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ON A GENERAL CLASS OF BIRKHOFF-REGULAR EIGENVALUE PROBLEMS

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1. Introduction. In this article we consider boundary eigenvalue problems of the form

$$\ell(y) = y^{(n)} + \sum_{\nu=1}^{n} p_\nu(x) y^{(n-\nu)} = \lambda y,\ x \in [0,1], \tag{1.1}$$

$$U_\nu(y) = U_{\nu 0}(y) + U_{\nu 1}(y) = 0,\ 1 \le \nu \le n. \tag{1.2}$$

We assume that $p_\nu \in L[0,1],\ 1 \le \nu \le n$, and that the boundary conditions (1.2) are normalized; this means that

$$U_{\nu 0}(y) = \alpha_\nu y^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu - 1} \alpha_{\nu j} y^{(j)}(0), \quad U_{\nu 1}(y) = \beta_\nu y^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu - 1} \beta_{\nu j} y^{(j)}(1), \tag{1.3}$$

where $\alpha_{\nu j}, \beta_{\nu j} \in \mathbb{C}$,

$$|\alpha_\nu| + |\beta_\nu| > 0 \ \text{ for } \ 1 \le \nu \le n,$$

$n - 1 \ge k_1 \ge k_2 \ge \cdots \ge k_n \ge 0$ with $k_{\nu+2} < k_\nu$ for $1 \le \nu \le n-2$, and where $k_0 := \sum_{\nu=1}^{n} k_\nu$ is minimal with respect to all equivalent boundary conditions. k_ν is called the *order* of U_ν.

The study of nonself-adjoint eigenvalue problems generated by n^{th}-order differential expressions (1.1) with smooth coefficients p_ν and by two-point boundary conditions (1.2) was originated by Birkhoff. In [2] Birkhoff proved asymptotic estimates for a fundamental system of solutions of (1.1), then, in [3], he introduced the class of (Birkhoff-)regular boundary conditions and he obtained sufficient conditions for the pointwise convergence of the expansion of a function f into a series of eigen- and associated functions (e.a.f.'s) of (1.1), (1.2). The corresponding series are called Birkhoff-series.

Some years later Tamarkin ([16]) and Stone ([15]) found under more general hypotheses that the expansion of a function $f \in L[0,1]$ into a series of e.a.f.'s of regular

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eigenvalue problems (1.1), (1.2) with $p_1 \equiv 0$ is for each interval $[a, b] \subset (0, 1)$ uniformly equiconvergent with the trigonometric Fourier expansion of f. Afterwards, continuing up to the present time, various questions concerning the spectral theory of differential operators generated by (1.1), (1.2) were studied very intensively, since there are many applications for regular problems (1.1), (1.2). In addition it has been shown by Salaff ([14]) and Minkin ([9]) that the boundary conditions of an arbitrary nonsingular self-adjoint eigenvalue problem (1.1), (1.2) are regular in the sense of Birkhoff. For further information and references on regular eigenvalue problems we refer the reader to the papers of Benzinger ([1]), Wermuth ([17]) and Kaufmann ([7]) (see also [5], [6]), who obtained interesting results more recently on the norm convergence, summability and pointwise convergence of Birkhoff-series which supplement and complete the classical investigations of Birkhoff, Tamarkin and Stone.

It is worthwhile to note that in all the papers mentioned above it is assumed that $p_1 \equiv 0$ or that p_1 is sufficiently smooth. The expansion problem for (1.1), (1.2) becomes more complicated if p_1 is not smooth, since the estimates for a fundamental system of solutions of (1.1) depend essentially on p_1—cf. [11], [13] where the equiconvergence theorem is generalized. Since the coefficients p_j are only integrable, the adjoint of $\ell(y)$ is only defined as *quasi-differential expression* (cf. [10, Section 15.2] and formula (2.4)).

Our main goal lies in the completion of the study of regular eigenvalue problems by investigating the norm convergence, pointwise convergence and summability of Birkhoff-series in the case $p_1 \not\equiv 0$. For this purpose we start from the asymptotic estimates for a fundamental system of solutions of (1.1) proved by Rykhlov ([11], [12]) and then we apply the techniques developed in [1], [13], [17], [7], [5] and [6]. We note that the methods used here can also be applied for the investigation of more general eigenvalue problems (cf. Remark 4.14). For convenience we confine ourselves in this article to the most important special case.

2. Preliminaries. In the following we confine ourselves to the case of even-order problems, $n = 2\mu$; in the case $n = 2\mu - 1$ we can obtain an analogous result by the same method.

We substitute $\lambda = \rho^n$ and consider the sectors

$$S_k = \{\rho \in \mathbb{C} : \frac{k\pi}{n} \le \arg \rho \le \frac{(k+1)\pi}{n}\},\ 0 \le k \le 2n-1.$$

Let $S \in \{S_0, \dots, S_{2n-1}\}$ be one of these sectors; then we can enumerate the n^{th} roots $\omega_1, \dots, \omega_n$ of 1 such that

$$\mathrm{Re}(\rho\omega_1) \le \cdots \le \mathrm{Re}(\rho\omega_\mu) \le 0 \le \mathrm{Re}(\rho\omega_{\mu+1}) \le \cdots \le \mathrm{Re}(\rho\omega_n) \text{ for } \rho \in S.$$

The following theorem was proved in [11], [12].

Theorem 2.1. *Let* $p_1 \in L_r[0, 1]$ *for some* $r \in [1, \infty]$. *In any sector* S, *equation* (1.1) *has* n *linearly independent solutions* $y_1(\cdot, \rho), \dots, y_n(\cdot, \rho)$ *which are holomorphic for*

$\rho \in S$ with $|\rho| > R_0$, where R_0 is a sufficiently large number, and which satisfy the asymptotic relations

$$y_j^{(m)}(x,\rho) := (\frac{\partial}{\partial x})^m y_j(x,\rho) = (\rho\omega_j)^m v(x) e^{\rho\omega_j x}\{1 + O(\eta(\rho))\} \tag{2.1}$$

for $0 \le m \le n-1,\ 1 \le j \le n$, where

$$v(x) = e^{-\frac{1}{n}\int_0^x p_1(\tau)d\tau}, \quad \eta(\rho) = \max_{x\in[0,1]} \eta(x,\rho),$$

$$\eta(x,\rho) = \sum_{1\le k<s\le n} \{|f_{ks}(x,\rho)| + |g_{ks}(x,\rho)| + \|f_{ks}(\cdot,\rho)\|_q + \|g_{ks}(\cdot,\rho)\|_q\} + \frac{1}{|\rho|}, \tag{2.2}$$

$$\|\cdot\|_q = \|\cdot\|_{L_q[0,1]}, \frac{1}{q} + \frac{1}{r} = 1,$$

$$f_{ks}(x,\rho) = \int_0^x e^{\rho(\omega_k-\omega_s)(x-t)} p_1(t)\,dt, \quad g_{ks}(x,\rho) = \int_x^1 e^{\rho(\omega_s-\omega_k)(x-t)} p_1(t)\,dt.$$

We will need the following classes of functions, with $\alpha \ge 0$ and $1 \le r \le \infty$:

$$H_r^\alpha[0,1] = \left\{ f \in L_r[0,1] | \omega(f,\delta)_r = \left\{ \begin{array}{ll} O(\ln^{-\alpha}\frac{1}{\delta}) & \text{for } \alpha > 0, \\ o(1) & \text{for } \alpha = 0, \end{array} \right\} \delta \to 0+ \right\},$$

where

$$\omega(f;\delta)_\infty = \sup_{0<h\le\delta} \sup_{t\in[0,1-h]} |f(t+h) - f(t)|;$$

$$\omega(f;\delta)_r = \sup_{0<h\le\delta} \Big(\int_0^{1-h} |f(t+h) - f(t)|^r\,dt \Big)^{1/r}, \quad 1 \le r < \infty.$$

In order to simplify the formulation of the results we set in this paper $L_\infty[0,1] := C[0,1]$ and $H_r^0[0,1] := L_r[0,1]$. For $\xi \in [0,1]$ and $\rho \in S$ let $w(\xi,\rho)$ be the Wronskian of $y_1(\xi,\rho), \dots, y_n(\xi,\rho)$ and let $w_{nk}(\xi,\rho), 1 \le k \le n$, be the algebraic complement of $y_k^{(n-1)}(\xi,\rho)$ with respect to $w(\xi,\rho)$. Further, we set

$$z_k(\xi,\rho) = \frac{w_{nk}(\xi,\rho)}{w(\xi,\rho)}, \quad \xi \in [0,1],\ \rho \in S,\ 1 \le k \le n. \tag{2.3}$$

In the sequel we use the abbreviation

$$[a] := a + O(\eta(x,\rho)) \quad (a \in \mathbb{C}, \rho \in S),$$

and the following *quasi derivatives*, which are closely related to the differential expression (1.1):

$$z^{\{0\}} := z,\ z^{\{k\}} := -\frac{d}{d\xi} z^{\{k-1\}} + p_k z^{\{0\}},\ k = 1, \dots, n. \tag{2.4}$$

Using formula (2.1), Lemma 5.1 and Lemma 5.4 we easily obtain the following estimates:

$$z_k^{\{m\}}(\xi,\rho) = (\rho\omega_k)^m \frac{[\omega_k]}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi} \tag{2.5}$$

for $\xi \in [0,1],\ 1 \le k \le n,\ 0 \le m \le n-1,\ \rho \in S$.

