{1}{2}|Du|^2)\, dx,$$

*where $F(t) = \int_0^t f(\tau)d\tau$, $\nu$ denotes the outwards normal at the boundary of $\Omega$ and $v_{\rho_0}^{(k)}$ is the vector field defined in Notations* 2.1.

**Proof.** Since $|P_1^k(x)| > \rho_0$ for every $x \in \Omega$, then b) of the previous lemma implies that $(dv_{\rho_0}^{(k)}(x)[Du] \cdot Du) \leq |Du|^2$ in $\Omega$. Now, notice that, since $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$, it must be that $P_1^k(Du) \not\equiv 0$ in $\Omega$. So, by c) of the previous lemma, there exists a non-negligible subset of $\Omega$ where the strict inequality

$$(dv_{\rho_0}^k(x)[Du] \cdot Du) < |Du|^2$$

holds. Therefore, we have

$$\int_\Omega (dv_{\rho_0}^k(x)[Du] \cdot Du)\, dx < \int_\Omega |Du|^2 dx$$

and so the assertion follows from Lemma 2.2 (with $v = v_{\rho_0}^{(k)}$) and a) of Lemma 2.3.

**3. Nonexistence results.** In this paragraph we shall use the inequality obtained in Lemma 2.5 to prove nonexistence results for Problem $(*)$ (see Introduction).

**Theorem 3.1.** *Let $n, k$ be two positive integers such that $1 \leq k \leq n-3$. Let $f$ be a continuous function which verifies the following condition: there exists $p \geq \frac{2(n-k)}{n-k-2}$ such that $tf(t) \geq pF(t)$, $\forall t \in \mathbb{R}$, where $F(t) = \int_0^t f(\tau)d\tau$. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with the following properties: there exists $\rho_0 > 0$ such that*

$$|P_1^k(x)| > \rho_0, \quad \forall x \in \Omega \quad \text{and } (v_{\rho_0}^{(k)} \cdot \nu) \geq 0 \quad \text{on } \partial\Omega.$$

*Then the problem*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*has only the solution $u \equiv 0$.*

**Proof.** Let us note that $u \equiv 0$ is a solution of the problem because in our assumptions $f(0) = 0$. Indeed we have: $f(t) \geq p\frac{F(t)}{t}$, $\forall t > 0$; hence, for $t \to 0^+$, we get $f(0) \geq pf(0)$. Analogously, for $t < 0$ we have $f(t) \leq p\frac{F(t)}{t}$; so, for $t \to 0^-$, we get $f(0) \leq pf(0)$. Since $p \neq 1$ (indeed $p > 2$), it follows that $f(0) = 0$.

Now, we show that $u \equiv 0$ is the unique solution of the boundary value problem. By contradiction, suppose that $u \not\equiv 0$ is solution. Then, since $(v_{\rho_0}^k \cdot \nu) \geq 0$ on the boundary of $\Omega$, by Lemma 2.5 we have

$$0 < \int_\Omega |Du|^2 dx + (n-k)\int_\Omega (F(u) - \frac{1}{2}|Du|^2) dx.$$

Hence, since $F(u) \leq \frac{uf(u)}{p}$,

$$0 < \frac{n-k}{p} \int_\Omega uf(u)\, dx - \frac{n-k-2}{2} \int_\Omega |Du|^2 dx.$$

Notice that $\int_\Omega uf(u) = \int_\Omega |Du|^2 dx$; so we get

$$0 < [\frac{1}{p} - \frac{n-k-2}{2(n-k)}] \int_\Omega |Du|^2 dx.$$

Since $\int_\Omega |Du|^2 dx > 0$ (because $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$ in $\Omega$), we have $[\frac{1}{p} - \frac{n-k-2}{2(n-k)}] > 0$. But this is in contradiction with the assumption $p \geq \frac{2(n-k)}{n-k-2}$; so the assertion is proved.

**3.2. Applications.** Here we give some examples of bounded domains $\Omega \subset \mathbb{R}^n$, satisfying the assumptions of the previous theorem. In particular, these examples prove Proposition 1.1.

Let $n$ and $k$ be two positive integers such that $1 \leq k \leq n-3$. We set

$$S_k(\rho) = \{x \in \mathbb{R}^n : |x| = \rho,\ P_2^k(x) = 0\}$$

(see Notations 2.1), and

$$T_k(\rho, \sigma) = \{x \in \mathbb{R}^n :\, \text{dist}\,(x, S_k(\rho)) < \sigma\}.$$

