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ON THE EXISTENCE AND STABILITY OF BOUNDED ALMOST PERIODIC AND PERIODIC SOLUTIONS OF A SINGULARLY PERTURBED NONAUTONOMOUS SYSTEM

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Abstract. The existence of solutions which are bounded on $\mathbb{R}$, almost periodic or periodic is considered for a nonautonomous, singularly perturbed system of ordinary differential equations. In addition, the stability properties of these solutions are characterized by the construction of manifolds of initial data, the solutions for which approach the given solutions as $t \to +\infty$ ($t \to -\infty$) at an exponential rate, α, independent of the small parameter. The key hypotheses are that certain linear systems have exponential dichotomies on $\mathbb{R}$. Applications are made to traveling wave solutions of reaction diffusion systems which are "forced" by a traveling wave input.

0. Introduction. Consider the singularly perturbed system of ordinary differential equations

$$\begin{aligned} x' &= F(t,x,y,\epsilon), \\ \epsilon y' &= G(t,x,y,\epsilon) \end{aligned} \tag{0.1}$$

where $\epsilon > 0$ is a small parameter, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. We are interested in the existence and stability properties of bounded (periodic or almost periodic) solutions of (0.1) in the case that F and G are bounded (periodic or almost periodic) in t, uniformly for (x,y) in compact sets. It is assumed that the reduced problem

$$x' = F(t,x,y,0), \qquad 0 = G(t,x,y,0) \tag{0.2}$$

at $\epsilon = 0$ has a bounded (periodic or almost periodic) "outer" solution which we take to be the trivial solution, that is, we suppose

$$F(t,0,0,0) \equiv G(t,0,0,0) \equiv 0$$

so that $(x,y) = (0,0)$ satisfies (0.2). Then expanding (0.1) about the trivial solution gives

$$\begin{aligned} x' &= A(t,\epsilon)x + B(t,\epsilon)y + f(t,x,y,\epsilon), \\ \epsilon y' &= C(t,\epsilon)x + D(t,\epsilon)y + g(t,x,y,\epsilon). \end{aligned} \tag{0.3}$$

One can think of, e.g., $A(t,\epsilon)$ as $\partial F/\partial x(t,0,0,\epsilon)$, but, in fact, it is really (0.3) which we study in this paper.

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We assume that the matrices A, B, C, D are continuous and bounded on $\mathbb{R} \times [0, \epsilon_0]$, for some $\epsilon_0 > 0$, and are continuous in ϵ, uniformly in $t \in \mathbb{R}$. The nonlinear terms f and g are assumed to have small Lipschitz constants, uniformly in t, when $|x|$ and $|y|$ are small and $|f(t, 0, 0, \epsilon)|$, $|g(t, 0, 0, \epsilon)|$ are small and tend to zero as $\epsilon \to 0+$, uniformly in $t \in \mathbb{R}$.

The principle assumptions are that the linear systems

$$z' = A(t, 0)z \tag{0.4}$$

and

$$w' = \epsilon^{-1} D(t, \epsilon) w \tag{0.5}$$

have exponential dichotomies on $\mathbb{R}$. For (0.4) there is a projection P and positive numbers α and K such that

$$\begin{aligned} |Z(t)PZ^{-1}(s)| &\leq Ke^{-\alpha(t-s)}, \qquad t \geq s, \\ |Z(t)(I-P)Z^{-1}(s)| &\leq Ke^{-\alpha(s-t)}, \qquad s \geq t, \end{aligned} \tag{0.6}$$

where $Z(t)$, $Z(0) = I$, is a fundamental matrix for (0.2). For (0.5), there is a continuous family of projections $Q(\epsilon)$, and positive constants K' and $\overline{\beta}$ such that

$$\begin{aligned} |W(t, \epsilon)Q(\epsilon)W^{-1}(s, \epsilon)| &\leq K'e^{-\overline{\beta}(t-s)/\epsilon}, \qquad t \geq s, \\ |W(t, \epsilon)(I - Q(\epsilon))W^{-1}(s, \epsilon)| &\leq K'e^{-\overline{\beta}(s-t)/\epsilon}, \qquad s \geq t, \end{aligned} \tag{0.7}$$

where $W(t, \epsilon)$, $W(0, \epsilon) = I$, is the fundamental matrix for (0.5). There are several circumstances under which (0.7) holds, one of which is that $D(t, \epsilon) \to D_0$ as $\epsilon \to 0+$, uniformly in $t \in \mathbb{R}$, where D_0 is a constant matrix with no purely imaginary or zero eigenvalues.

It is also assumed that $C(t, 0) \equiv 0$. In the study of periodic (almost periodic) solutions of (0.1), it is natural to assume that A, B, C, D, f, g are periodic of the same period (almost periodic) in t.