It is well known ([10, page 15], [11]) that ρ^n with $\rho \in S$ is an eigenvalue of (1.1) and (1.2) if and only if ρ is a zero of the characteristic determinant

$$D(\rho) := \begin{bmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \dots & \dots & \dots \\ U_n(y_1) & \dots & U_n(y_n) \end{bmatrix}, \quad \Delta(\rho) := \det D(\rho). \tag{2.6}$$

For the formulation of the asymptotic estimates for $\Delta(\rho)$ we introduce the following notation:

$$\theta_{ij} := \begin{vmatrix} \alpha_1\omega_1^{k_1} & \cdots & \alpha_1\omega_{\mu-1}^{k_1} & \gamma_1^i\omega_\mu^{k_1} & \gamma_1^j\omega_{\mu+1}^{k_1} & \cdots & \beta_1\omega_n^{k_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_n\omega_1^{k_n} & \cdots & \alpha_n\omega_{\mu-1}^{k_n} & \gamma_n^i\omega_\mu^{k_n} & \gamma_n^j\omega_{\mu+1}^{k_n} & \cdots & \beta_n\omega_n^{k_n} \end{vmatrix}$$

with

$$\gamma_\nu^\ell := \begin{cases} \alpha_\nu,\ \ell = 0 \\ \beta_\nu,\ \ell = 1 \end{cases} \quad (1 \le \nu \le n).$$

If we substitute the estimates (2.1) into the boundary forms (1.2) we obtain

$$\begin{aligned} U_\nu(y_j) &= (\rho\omega_j)^{k_\nu}[\alpha_\nu] \quad \text{for} \quad j \le \mu-1, \\ U_\nu(y_\mu) &= (\rho\omega_\mu)^{k_\nu}\{[\alpha_\nu] + e^{\rho\omega_\mu}v(1)[\beta_\nu]\}, \\ U_\nu(y_{\mu+1}) &= (\rho\omega_{\mu+1})^{k_\nu}\{[\alpha_\nu] + e^{\rho\omega_{\mu+1}}v(1)[\beta_\nu]\} = (\rho\omega_{\mu+1})^{k_\nu}\{[\alpha_\nu] + e^{-\rho\omega_\mu}v(1)[\beta_\nu]\}, \\ U_\nu(y_j) &= (\rho\omega_j)^{k_\nu}v(1)e^{\rho\omega_j}[\beta_\nu] \quad \text{for} \quad j \ge \mu+2. \end{aligned} \tag{2.7}$$

Now we insert these estimates into (2.6), and then we transform $\Delta(\rho)$ as follows:

($\mathbf{T_1}$) for $1 \le \nu \le n$ we divide the ν^{th} row of $\Delta(\rho)$ by ρ^{k_ν},

($\mathbf{T_2}$) for $\mu+1 \le j \le n$ we divide the j^{th} column of $\Delta(\rho)$ by $e^{\rho\omega_j}v(1)$.

These transformations yield

$$\Delta(\rho) = \rho^{k_0} e^{\sum\limits_{j=\mu+1}^{n}\omega_j}\{v(1)\}^\mu\{[\theta_{01}] + (\frac{1}{v(1)}[\theta_{00}] + v(1)[\theta_{11}])e^{\rho\omega_\mu} + [\theta_{10}]e^{2\rho\omega_\mu}\},$$

$k_0 = \sum_{\nu=1}^n k_\nu$. In this article we assume that (1.1), (1.2) is (*Birkhoff-*)*regular;* this means that $\theta_{01} \neq 0 \neq \theta_{10}$. We note that $|\theta_{01}| = |\theta_{10}|$ (which is proved as with [10, Section 4.8]). Hence,

$$\Delta(\rho) = \rho^{k_0} e^{\rho\sum\limits_{j=\mu+1}^{n}\omega_j}\{v(1)\}^\mu\theta_{01}(1+T(\rho))[1], \quad \rho \in S, \tag{2.8}$$

where $T(\rho)$ has the form

$$T(\rho) = \sum_{i=1}^{k} [c_i] e^{\rho d_i} \tag{2.9}$$

with $k \in \mathbb{N}$, $c_i \in \mathbb{C}$, $d_i \in \mathbb{C} \setminus \{0\}$ for $1 \le i \le k$ and with $\operatorname{Re}(\rho d_i) \le 0$ for $\rho \in S$ and $1 \le i \le k$.

On account of the special form of the exponential sum $T(\rho)$, an application of [16, Lemma, page 26] yields:

Remark 2.1. Let $\sigma = \{\rho \in \mathbb{C} : \rho^n \text{ is eigenvalue of (1.1), (1.2)}\}$ and let $\delta > 0$. Then there exist constants $c(\delta) > 0$ and $R_\delta > 0$ with

$$|1 + T(\rho)| \ge c(\delta) \quad \text{for} \quad \rho \in S(\delta) := \{\rho \in S : \operatorname{dist}(\rho, \sigma) \ge \delta\} \tag{2.10}$$

with $|\rho| > R_\delta$. If $T(\rho)$ has the form (2.9) and if $a \in \mathbb{C}$, we write in the sequel

$$[[a]] := (a + T(\rho))[1].$$

Remark 2.2. (i) On account of (2.8) we have

$$\Delta(\rho) = \rho^{k_0} e^{\rho \sum\limits_{j=\mu+1}^{n} \omega_j} \{v(1)\}^{\mu} \theta_{01} [[1]] \quad \text{for} \quad \rho \in S. \tag{2.11}$$

(ii) If $T(\rho)$ is a function of the form (2.9) which satisfies (2.10), then we have

$$\frac{1}{1 + T(\rho)} - 1 = \frac{-T(\rho)}{1 + T(\rho)} = [[0]] \quad \text{for} \quad \rho \in S(\delta),\ |\rho| > R_\delta.$$

Hence,

$$\frac{1}{\Delta(\rho)} = \rho^{-k_0} e^{-\rho \sum\limits_{j=\mu+1}^{n} \omega_j} \{v(1)\}^{-\mu} \frac{[[1]]}{\theta_{01}} \quad \text{for} \quad \rho \in S(\delta),\ |\rho| > R_\delta. \tag{2.12}$$

3. Green's function. In this section we assume that $\rho \in S(\delta), \delta > 0$, and $f \in L[0,1]$. Using the solutions (2.1) of equation (1.1) we obtain by variation of constants (see [10, Section 3.7]). The general solution of the inhomogeneous equation

$$\ell(y) = \lambda y + f(x) \tag{3.1}$$

has the form

$$y(x, \rho) = \sum_{j=1}^{n} c_j y_j(x, \rho) + \int_0^1 g(x, \xi, \rho) f(\xi)\, d\xi, \tag{3.2}$$

where

$$g(x, \xi, \rho) = \begin{cases} \sum_{k=1}^{\mu} y_k(x, \rho) z_k(\xi, \rho) & \text{if} \quad x \ge \xi, \\ -\sum_{k=\mu+1}^{n} y_k(x, \rho) z_k(\xi, \rho) & \text{if} \quad x < \xi. \end{cases} \tag{3.3}$$