*If* $0 < \sigma < \rho$, *then the bounded domain* $\Omega = T_k(\rho, \sigma)$ *satisfies the assumptions of Theorem* 3.1 *with* $\rho_0 = \rho - \sigma > 0$. In fact, if $x \in T_k(\rho, \sigma)$ we have $|P_2^k(x)|^2 + (|P_1^k(x)| - \rho)^2 < \sigma^2$; therefore $(|P_1^k(x)| - \rho)^2 < \sigma^2$ and so $|P_1^k(x)| > \rho - \sigma = \rho_0$ for every $x \in T_k(\rho, \sigma)$. We prove that $(v_{\rho_0}^{(k)} \cdot \nu) \geq 0$ on the boundary of $T_k(\rho, \sigma)$: let us note that the boundary of $T_k(\rho, \sigma)$ has equation

$$|P_2^k(x)|^2 + (|P_1^k(x)| - \rho)^2 = \sigma^2$$

and the outward normal is $\nu(x) = \frac{1}{\sigma}(x - \rho \frac{P_1^k(x)}{|P_1^k(x)|})$. Hence we have to prove that $|P_2^k(x)|^2 + (|P_1^k(x)| - \rho)^2 = \sigma^2$ implies

$$([\varphi(|P_1^k(x)|) P_1^k(x) + P_2^k(x)] \cdot [x - \rho \frac{P_1^k(x)}{|P_1^k(x)|}]) \geq 0,$$

that is,

$$\varphi(|P_1^k(x)|) |P_1^k(x)| \cdot (|P_1^k(x)| - \rho) + |P_2^k(x)|^2 \geq 0.$$

Since $|P_2^k(x)|^2 = \sigma^2 - (|P_1^k(x)| - \rho)^2$, we have

$$\begin{aligned}
&\varphi(|P_1^k(x)|)|P_1^k(x)|(|P_1^k(x)| - \rho) + |P_2^k(x)|^2 \\
&= \varphi(|P_1^k(x)|)|P_1^k(x)|(|P_1^k(x)| - \rho) + (\sigma - |P_1^k(x)| + \rho)(\sigma + |P_1^k(x)| - \rho) \\
&= \varphi(|P_1^k(x)|)|P_1^k(x)|\sigma + \varphi(|P_1^k(x)|)|P_1^k(x)|(|P_1^k(x)| - \rho - \sigma) \\
&+ (\sigma - |P_1^k(x)| + \rho)(\sigma + |P_1^k(x)| - \rho) = \varphi(|P_1^k(x)|)|P_1^k(x)|\sigma \\
&+ (\sigma - |P_1^k(x)| + \rho)(\sigma + |P_1^k(x)| - \rho - \varphi(|P_1^k(x)|)|P_1^k(x)|).
\end{aligned}$$

As $\rho-\sigma \le |P_1^k(x)| \le \rho+\sigma$ for every $x\in \partial T_k(\rho,\sigma)$ and $\varphi(t)\ge 0$ for every $t\ge \rho_0 = \rho-\sigma$, it suffices to prove that $t\varphi(t)\le \sigma+t-\rho$ for every $t>\rho-\sigma$, i.e., $t\varphi(t)\le t-\rho_0$ for every $t\ge\rho_0$; but this is true because $t\varphi(t)-t+\rho_0=0$ for $t=\rho_0$ and moreover

$$\frac{d}{dt}[t\varphi(t)-t+\rho_0] = -k\varphi(t)\le 0 \text{ for } t\ge\rho_0.$$

Therefore we have that *Problem* $(*)$ *has no solution for* $\Omega = T_k(\rho,\sigma)$ *with* $0<\sigma<\rho$ *and* $f(t)=t|t|^{p-2}$ *with* $p\ge \frac{2(n-k)}{n-k-2}$.

On the contrary, if $2<p<\frac{2(n-k)}{n-k-2}$, we can use the symmetry of the domain $\Omega = T_k(\rho,\sigma)$ in order to obtain solutions of $(*)$. Indeed, let us note that $T_k(\rho,\sigma)$ has radial symmetry with respect to the co-ordinates $x_1\ldots x_{k+1}$, i.e.,

$$x\in\Omega \ \text{ if and only if } \ (0\ldots 0,\ |P_1^k(x)|,\ P_2^k(x))\in\Omega.$$

So, it is natural to look for solutions in the space of the functions having the same symmetry: we say that a function $u$ has radial symmetry with respect to $x_1\ldots x_{k+1}$ if

$$u(x) = u(0\ldots 0,\ |P_1^k(x)|,\ P_2^k(x)),\quad \forall x\in\Omega.$$

We shall denote $H_{0,r}^{1,2}(\Omega)$ the subspace of $H_0^{1,2}(\Omega)$ constituted by the radial functions. Since