Now we choose the functions $c_j(= c_j(\rho))$ such that $y(\cdot,\rho)$ satisfies the boundary conditions (1.2). First we substitute (3.2) into (1.2), then we solve the equations for the functions c_j by Cramer's rule, and finally we insert these solutions into (3.2). This procedure yields (see [10, page 37])

$$y(x,\rho) = \int_0^1 G(x,\xi,\rho) f(\xi)\, d\xi, \tag{3.4}$$

where the Green's function G is defined by

$$G(x,\xi,\rho) = \frac{1}{\Delta(\rho)} H(x,\xi,\rho), \quad x,\xi \in [0,1],\ \rho \in S(\delta) \tag{3.5}$$

with

$$H(x,\xi,\rho) = \begin{vmatrix} y_1(x,\rho) & \dots & y_n(x,\rho) & g(x,\xi,\rho) \\ & & & U_1(g)(\xi,\rho) \\ & D(\rho) & & \vdots \\ & & & U_n(g)(\xi,\rho) \end{vmatrix}. \tag{3.6}$$

Substituting (2.1) into (3.3) we get

$$g(x,\xi,\rho) = v(x) \begin{cases} \sum_{k=1}^{\mu} e^{\rho\omega_k x}[1] z_k(\xi,\rho) & \text{for} \quad x \geq \xi, \\ -\sum_{k=\mu+1}^{n} e^{\rho\omega_k x}[1] z_k(\xi,\rho) & \text{for} \quad x < \xi. \end{cases} \tag{3.7}$$

Applying the boundary conditions (1.2) we infer

$$U_\nu(g)(\xi,\rho) = -\sum_{k=\mu+1}^{n} (\rho\omega_k)^{k_\nu}[\alpha_\nu] z_k(\xi,\rho) + v(1) \sum_{k=1}^{\mu} (\rho\omega_k)^{k_\nu}[\beta_\nu] z_k(\xi,\rho) \tag{3.8}$$

for $1 \leq \nu \leq n$, $\xi \in [0,1]$ and $\rho \in S$.

For the formulation of the asymptotic estimates of $G(x,\xi,\rho)$ we introduce some further abbreviations.

For $1 \leq j \leq n$ let $\theta_{j*k}(\rho)$, $\mu+1 \leq k \leq n$, and $\theta_{j*k}(\rho)$, $1 \leq k \leq \mu$, be the determinants of the matrices, obtained by replacing the j^{th} column of the matrix

$$\begin{bmatrix} \alpha_1\omega_1^{k_1} & \dots & \alpha_1\omega_\mu^{k_1} & \beta_1\omega_{\mu+1}^{k_1} & \dots & \beta_1\omega_n^{k_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_n\omega_1^{k_n} & \dots & \alpha_n\omega_\mu^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \dots & \beta_n\omega_n^{k_n} \end{bmatrix}$$

by

$$a_k = (\alpha_1\omega_k^{k_1}, \dots, \alpha_n\omega_k^{k_n})^T \quad \text{and} \quad b_k = (\beta_1\omega_k^{k_1}, \dots, \beta_n\omega_k^{k_n})^T,$$

respectively.

Combining the preceding results we get the following estimates for the Green's function G of (1.1), (1.2), which are fundamental for the proof of expansion theorems.

Theorem 3.1. *Let $\delta > 0$; then the Green's function G of a regular boundary eigenvalue problem* (1.1), (1.2) *has for $\rho \in S(\delta)$ and $x, \xi \in [0,1]$ the representation*

$$G(x,\xi,\rho) = g(x,\xi,\rho) + \sum_{j=1}^{n} \varphi_j(x,\rho)\psi_j(x,\rho). \tag{3.9}$$

$g(x,\xi,\rho)$ satisfies (3.3) *and* (3.7),

$$\varphi_j(x,\rho) = \begin{cases} y_j(x,\rho) & \text{for } 1 \le j \le \mu, \\ y_j(x,\rho)e^{-\rho\omega_j}\frac{1}{v(1)} & \text{for } \mu+1 \le j \le n, \end{cases} \tag{3.10}$$

and

$$\psi_j(\xi,\rho) = \frac{1}{\theta_{01}}\Big(\sum_{k=\mu+1}^{n} A_{jk}(\rho)z_k(\xi,\rho) - v(1)\sum_{k=1}^{\mu} A_{jk}(\rho)e^{\rho\omega_k}z_k(\xi,\rho)\Big), \tag{3.11}$$

where

$$A_{jk}(\rho) = [[\theta_{j*k}]]. \tag{3.12}$$

Proof. According to (3.6) $G(x,\xi,\rho) = \frac{1}{\Delta(\rho)}H(x,\xi,\rho)$, where $\Delta(\rho)$ satisfies (2.11) and where all the entries of $H(x,\xi,\rho)$ have been estimated previously. We substitute these estimates into $H(x,\xi,\rho)$ and then we transform the $(\nu+1)^{st}$ row, $1 \le \nu \le n$, and the j^{th} column, $1 \le j \le n$ of $H(x,\xi,\rho)$ in the same way that we have transformed the ν^{th} row and the j^{th} column of $\Delta(\rho)$ (see also §2, ($\mathbf{T_1}$), ($\mathbf{T_2}$)).

Expanding the resulting determinant with respect to the first row we obtain with the preceding abbreviations the assertions (3.9)–(3.11). We omit details since the proof is similar to the proof for the case $p_1 = 0$.

4. Expansion theorems. For the proof of the expansion theorems we use the contour integration method (see also [10, Chapter II]). Let

$$\hat{S} := S_0 \bigcup S_{2n-1} = \{\rho \in \mathbb{C} : -\frac{\pi}{n} \le \arg\rho \le \frac{\pi}{n}\}$$

and let γ_m be a circular arc in $\hat{S}$ of radius $R_m > 0$, centered at the origin. On account of the distribution of the eigenvalues of (1.1), (1.2) (see [13]) we can choose $\delta > 0$ and a strictly increasing sequence $(R_m)_{m\in\mathbb{N}}$ such that the corresponding sequence $(\gamma_m)_{m\in\mathbb{N}}$ of arcs is contained in $\hat{S}(\delta) = S_0(\delta) \cup S_{2n-1}(\delta)$. Then for any summable function f,

$$I_{R_m}(f)(x) := \frac{-1}{2\pi i}\int_{\gamma_m}\int_0^1 G(x,\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \tag{4.1}$$

is a partial sum of the expansion of f in eigenfunctions and (possibly in the case of multiple eigenvalues) associated functions of (1.1), (1.2).

Using the asymptotic estimates of Section 3 we shall derive a precise representation for

$$\begin{aligned} I^j_{R_m}(f)(x) &:= \frac{-1}{2\pi i}\int_{\gamma^j_m}\int_0^1 g(x,\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \\ &+ \frac{-1}{2\pi i}\int_{\gamma^j_m}\int_0^1 (G-g)(x,\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \\ &=: I^{j,0}_{R_m}(f)(x) + I^{j,1}_{R_m}(f)(x),\; j\in\{0,2n-1\},\quad m\in\mathbb{N}. \end{aligned}$$

Here $\gamma^j_m := \gamma_m \cap S_j$. In the following we choose $S = S_0$.

Since $I^0_{R_m}(f)$ and $I^{2n-1}_{R_m}(f)$ have an analogous form, it will be sufficient to give a detailed description of $I^0_{R_m}(f)$.

With the preceding abbreviations and the estimates (3.12) we infer from Theorem 3.1:

Theorem 4.1. *Let $f\in L[0,1]$. For $x\in[0,1]$*

$$\begin{aligned} I^{0,1}_{R_m}(f)(x) &= -\frac{v(x)}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=\mu+1}^{n}\int_{\gamma^0_m}\frac{[[\theta_{j*k}]]}{\theta_{01}}e^{\rho\omega_j x}\int_0^1 z_k(\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \\ &+ \frac{v(x)}{2\pi i}v(1)\sum_{j=1}^{\mu}\sum_{k=1}^{\mu}\int_{\gamma^0_m}\frac{[[\theta_{j*k}]]}{\theta_{01}}e^{\rho\omega_j x}\int_0^1 e^{\rho\omega_k} z_k(\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \\ &- \frac{v(x)}{2\pi i}\frac{1}{v(1)}\sum_{j=\mu+1}^{n}\sum_{k=\mu+1}^{n}\int_{\gamma^0_m}\frac{[[\theta_{j*k}]]}{\theta_{01}}e^{\rho\omega_j (x-1)}\int_0^1 z_k(\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho \\ &+ \frac{v(x)}{2\pi i}\sum_{j=\mu+1}^{n}\sum_{k=1}^{\mu}\int_{\gamma^0_m}\frac{[[\theta_{j*k}]]}{\theta_{01}}e^{\rho\omega_j (x-1)}\int_0^1 e^{\rho\omega_k} z_k(\xi,\rho)f(\xi)\,d\xi\; n\rho^{n-1}\,d\rho. \end{aligned}$$

The analogous estimates are valid for the sector S_{2n-1}.

Let us denote by $J^{0,1}_{R_m}(f)(x)$ the expression obtained from the right-hand sides of the formulae in Theorem 4.1 by deleting the double square brackets and by substituting instead of $z_k(\xi,\rho)$ the function $\frac{\omega_k}{n\rho^{n-1}}\frac{1}{v(\xi)}e^{-\rho\omega_k\xi}$; similarly we define $J^{\alpha,\beta}_{R_m}$ for $\alpha\in\{0,2n-1\}$ and $\beta\in\{0,1\}$. Then we infer from [15, Lemmas IV',… ,VI', page 754] and Lemma 5.9:

Lemma 4.2. *If $p_1\in H^\alpha_q[0,1]$, $f\in H^\beta_r[0,1]$, $\frac{1}{q}+\frac{1}{r}=1, \alpha+\beta>1$, then*

$$\lim_{m\to\infty}(I^{j,1}_{R_m}(f) - J^{j,1}_{R_m}(f))(x) = 0, \quad j\in\{0,2n-1\},$$

uniformly for $x\in[0,1]$.

Combining [15, Lemmas IV' and V'] with Theorem 4.1 we get

Lemma 4.3. *For* $f \in L[0,1]$, $p_1 \in L[0,1]$,

$$\lim_{m\to\infty} I^{j,1}_{R_m}(f)(x) = 0, \qquad j \in \{0, 2n-1\},$$

uniformly on each compact set $K \subset (0,1)$.

Let us denote by $g_0(x,\xi,\rho)$ the expression obtained from $g(x,\xi,\rho)$ (cf. formula (3.7)) by replacing $z_k(\xi,\rho)$ therein by $\frac{\omega_k}{n\rho^{n-1}}\frac{1}{v(\xi)}e^{-\rho\omega_k\xi}$ and by deleting the square bracket. Then we infer from Lemma 5.10 by elementary calculations

Lemma 4.4. *If* $p_1 \in H^\alpha_q[0,1]$, $f \in H^\beta_r[0,1]$, $\frac{1}{q}+\frac{1}{r}=1$, $\alpha+\beta>1$, *then for* $x \in [0,1]$ *and* $j \in \{0, 2n-1\}$

$$I^{j,0}_{R_m}(f) - J^{j,0}_{R_m}(f) = \frac{-1}{2\pi i}\int_{\gamma^j_m}\int_0^1 (g-g_0)(x,\xi,\rho)f(\xi)\,d\xi\, n\rho^{n-1}\,d\rho = o(1), \tag{4.2}$$

where $o(1)$ *tends to zero uniformly for* $x \in [0,1]$ *as* $m \to \infty$.

Remark 4.5. (i) Let $p_1 \in H^\alpha_q[0,1]$, $f \in H^\beta_r[0,1]$, $\frac{1}{q}+\frac{1}{r}=1$, $\alpha+\beta>1$, then for $x \in [0,1]$ we easily obtain (see also [17, Satz 2])

$$\begin{aligned} J^{0,0}_{R_m}(f) + J^{2n-1,0}_{R_m}(f) &= -\frac{1}{2\pi i}\int_{\gamma_m}\int_0^1 g_0(x,\xi,\rho)f(\xi)\,d\xi\, n\rho^{n-1}\,d\rho \\ &= \int_0^1 \frac{v(x)}{v(\xi)} f(\xi)\frac{\sin(R_m(x-\xi))}{\pi(x-\xi)}\,d\xi. \end{aligned} \tag{4.3}$$

From Lemmas 4.3 and 4.4 we deduce that $I_{R_m}(f)$ and the generalized Dirichlet integral in (4.3) (and hence, according to [15], $I_{R_m}(f)$ and the corresponding partial sums of the trigonometric Fourier series of f) are *uniformly equiconvergent* on every compact set $K \subset (0,1)$ (see also [13] for details).

(ii) If $a \in (0,1)$, f and p_1 satisfy the conditions of Lemma 4.4 and if $\frac{f}{v}$ satisfies hypothesis $H(a)$ (cf. Notation 4.8), then we infer from Lemmas 4.3, 4.4 and 5.3 the following result on the *pointwise convergence* of $I_{R_m}(f)$:

$$\lim_{m\to\infty} I_{R_m}(f)(a) = \frac{1}{2}\{f(a+)+f(a-)\}.$$

The object of the next part of this section is to evaluate

$$\lim_{m\to\infty} I_{R_m}(f)(x) \qquad \text{for} \quad x \in \{0,1\}.$$

For this purpose we use Lemma 4.2 and the methods of [5, Section 4].

Using Theorem 4.1 and Lemma 4.2 we get (cf. [5, page 1192]).

Theorem 4.6. *If* $p_1 \in H_q^\alpha[0,1]$, $f \in H_r^\beta[0,1]$, $\frac{1}{q}+\frac{1}{r}=1$, $\alpha+\beta>1$ *and* $m \to \infty$, *then*

$$\begin{aligned} I_{R_m}^{0,1}(f)(0) = o(1) &- \frac{1}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=\mu+1}^{n}\frac{\theta_{j*k}}{\theta_{01}}\int_0^1 \frac{f(\xi)}{-\xi v(\xi)}e^{-\rho\omega_k\xi}\Big|_{\rho=R_m}^{\rho=R_m e^{\frac{i\pi}{n}}}\,d\xi \\ &+ \frac{1}{2\pi i}\sum_{j=1}^{\mu}\sum_{k=1}^{\mu}\frac{v(1)\theta_{j*k}}{\theta_{01}}\int_0^1 \frac{f(\xi)}{(1-\xi)v(\xi)}e^{\rho\omega_k(1-\xi)}\Big|_{\rho=R_m}^{\rho=R_m e^{\frac{i\pi}{n}}}\,d\xi \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} I_{R_m}^{0,1}(f)(1) = o(1) &- \frac{1}{2\pi i}\sum_{j=\mu+1}^{n}\sum_{k=\mu+1}^{n}\frac{\theta_{j*k}}{\theta_{01}}\int_0^1 \frac{f(\xi)}{-\xi v(\xi)}e^{-\rho\omega_k\xi}\Big|_{\rho=R_m}^{\rho=R_m e^{\frac{i\pi}{n}}}\,d\xi \\ &+ \frac{1}{2\pi i}\sum_{j=\mu+1}^{n}\sum_{k=1}^{\mu}\frac{v(1)\theta_{j*k}}{\theta_{01}}\int_0^1 \frac{f(\xi)}{(1-\xi)v(\xi)}e^{\rho\omega_k(1-\xi)}\Big|_{\rho=R_m}^{\rho=R_m e^{\frac{i\pi}{n}}}\,d\xi. \end{aligned} \tag{4.5}$$

The analogous estimates are valid for $I_{R_m}^{2n-1,1}(f)$.

The following result of Wermuth is the key for the evaluation of the pointwise limits $\lim\limits_{m\to\infty} I_{R_m}(f)(a)$.

Theorem 4.7 (Wermuth ([18])). *Let* $f \in L[0,1]$, $\alpha \in \mathbb{C}\setminus\{0\}$, $\operatorname{Im}\alpha \ge 0$, $A \in \mathbb{C}$ *and* $x_0 \in (0,1)$. *If*

(*i*) $$\lim_{x\to 0+}\frac{1}{x}\int_0^x f(\xi)\,d\xi = A$$

and

(*ii*) $$\lim_{x\to 0+}\int_x^{x_0}\frac{|f(\xi)-f(\xi+x)|}{\xi}\,d\xi = 0,$$

then

$$\lim_{R\to\infty}\int_0^1 f(x)\frac{e^{iRx}-e^{\alpha iRx}}{x}\,dx = A\ln\alpha,$$

where $0 \le \operatorname{Im}\,\ln\alpha \le \pi$.