$$0<\rho-\sigma<|P_1^k(x)|<\rho+\sigma,\quad \forall x\in T_k(\rho,\sigma),$$

we have that $H_{0,r}^{1,2}(\Omega)$ is embedded in $L^p(\Omega)$ for every $p\le\frac{2(n-k)}{n-k-2}$ and the embedding is compact for $p<\frac{2(n-k)}{n-k-2}$; it follows that Problem $(*)$, for $\Omega=T_k(\rho,\sigma)$ with $0<\sigma<\rho$ and $f(t)=t|t|^{p-2}$ with $2<p<\frac{2(n-k)}{n-k-2}$, has infinitely many radial solutions and at least one of them is positive. Finally, let us note that for $0<\sigma<\rho$ the domain $\Omega=T_k(\rho,\sigma)$ is homotopically equivalent to the $k$-dimensional sphere $S_k$ (hence it has nontrivial topology in the sense of [1]). So, Proposition 1.1 is proved. Notice that an open question is whether, for $2<p<\frac{2(n-k)}{n-k-2}$, analogous existence results hold for bounded domains homotopically equivalent to $S_k$, without radial symmetry properties.

**Examples 3.3.** Here we examine some types of functions $f$ for which the assumptions of Theorem 3.1 are satisfied. For instance, let

$$f(t) = t(|t|^{p-2}+\lambda|t|^{q-2})$$

with $p\ge\frac{2(n-k)}{n-k-2}$ $(1\le k\le n-3)$, $q>1$ and $\lambda\in\mathbb{R}$. If $\frac{2(n-k)}{n-k-2}\le q\le p$, the assumptions of Theorem 3.1 are satisfied for every $\lambda\in\mathbb{R}$, because we have obviously

$$tf(t)\ge q\ F(t),\quad \forall t\in\mathbb{R} \text{ and } \forall\lambda\in\mathbb{R}.$$

If $1<q<\frac{2(n-k)}{n-k-2}$, the function $f$ satisfies the assumptions of Theorem 3.1 if and only if $\lambda\le 0$. In fact, for $\lambda\le 0$ we have

$$tf(t)\ge pF(t),\quad \forall t\in\mathbb{R};$$

conversely, if there exists $p'\ge\frac{2(n-k)}{n-k-2}$ such that $tf(t)\ge p'F(t)$, $\forall t\in\mathbb{R}$, i.e.,

$$|t|^p+\lambda|t|^q\ge p'(\frac{|t|^p}{p}+\lambda\frac{|t|^q}{q}),\quad \forall t\in\mathbb{R},$$

we can deduce that $\lambda \geq \frac{p'}{q}\lambda$; so $\lambda \leq 0$, because $p' > q$. Finally, if $q > p$ the assumptions of Theorem 3.1 are satisfied if and only if $\lambda \geq 0$. In fact, for $\lambda \geq 0$ we have

$$tf(t) \geq pF(t), \quad \forall t \in \mathbb{R};$$

conversely, if there exists $p' \geq \frac{2(n-k)}{n-k-2}$ such that

$$tf(t) \geq p'F(t), \quad \forall t \in \mathbb{R},$$

we infer that $p \geq p'$ and $\lambda \geq \lambda\frac{p'}{q}$; so $\lambda \geq 0$ because $p' \leq p < q$.On the other hand, let us note that, for $\lambda < 0$ small enough and $q > p$, existence results of positive solutions for Problem $(*)$ with

$$f(t) = t(|t|^{p-2} + \lambda|u|^{q-2})$$

have been obtained in [16], [17], [18], [19].

For functions $f$ of the type $f(t) = t|t|^{p-2} + \lambda t$, from Lemma 2.5 we obtain the following nonexistence result.

**Proposition 3.4.** *Let $n, k$ be two positive integers such that $1 \leq k \leq n-3$. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain satisfying the same assumptions of Theorem* 3.1. *Let*

$$f(t) = t|t|^{p-2} + \lambda t \text{ with } p > \frac{2(n-k)}{n-k-2} \text{ and } \lambda \in \mathbb{R}.$$

*Then there exists $\lambda^* > 0$ ($\lambda^*$ depends on $\Omega, p, n, k$) such that Problem $(*)$ has no solution for $\lambda \leq \lambda^*$.*

**Proof.** Let $u \not\equiv 0$ in $\Omega$ be a solution of Problem $(*)$ with $f(t) = t|t|^{p-2} + \lambda t$. Then, by Lemma 2.5 we have

$$0 < \int_\Omega (\frac{|u|^p}{p} + \lambda\frac{u^2}{2})\, dx - \frac{n-k-2}{2(n-k)} \int_\Omega |Du|^2 dx.$$

Since $\int_\Omega |Du|^2 dx = \int_\Omega |u|^p dx + \lambda \int_\Omega u^2 dx$, we obtain that

$$\Big[\frac{n-k-2}{2(n-k)} - \frac{1}{p}\Big] \int_\Omega |u|^p dx < \lambda\Big[\frac{1}{2} - \frac{n-k-2}{2(n-k)}\Big] \int_\Omega u^2 dx.$$