This theorem generalizes the well-known convergence criterion of Lebesgue for trigonometric Fourier series (cf. [19,b), I, page 65]) which is known to include the criteria of Dini, Dirichlet-Jordan, de la Vallée Poussin and W.H. Young (cf. [19, a), page 29 ff.]).

Notation 4.8. Let $f \in L[0,1]$ and $0 \le a \le 1$. We say that f satisfies the hypothesis $H(a)$ if

$$f_1 : [0,1] \to \mathbf{R}, \quad x \mapsto \begin{cases} f(x+a) & \text{for } 0 \le x \le 1-a \\ 0 & \text{for } 1-a < x \le 1 \end{cases}$$

f satisfies hypotheses (i) and (ii) of Theorem 4.7 with

$$A = \lim_{x\to 0+} \frac{1}{x}\int_0^x f(a+\xi)\,d\xi \quad (=: f(a+) \text{ if } a<1 \text{ and } f(a+) \text{ is not yet defined})$$

and if

$$f_2 : [0,1] \to \mathbf{R}, \quad x \mapsto \begin{cases} f(a-x) & \text{for } 0 \le x \le a \\ 0 & \text{for } 1-a < x \le 1 \end{cases}$$

satisfies hypotheses (i) and (ii) of Theorem 4.7 with

$$A = \lim_{x\to 0+} \frac{1}{x}\int_0^x f(a-\xi)\,d\xi \quad (=: f(a-)).$$

The following theorem reveals the behaviour of the expansion of f at the boundary of $[0,1]$.

Theorem 4.9. *If $p_1 \in H_q^\alpha[0,1]$, $f \in H_r^\beta[0,1]$, $\frac{1}{q}+\frac{1}{r}=1$, $\alpha+\beta>1$, and $\frac{f}{v}$ satisfies hypotheses $H(0)$ and $H(1)$, then*

$$\lim_{m\to\infty} I_{R_m}(f)(0) = f(0+) + \delta_{k_n,0}\frac{1}{2}\Big(\frac{B_n^0}{\theta_{01}} + \frac{\tilde{B}_n^0}{\tilde{\theta}_{01}}\Big)U_n(f) + \delta_{k_{n-1},0}\frac{1}{2}\Big(\frac{B_{n-1}^0}{\theta_{01}} + \frac{\tilde{B}_{n-1}^0}{\tilde{\theta}_{01}}\Big)U_{n-1}(f)$$

and

$$\lim_{m\to\infty} I_{R_m}(f)(1) = f(1-) + \delta_{k_n,0}\frac{1}{2}\Big(\frac{B_n^1}{\theta_{01}} + \frac{\tilde{B}_n^1}{\tilde{\theta}_{01}}\Big)U_n(f) + \delta_{k_{n-1},0}\frac{1}{2}\Big(\frac{B_{n-1}^1}{\theta_{01}} + \frac{\tilde{B}_{n-1}^1}{\tilde{\theta}_{01}}\Big)U_{n-1}(f),$$

where δ_{ij} is the Kronecker symbol and

$$B_j^\kappa := \begin{vmatrix} 1-\delta_{\kappa 1} & \dots & 1-\delta_{\kappa 1} & \delta_{\kappa 1} & \dots & \delta_{\kappa 1} & 0 \\ \alpha_1\omega_1^{k_1} & \dots & \alpha_1\omega_\mu^{k_1} & \beta_1\omega_{\mu+1}^{k_1} & \dots & \beta_1\omega_n^{k_1} & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1}\omega_1^{k_{n-1}} & \dots & \alpha_{n-1}\omega_\mu^{k_{n-1}} & \beta_{n-1}\omega_{\mu+1}^{k_{n-1}} & \dots & \beta_{n-1}\omega_n^{k_{n-1}} & \delta_{jn-1} \\ \alpha_n\omega_1^{k_n} & \dots & \alpha_n\omega_\mu^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \dots & \beta_n\omega_n^{k_{n-1}} & \delta_{jn} \end{vmatrix}$$

for $j \in \{n-1,n\}$ and $\kappa \in \{0,1\}$. $\tilde{B}_j^\kappa$ and $\tilde{\theta}_{01}$ are the determinants corresponding to B_j^κ and θ_{01}, respectively, for the sector S_{2n-1}.

Proof. From Lemma 4.2, Lemma 4.4 and the definition of $J_{R_m}^{k,j}(f)$ (see the sentence following Theorem 4.1) we infer that under the hypothesis of Lemma 4.2

$$\lim_{m\to\infty}\{I_{R_m}(f) - (J_{R_m}^{2n-1,0}(f) + J_{R_m}^{2n-1,1}(f) + J_{R_m}^{0,0}(f) + J_{R_m}^{0,1}(f))\}(x) = 0 \tag{4.6}$$

uniformly for $x \in [0,1]$.

Starting from the detailed representation of the terms $J^{k,j}_{R_m}(f)(x)$ proved in (4.3) and Theorem 4.6 we can apply Theorem 4.6, Lemma 5.2 and Lemma 5.3 to evaluate $\lim\limits_{m\to\infty} J^{k,j}_{R_m}(f)(x)$ for $x \in \{0,1\}$, $k \in \{2n-1, 0\}$ and $j \in \{0,1\}$; we omit details since the remaining part of the proof is a word-by-word repetition of the proof of [5, Theorem 4.10].

Notation 4.10. (i) We say that f and p_1 satisfy *Hypothesis A* if

$$p_1 \in H^{\alpha}_q[0,1],\ f \in H^{\beta}_r[0,1],\ \frac{1}{q} + \frac{1}{r} = 1 \quad \text{and} \quad \alpha + \beta > 1,$$

where $q, r \in [1,\infty]$ (and $L_\infty[0,1] := C[0,1]$, $H^0_r[0,1] := L_r[0,1]$).

(ii) Let

$$\ln^+ a = \begin{cases} \ln a & \text{for } a > 1 \\ 0 & \text{for } 0 \le a \le 1. \end{cases}$$

We say that f satisfies *Hypothesis B* if either $1 < r < \infty$ and $f \in L_r[0,1]$ or $r = 1, f \in L[0,1]$ and $f \ln^+ |f| \in L[0,1]$.

Combining the estimates obtained here with the results obtained in [6, Sections 3, 4 and 5] we can state the following theorems on norm convergence, uniform convergence and summability of Birkhoff series. The assertions of the two following theorems are an immediate consequence of formula (4.6) and of the proofs of [6, Theorem 3.2 and Theorem 4.2].

Theorem 4.11. *If f and p_1 satisfy Hypothesis A and if f satisfies Hypothesis B, then*

$$\lim_{m\to\infty} \|f - I_{R_m}(f)\|_r = 0.$$

Theorem 4.12. *If $f \in C[0,1] \cap BV[0,1]$ and p_1 satisfy Hypothesis A and f satisfies the boundary conditions of order zero of a regular problem* (1.1), (1.2), *then for $m \to \infty$, $I_{R_m}(f)$ tends to f uniformly on the interval* $[0,1]$.

According to Stone ([15]) the Riesz typical means of order $\ell > 0$ of the partial sums of the expansion of f into a series of e.a.f.'s of (1.1), (1.2) take on the form

$$S_{R_m}(f)(x) = -\int_0^1 \Big(\frac{1}{2\pi i}\int_{\gamma_m} \Big(1 - (\frac{\rho}{R_m})^{4n}\Big)^{\ell} G(x,\xi,\rho) n\rho^{n-1}\, d\rho\Big) f(\xi)\, d\xi. \tag{4.7}$$

The following theorem generalizes the theorem of M. Riesz and Chapman in the theory of Fourier series.

Theorem 4.13. *Let $f \in C[0,1]$ and p_1 satisfy Hypothesis A and let f satisfy the boundary conditions of order zero of a regular problem* (1.1), (1.2). *Then the expansion* (4.7) *formed for $\ell > 0$ tends to f uniformly on the interval* $[0,1]$.

Since $S_{R_m}(f)(x)$ and $I_{R_m}(f)(x)$ do only differ by the bounded factor $(1-(\frac{\rho}{R_m})^{4n})^{\ell}$, we can use Lemma 5.9 and Lemma 5.10 to show all *error terms* are tending uniformly to zero—this means that the limit of (4.7) does not change if we remove in our estimates for $G(x,\xi,\rho)$ all square and double square brackets. Consequently the assertion of Theorem 4.13 is obtained as with the proof of [6, Lemma 5.1].

Remark 4.14. (i) The methods of this paper can be used for the investigation of more general types of problems. For example:

(a) we can replace the boundary conditions (1.2) by regular two-point or multi-point boundary conditions where the coefficients are polynomials in λ,
(b) we can replace the differential equation (1.1) by a differential equation with a weight function r which is a nonzero step function (cf. [5], [6]),
(c) it is possible to generalize the methods of this paper for the investigation of eigenvalue problems for bundles or for first-order systems of equations; the corresponding results will be published elsewhere.

(ii) Using the results of [13] we get an estimate on the order of convergence of (4.6).

5. Auxiliary results. Let Ω be the Vandermonde determinant of $\omega_1, \omega_2, \dots, \omega_n$ and let Ω_{sj}, $1 \le j \le n$, $1 \le s \le n$, be the algebraic complement of ω_j^{s-1} with respect to Ω.

Lemma 5.1 ([10, page 86]; [5, Lemma 5.1])**.**

$$\frac{\Omega_{sj}}{\Omega} = \frac{1}{n}\omega_j^{n-s+1}, \ \ 1 \le j \le n, \ 1 \le s \le n. \tag{5.1}$$

Proof. Let $1 \le s \le n$ be fixed. The system of equations

$$\sum_{j=1}^{n} \omega_j^k \frac{\Omega_{sj}}{\Omega} \ = \ \delta_{k,s-1}, \quad 0 \le k \le n-1,$$

has a unique solution since the determinant of its coefficients is $\Omega \neq 0$. In consequence of

$$\sum_{j=1}^{n} \omega_j^k \frac{\omega_j^{n-s+1}}{n} \ = \ \delta_{k,s-1}, \quad 0 \le k \le n-1, \ 1 \le s \le n$$

this solution is defined by (5.1).

For the evaluation of the limits of the integrals in Theorem 4.6 we use Wermuth's criterion (cf. also [5, Lemma 5.3]) and the notation introduced in Section 2.

Lemma 5.2. *Let $f \in L[0,1]$ and $\alpha = e^{i\pi/n}$.*

(i) *If f satisfies hypothesis $H(1)$ (compare Notation 4.8), then for $1 \le k \le \mu$*

$$\lim_{R\to\infty} \int_0^1 \frac{e^{R\alpha\omega_k(1-\xi)} - e^{R\omega_k(1-\xi)}}{1-\xi} f(\xi)\, d\xi = -\frac{i\pi}{n} f(1-).$$

(ii) *If f satisfies hypothesis $H(0)$, then for $\mu+1 \le k \le n$*

$$\lim_{R\to\infty} \int_0^1 \frac{e^{R\alpha\omega_k(-\xi)} - e^{R\omega_k(-\xi)}}{-\xi} f(\xi)\, d\xi = \frac{i\pi}{n} f(0+).$$

Proof. Substituting $1-\xi=t$ and applying Theorem 4.7 we get (i) from

$$\begin{aligned}&\lim_{R\to\infty}\int_0^1 \frac{e^{R\alpha\omega_k(1-\xi)}-e^{R\omega_k(1-\xi)}}{1-\xi}f(\xi)\,d\xi\\ =&\lim_{R\to\infty}\int_0^1 \frac{1}{t}(e^{iR(\frac{\alpha\omega_k}{i})t}-e^{iRt}+e^{iRt}-e^{iR(\frac{\omega_k}{i})t})f(1-t)\,dt\\ =&f(1-)(-\ln(\alpha\frac{\omega_k}{i})+\ln\frac{\omega_k}{i})=-f(1-)\ln\alpha=\frac{i\pi}{n}f(1-).\end{aligned}$$

We note that $\operatorname{Re}\omega_k\le 0$ and $\operatorname{Re}(\alpha\omega_k)\le 0$ for $1\le k\le\mu$; hence $\operatorname{Im}\frac{\omega_k}{i}\ge 0$ and $\operatorname{Im}(\alpha\frac{\omega_k}{i})\ge 0$ for $1\le k\le\mu$, and Theorem 4.7 can be applied.

The following lemma is used to determine $\lim_{m\to\infty} I^{j,0}_{R_m}(f)(x)$ (cf. [5, Lemma 5.4]).

Lemma 5.3. *Let $j\in\{0,2n-1\}$ and $f\in L[0,1]$.*

(i) *If $\frac{f}{v}$ satisfies hypothesis $H(1)$, then*

$$\lim_{m\to\infty} J^{j,0}_{R_m}(f)(1)=\lim_{m\to\infty}-\frac{1}{2\pi i}\int_0^1\int_{\gamma^j_m} g_0(1,\xi,\rho)f(\xi)n\rho^{n-1}\,d\rho\,d\xi=\frac{1}{4}f(1-).$$

(ii) *If $\frac{f}{v}$ satisfies hypothesis $H(0)$, then*

$$\lim_{m\to\infty} J^{j,0}_{R_m}(f)(0)=\lim_{m\to\infty}-\frac{1}{2\pi i}\int_0^1\int_{\gamma^j_m} g_0(0,\xi,\rho)f(\xi)n\rho^{n-1}\,d\rho\,d\xi=\frac{1}{4}f(0+).$$

(iii) *If $a\in(0,1)$ and if $\frac{f}{v}$ satisfies hypothesis $H(a)$, then*

$$\begin{aligned}\lim_{m\to\infty} J^{j,0}_{R_m}(f)(a)&=\lim_{m\to\infty}-\frac{1}{2\pi i}\int_0^1\int_{\gamma^j_m} g_0(a,\xi,\rho)f(\xi)n\rho^{n-1}\,d\rho\,d\xi\\&=\frac{1}{4}\{f(a-)+f(a+)\}.\end{aligned}$$

Proof of (i). Because

$$g_0(x,\xi,\rho)=\frac{1}{n\rho^{n-1}}\frac{v(x)}{v(\xi)}\begin{cases}\sum_{k=1}^{\mu}\omega_k e^{\rho\omega_k(x-\xi)} & \text{for } x\ge\xi,\\ -\sum_{k=\mu+1}^{n}\omega_k e^{\rho\omega_k(x-\xi)} & \text{for } x<\xi,\end{cases}$$

we get

$$\begin{aligned}I:&=\lim_{m\to\infty}-\frac{1}{2\pi i}\int_0^1\int_{\gamma^0_m} g_0(1,\xi,\rho)f(\xi)n\rho^{n-1}\,d\xi\\&=\lim_{m\to\infty}-\frac{1}{2\pi i}\sum_{k=1}^{\mu}\int_0^1\int_{\gamma^0_m}\frac{v(1)}{v(\xi)}\omega_k e^{\rho\omega_k(1-\xi)}f(\xi)\,d\rho\,d\xi.\end{aligned}$$

Integrating with respect to ρ and applying Lemma 5.2 (i), we obtain from $\mu=\frac{n}{2}$

$$I=-\frac{1}{2\pi i}\mu(-\frac{i\pi}{n}f(1-0)\frac{v(1)}{v(1)})=\frac{1}{4}f(1-).$$

(ii) and (iii) are proved in the same way.

Lemma 5.4. *For the functions $z_k(\xi,\rho)$, $1 \le k \le n$, defined by (2.3), the following formulae are valid:*

$$\begin{aligned} z_k^{\{\nu\}}(\xi,\rho) &= \frac{w_{n-\nu,k}(\xi,\rho)}{w(\xi,\rho)}, \quad \nu = 0,1,\dots,n-1, \\ z_k^{\{n\}}(\xi,\rho) &= \lambda z_k(\xi,\rho), \end{aligned} \tag{5.2}$$

$k = 1,2,\dots,n$, $\xi \in [0,1]$, $\rho \in S$; *where $w_{sk}(\xi,\rho)$ is the algebraic complement of $y_k^{(s-1)}(\xi,\rho)$ with respect to $w(\xi,\rho)$.*

Proof. Using the definition (2.3) of the functions $z_k(\xi,\rho)$, the definition of the quasi derivatives (2.4), the rule of differentiating of determinants, and taking into consideration that the functions $y_j(\xi,\rho)$, $j = 1,2,\dots,n$, are solutions of equation (1.1), we easily obtain the statement of Lemma 5.4 (cf. [13, Lemma 1.1]).