Set $\lambda_1 = \min\{\int_\Omega |Du|^2 dx : u \in H_0^{1,2}(\Omega),\ \int_\Omega u^2 dx = 1\}$ ($\lambda_1$ is the first eigenvalue of $-\Delta$ in $H_0^{1,2}(\Omega)$). Then we have

$$\begin{aligned}\lambda_1 \int_\Omega u^2 dx \leq \int_\Omega |Du|^2 dx &= \int_\Omega |u|^p dx + \lambda \int_\Omega u^2 dx \\ &< \lambda \int_\Omega u^2 dx \Big[\frac{(p-2)(n-k)}{n-k-2}\Big]\Big[p - \frac{2(n-k)}{n-k-2}\Big]^{-1}.\end{aligned}$$

Since $\int_\Omega u^2 dx > 0$, it follows that

$$\lambda > \lambda_1 \cdot \frac{n-k-2}{n-k} \cdot \frac{p - \frac{2(n-k)}{n-k-2}}{p-2}.$$

So, the assertion is obtained for

$$\lambda^* = \lambda_1 \cdot \frac{n-k-2}{n-k} \cdot \frac{p - \frac{2(n-k)}{n-k-2}}{p-2} > 0.$$

Let us note that $\lambda^* < \lambda_1$, according with well known bifurcation results (see Remark 3.5).

**Remark 3.5.** Well known bifurcation results (see [3], [15], [28], [29]) guarantee that every eigenvalue $\lambda_i$ of the operator $-\Delta$ in $H_0^{1,2}(\Omega)$ $(0 < \lambda_1 < \lambda_2 \leq \ldots)$ is a bifurcation point for the problem

$$\begin{cases} \Delta u + u|u|^{p-2} + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover the bifurcation branches are unbounded in $\mathbb{R} \times L^\infty(\Omega)$. Proposition 3.4 states that, if $\Omega$ satisfies the assumptions of Theorem 3.1 and $p > \frac{2(n-k)}{n-k-2}$, then the projection of every bifurcation branch on the $\lambda$-axis is contained in $]\lambda^*, +\infty[$ for some $\lambda^* > 0$.

Notice that $\lambda^* < \lambda_1$ (see the proof of Proposition 3.4) according to the fact that $\lambda_1$ is a bifurcation point. If $p = \frac{2(n-k)}{n-k-2}$, then the projection of every bifurcation branch is contained in $]0, +\infty[$ because the function $f(t) = t|t|^{p-2} + \lambda t$ satisfies the assumptions of Theorem 3.1 for $\lambda \leq 0$. Finally, if $2 < p < \frac{2(n-k)}{n-k-2}$ and $\Omega$ has radial symmetry with respect to $x_1 \ldots x_{k+1}$, then it is well known that the projections of the bifurcation branches cover the $\lambda$-axis; moreover for every $\lambda \in \mathbb{R}$ there exist infinitely many radial solutions of Problem $(*)$ with $f(t) = t|t|^{p-2} + \lambda t$.

**Remark 3.6.** Let us note that the class of the domains $\Omega$, for which the nonexistence results of [26] hold, is different from that of the domains which satisfy the assumptions of Theorem 3.1. However, it should be noted that the domains $\Omega = T_k(\rho, \sigma)$ (see Applications 3.2) belong to both the classes. For domains of this type Theorem 3.1 gives stronger results than those stated in [26]. Indeed, in the case $f(t) = t|t|^{p-2}$ and $\Omega = T_k(\rho, \sigma)$ with $0 < \sigma < \rho$, the results of [26] only imply that:

a) for $p > \frac{2n}{n-k-2}$, Problem $(*)$ has no solution (for every $\rho$ and $\sigma$);

b) for $\frac{2(n-k)}{n-k-2} < p \leq \frac{2n}{n-k-2}$, Problem $(*)$ has no solution if the ratio $\frac{\sigma}{\rho}$ is small enough.

For $p = \frac{2(n-k)}{n-k-2}$, nothing can be deduced by the results of [26]. On the contrary, Theorem 3.1 allows us to say that Problem $(*)$ with $\Omega = T_k(\rho, \sigma)$ and $f(t) = t|t|^{p-2}$ has no solutions for $p \geq \frac{2(n-k)}{n-k-2}$ (for every $\rho$ and $\sigma$ with $0 < \sigma < \rho$), while it has infinitely many radial solutions for $2 < p < \frac{2(n-k)}{n-k-2}$ (see Applications 3.2). Moreover, let us note that in this paper it is not assumed that $F(t) = \int_0^t f(\tau)d\tau \geq 0$, $\forall t \in \mathbb{R}$ (as we do in [26]); so it is possible to consider more general functions $f$, for example $f(t) = t|t|^{p-2} + \lambda t$ (also with $\lambda < 0$), which occur in bifurcation problems (see Proposition 3.4 and Remark 3.5).

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