In the sequel we shall need some results from [4, 8], which are stated below as lemmas.

Lemma 5.5. *If $f(x) \in L_p[0,1]$ $(1 \le p \le \infty)$, then* ([4])

$$E_\nu(f)_p \le C\omega(f;\frac{1}{\nu})_p,$$

where $E_\nu(f)_p$ is the best approximation to $f(x)$ in the metric of $L_p[0,1]$ by algebraic polynomials of degree $\ell \le \nu \in \mathbb{N}$.

Lemma 5.6. *If Q_ν is an arbitrary algebraic polynomial of degree ν, $1 \le p < q \le \infty$, s is a natural number, then there exist a constant $C(p,q)$, depending only on p and q, and a constant $C(s)$, depending only on s, such that* ([8])

a) $\|Q_\nu\|_q \le C(p,q)\nu^{2(\frac{1}{p}-\frac{1}{q})}\|Q_\nu\|_p$;

b) $\|Q_\nu^{(s)}\|_p \le C(s)\nu^{2s}\|Q_\nu\|_p$.

Using Lemmas 5.5 and 5.6 we obtain

Lemma 5.7 ([13, Lemma 1.5])**.** *If $p_1 \in H_q^\alpha[0,1]$, then for sufficiently large $|\rho|$ and any $\ell \in \mathbb{N}$*

$$\eta(\rho) = O\Big((\Psi_0(\ell) + \frac{\ell^2}{|\rho|})\sum_{1\le j<s\le n}\chi_{js}(\rho,r) + \frac{\ell^{\frac{2}{q}}}{|\rho|}\Big),$$

where

$$\Psi_0(\ell) = \frac{1}{\ln^\alpha \ell} \quad \text{for} \quad \alpha > 0,\ \Psi_0(\ell) = o(1) \quad \textit{for} \quad \alpha = 0$$

and

$$\chi_{js}(\rho,r) = \left(\frac{1-\exp(r\ Re\ \rho(\omega_j-\omega_s))}{Re\ \rho(\omega_s-\omega_j)}\right)^{\frac{1}{r}}.$$

Lemma 5.8 ([13, Lemma 1.6])**.** *If $\alpha_1, \alpha_2 \geq 1$, β_1 and β_2 are nonzero complex numbers, $\zeta_R = \{z \in \mathbb{C} : z = R\exp(i\varphi), \varphi_1 \leq \varphi \leq \varphi_2\}$, $R > 0$, $\operatorname{Re} \beta_1 z \geq 0$ and $\operatorname{Re} \beta_2 z \geq 0$ for $z \in \zeta_R$, then the integral*

$$J_R = \int_{\zeta_R} \prod_{s=1}^{2} \left(\frac{1 - \exp(-\alpha_s \, Re \, \beta_s z)}{Re \, \beta_s z} \right)^{\frac{1}{\alpha_s}} |dz|$$

satisfies the following estimates for sufficiently large R:

1) $J_R \leq C$, *when* $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} > 1$;
2) $J_R \leq C \ \ln R$, *when* $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$;
3) $J_R \leq C \ R^{1-(\frac{1}{\alpha_1}+\frac{1}{\alpha_2})}$, *when* $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} < 1$.

The following lemmas are central for evaluating the error terms in Section 4. For the formulation of these lemmas we set $\sigma_R = \{\rho \in S | \ |\rho| = R\}$.

Lemma 5.9. *Let $p_1 \in H_q^\alpha[0,1]$, $f \in H_r^\beta[0,1]$, $\frac{1}{q} + \frac{1}{r} = 1$, $\alpha + \beta > 1$. If $M(x,\rho)$ is uniformly bounded for $x \in [0,1]$, $\rho \in S$ and if $N(x,\rho) = O(\eta(\rho))$ for $x \in [0,1]$, $\rho \in S$, then the following asymptotics formulae are valid as $R \to \infty$ uniformly with respect to x :*

(i) $\int_{\sigma_R} M(x,\rho) \int_0^1 (z_k(\xi,\rho) - \frac{\omega_k}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi}) f(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho = o(1)$ *for* $k = \mu+1, \dots, n$;
(ii) $\int_{\sigma_R} M(x,\rho) \int_0^1 e^{\rho\omega_k} (z_k(\xi,\rho) - \frac{\omega_k}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi}) f(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho = o(1)$ *for* $k = 1, \dots, \mu$;
(iii) $\int_{\sigma_R} N(x,\rho) \int_0^1 z_k(\xi,\rho) f(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho = o(1)$ $k = \mu+1, \dots, n$;
(iv) *for* $\int_{\sigma_R} N(x,\rho) \int_0^1 e^{\rho\omega_k} z_k(\xi,\rho) f(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho = o(1)$ *for* $k = 1, \dots, \mu$.

Proof of (i). For simplicity we confine ourselves to the case $\alpha > 0, \beta > 0$; the proof is similar for the case $\alpha = 0$ or $\beta = 0$. Let $F_\ell(\xi)$ be the algebraic polynomial of best approximation to $f(\xi)$ in the metric of $L_r[0,1]$ and of degree $\leq \ell$. Obviously,

$$\begin{aligned} A_R &:= \int_{\sigma_R} M(x,\rho) \int_0^1 (z_k(\xi,\rho) - \frac{\omega_k}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi}) f(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho \\ &= \int_{\sigma_R} M(x,\rho) \int_0^1 (z_k(\xi,\rho) - \frac{\omega_k}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi}) f_\ell(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho \\ &\quad + \int_{\sigma_R} M(x,\rho) \int_0^1 (z_k(\xi,\rho) - \frac{\omega_k}{n\rho^{n-1}} \frac{1}{v(\xi)} e^{-\rho\omega_k\xi}) F_\ell(\xi)\, d\xi \ \ n\rho^{n-1}\, d\rho \\ &=: A_{1R} + A_{2R}, \end{aligned}$$

where $f_\ell = f - F_\ell$.

First we consider A_{1R}. Using formula (2.5), Lemma 5.5 and the definition of the class $H_r^\beta[0,1]$ we obtain

$$\begin{aligned} |A_{1R}| &\leq C \int_{\sigma_R} \int_0^1 \eta(\rho) |e^{-\rho\omega_k\xi} \, f_\ell(\xi)|\, d\xi\, |d\rho| \\ &\leq C \int_{\sigma_R} \eta(\rho) \|f_\ell\|_r (\int_0^1 e^{-q\operatorname{Re}(\rho\omega_k\xi)} d\xi)^{\frac{1}{q}} \, |d\rho| \leq C \int_{\sigma_R} \eta(\rho) \frac{1}{\ln^\beta \ell} \kappa_k(\rho,q)\, |d\rho|, \end{aligned}$$

where

$$\kappa_k(\rho,q)=(\frac{1-e^{-q\mathrm{Re}\rho\omega_k}}{\mathrm{Re}\ \ \rho\omega_k})^{\frac{1}{q}}.$$

In view of Lemmas 5.7 and 5.8 we find

$$\begin{aligned}|A_{1R}|&\le C\int_{\sigma_R}(\frac{1}{\ln^{\alpha+\beta}\ell}\kappa_k(\rho,q)\sum_{1\le j<s\le n}\chi_{js}(\rho,r)\\ &\quad+\frac{\ell^2}{|\rho|\ln^{\beta}\ell}\ \ \kappa_k(\rho,q)\sum_{1\le j<s\le n}\chi_{js}(\rho,r)+\frac{\ell^{\frac{2}{q}}}{|\rho|\ln^{\beta}\ell}\ \ \kappa_k(\rho,q))\,|d\rho|\\ &\le C\left(\frac{\ln R}{\ln^{\alpha+\beta}\ell}+\frac{\ell^2\ \ \ln R}{R\ln^{\beta}\ell}+\begin{cases}\frac{\ell^2\ln R}{R\ \ln^{\beta}\ell} & \text{for } q=1\\ \frac{\ell^{\frac{2}{q}}}{R^{\frac{1}{q}}\ \ln^{\beta}\ell} & \text{for } q>1\end{cases}\right).\end{aligned}$$

Because ℓ is a parameter, we assume $\ell=int(R^{\frac{1}{4}})$, where $int(x)$ denotes the integral part of x, and obtain

$$A_{1R}=o(1)\quad\text{as}\quad R\to\infty. \tag{5.3}$$

Now we consider A_{2R}. Using formula (5.2), the definition of $v(\xi)$ and the definition (2.4) of the quasi derivatives, integrating once by parts, and applying formula (2.5) we infer

$$\begin{aligned}B_R(\rho)&:=\int_0^1(z_k(\xi,\rho)-\frac{\omega_k}{n\rho^{n-1}}\frac{1}{v(\xi)}e^{-\rho\omega_k\xi})F_\ell(\xi)\,d\xi\\
&=\frac{1}{\lambda}\int_0^1 z_k^{\{n\}}(\xi,\rho)F_\ell(\xi)\,d\xi-\frac{\omega_k}{n\rho^{n-1}}\int_0^1 e^{-\rho\omega_k\xi}\frac{1}{v(\xi)}F_\ell(\xi)\,d\xi\\
&=-\frac{1}{\lambda}\int_0^1 F_\ell(\xi)dz_k^{\{n-1\}}(\xi,\rho)+\frac{1}{\lambda}\int_0^1 p_n(\xi)z_k^{\{0\}}(\xi,\rho)F_\ell(\xi)d\xi\\
&+\frac{1}{n\lambda}\int_0^1\frac{1}{v(\xi)}F_\ell(\xi)de^{-\rho\omega_k\xi}\\
&=-\frac{1}{\lambda}F_\ell(1)z_k^{\{n-1\}}(1,\rho)+\frac{1}{\lambda}F_\ell(0)z_k^{\{n-1\}}(0,\rho)+\frac{1}{\lambda}\int_0^1 z_k^{\{n-1\}}(\xi,\rho)F'_\ell(\xi)\,d\xi\\
&+\frac{1}{\lambda}\int_0^1 p_n(\xi)z_k(\xi,\rho)F_\ell(\xi)d\xi+\frac{1}{n\lambda}\frac{1}{v(1)}F_\ell(1)e^{-\rho\omega_k}-\frac{1}{n\lambda}F_\ell(0)\\
&-\frac{1}{n\lambda}\int_0^1 e^{-\rho\omega_k\xi}\frac{F'_\ell(\xi)}{v(\xi)}\,d\xi-\frac{1}{n\lambda}\int_0^1 e^{-\rho\omega_k\xi}F_\ell(\xi)\frac{p_1(\xi)}{nv(\xi)}d\xi\\
&=\frac{1}{\lambda}F_\ell(1)(\frac{1}{nv(1)}e^{-\rho\omega_k}-z_k^{\{n-1\}}(1,\rho))-\frac{1}{\lambda}F_\ell(0)(\frac{1}{n}-z_k^{\{n-1\}}(0,\rho))\\
&-\frac{1}{\lambda}\int_0^1 F'_\ell(\xi)(\frac{1}{nv(\xi)}e^{-\rho\omega_k\xi}-z_k^{\{n-1\}}(\xi,\rho))\,d\xi+\frac{1}{\lambda}\int_0^1 p_n(\xi)z_k(\xi,\rho)F_\ell(\xi)\,d\xi\\
&-\frac{1}{n^2\lambda}\int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}\,d\xi.\end{aligned}$$

Now applying Lemma 5.6 we find

$$
\begin{aligned}
&|B_R(\rho)| \\
&\le C(\frac{1}{|\rho|^n}|F_\ell(1)|\eta(\rho) + \frac{1}{|\rho|^n}|F_\ell(0)|\eta(\rho) + \frac{1}{|\rho|^n}\int_0^1 |F_\ell'(\xi)e^{-\rho\omega_k\xi}|\eta(\rho)\,d\xi \\
&+ \frac{1}{|\rho|^{2n-1}}\int_0^1 |p_n(\xi)F_\ell(\xi)e^{-\rho\omega_k\xi}|\,d\xi + \frac{1}{|\rho|^n}|\int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}\,d\xi|) \\
&\le C(\frac{1}{|\rho|^n}(\ell^{\frac{2}{r}}\eta(\rho) + \ell^2\eta(\rho)(\int_0^1 e^{-q\mathrm{Re}(\rho\omega_k\xi)}d\xi)^{\frac{1}{q}} + \frac{\|F_\ell\|_\infty}{|\rho|^{n-1}} \\
&+ |\int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}d\xi|)) \\
&\le C(\frac{1}{|\rho|^n}(\ell^{\frac{2}{r}}\eta(\rho) + \ell^2\eta(\rho)\kappa_k(\rho,q) + \frac{\ell^{\frac{2}{r}}}{|\rho|^{n-1}} + |\int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}\,d\xi|)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|A_{2R}| &= \Big|\int_{\sigma_R} M(x,\rho)B_R(\rho)n\rho^{n-1}\,d\rho\Big| \qquad (5.4)\\
&\le C\int_{\sigma_R}(\frac{1}{|\rho|}(\ell^{\frac{2}{r}}\eta(\rho) + \ell^2\eta(\rho)\kappa_k(\rho,q) + \Big|\int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}\,d\xi\Big| + \frac{1}{|\rho|^{n-1}}))\,|d\rho|.
\end{aligned}
$$

For the evaluation of

$$E_\ell(\rho) := \int_0^1 F_\ell(\xi)\frac{p_1(\xi)}{v(\xi)}e^{-\rho\omega_k\xi}\,d\xi$$

we use the same method as we have already used for evaluating A_R (applying the best approximation polynomials, integrating by parts and so on). We find (cf. [13, formula (2.7)])

$$E_\ell(\rho) = O(\frac{1}{\ln^\alpha \ell} + \frac{\ell^2}{|\rho|}).$$

Substituting this estimate into (5.4), using Lemmas 5.7 and 5.8, choosing the parameter ℓ as $int(R^{\frac{1}{4}})$ we obtain

$$A_{2R} = o(1) \quad \text{as} \quad R \to \infty.$$

Consequently

$$A_R = A_{1R} + A_{2R} = o(1) \quad \text{as} \quad R \to \infty.$$

Formulas (ii), (iii), (iv) are obtained in the same way. Hence Lemma 5.9 has been proved.

In a similar way to the proof of Lemma 5.9 we can prove

Lemma 5.10. *Let $p_1 \in H_q^\alpha[0,1]$, $f \in H_r^\beta[0,1]$, $\frac{1}{q}+\frac{1}{r}=1$, $\alpha+\beta>1$. If $M(x,\rho)$ is a bounded function with respect to $x \in [0,1]$, $\rho \in S$ and $N(x,\rho)=O(\eta(\rho))$ for $x \in [0,1]$, $\rho \in S$, then the following asymptotic formulae are valid uniformly with respect to x as $R \to \infty$:*

$$(i) \qquad \int_{\sigma_R} M(x,\rho)\int_0^x e^{\rho\omega_k x}\Big(z_k(\xi,\rho)-\frac{\omega_k}{n\rho^{n-1}}\frac{1}{v(\xi)}e^{-\rho\omega_k\xi}\Big)f(\xi)\,d\xi n\rho^{n-1}d\rho = o(1)$$

for $k=1,\dots,\mu$;

$$(ii) \qquad \int_{\sigma_R} M(x,\rho)\int_x^1 e^{\rho\omega_k x}\Big(z_k(\xi,\rho)-\frac{\omega_k}{n\rho^{n-1}}\frac{1}{v(\xi)}e^{-\rho\omega_k\xi}\Big)f(\xi)\,d\xi \;\; n\rho^{n-1}\,d\rho = o(1)$$

for $k=\mu+1,\dots,n$;

$$(iii) \qquad \int_{\sigma_R} N(x,\rho)\int_0^x e^{\rho\omega_k x} z_k(\xi,\rho) f(\xi)\,d\xi \;\; n\rho^{n-1}\,d\rho = o(1)$$

for $k=1,\dots,\mu$;

$$(iv) \qquad \int_{\sigma_R} N(x,\rho)\int_x^1 e^{\rho\omega_k x} z_k(\xi,\rho) f(\xi)\,d\xi \;\; n\rho^{n-1}\,d\rho = o(1)$$

for $k=\mu+1,\dots,n$.

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