Our main results, Theorems 1.4 and 3.1, establish the existence of a continuous family of bounded (periodic, almost periodic) solutions $(x^*(t, \epsilon), y^*(t, \epsilon))$ of (0.1) which tend to 0 uniformly in t as $\epsilon \to 0+$ (Theorem 1.4) and, if $C = \epsilon\overline{C}$, $g = \epsilon\bar{g}$, where C is bounded, $\bar{g}$ satisfies the same assumptions as does f, then $(x^*(t, \epsilon), y^*(t, \epsilon))$ has a k-dimensional stable manifold $S(\sigma, \epsilon)$, $k = \dim \mathcal{R}(P) + \dim \mathcal{R}(Q(\epsilon))$, consisting of initial conditions for (0.3) at $t = \sigma$ such that the corresponding solutions approach $(x^*(t, \epsilon), y^*(t, \epsilon))$ as $t \to +\infty$ at the exponential rate $\bar{\alpha} < \alpha$, where $\bar{\alpha}$ is independent of ϵ and is related to α in (0.6), and an $(n+m-k)$-dimensional unstable manifold $U(\sigma, \epsilon)$, consisting of initial values at $t = \sigma$ of solutions which approach $(x^*(t, \epsilon), y^*(t, \epsilon))$ as $t \to -\infty$ at the exponential rate $\bar{\alpha}$. It will be clear that our analysis leads immediately to series expansions for $(x^*(t, \epsilon), y^*(t, \epsilon))$ in the case that A, B, C, D, f, g have convergent series expansions in powers of ϵ with coefficients which are bounded (periodic or almost periodic) functions of t.

In a sense, for our problem, the singular nature of (0.3) does not play a role. This might be expected as there are no boundaries to bring out the singular character of the equation; we are looking for bounded solutions on $\mathbb{R}$. Basically, we find these solutions by regular perturbation techniques as in [5, Ch. IV]. Chang [3] considers the case that (0.1) is almost periodic and obtains the existence of the family of almost periodic solutions $(x^*(t,\epsilon)), y^*(t,\epsilon))$ but does not consider their stability properties. Work which is somewhat related to the subject of this paper but which influenced significantly the final form it took are the paper of Sacker and Sell [8] and the treatment of perturbed noncritical linear systems by Hale [5, Ch. IV]. Sacker and Sell assume the existence of a family of solutions $(x^*(t,\epsilon), y^*(t,\epsilon))$ and consider the conditional stability of this branch. More precisely, they give sufficient conditions for the existence of manifolds of initial data for which solutions approach $(x^*(t,\epsilon), y^*(t,\epsilon))$ as $t \to \infty$ ($t \to -\infty$) at an exponential rate α/ϵ, $\alpha > 0$. This work builds on earlier results of Hoppensteadt [6]. It also applies in the situation considered here as we make more restrictive hypotheses than in [8].

At least in spirit, the work of Flatto and Levinson [4] and Anasov [1] is related to ours. These authors considered the autonomous system (0.1) in the case that the reduced problem (0.2) has a periodic orbit. They show that a family of periodic solutions of (0.1) exists, and that these solutions approach the periodic solution of the reduced problem as $\epsilon \to 0+$, under suitable hypotheses. Our methods do not apply to this situation since in this case, the variational equation (0.4) does not satisfy an exponential dichotomy (0.6). A motivating example for this work is the singularly perturbed second order system

$$\epsilon D u'' + u' - f(t,u,\epsilon) = 0. \tag{0.8}$$

Suppose that for $\epsilon = 0$, (0.8) has a bounded (periodic or almost periodic) solution $u_0(t)$. The question that naturally arises is whether there is a continuous family of solutions of (0.8), $u_\epsilon(t)$, for $\epsilon > 0$ sufficiently small, which are bounded (periodic or almost periodic) and satisfy $u_\epsilon \to u_0$ uniformly as $\epsilon \to 0$. If such a family exists, then how are the stability properties of u_ϵ related to the stability properties of u_0, as a solution of the reduced problem

$$u' = f(t,u,0)$$

and the properties of the matrix D?

In order to answer these questions, assume that u_0'' exists and is bounded (periodic or almost periodic) and introduce the change of variables

$$u = u_0 + \bar{u}.$$

Then $\bar{u}$ satisfies

$$\epsilon D\bar{u}'' + \bar{u}' - A(t,\epsilon)\bar{u} = r(t,\bar{u},\epsilon)$$

where

$$A(t,\epsilon) = \frac{\partial f}{\partial u}(t,u_0(t),\epsilon)$$

and

$$r(\bar{u}, t, \epsilon) = f(t, u_0(t) + \bar{u}, \epsilon) - f(t, u_0(t), \epsilon) - A(t, \epsilon)\bar{u} \\ + f(t, u_0(t), \epsilon) - f(t, u_0(t), 0) - \epsilon D u_0''(t).$$

Finally, setting

$$x = \epsilon D\bar{u}' + \bar{u}, \qquad y = \epsilon D\bar{u}',$$

we obtain the system

$$\begin{aligned} x' &= A(t,\epsilon)x - A(t,\epsilon)y + r(t, x-y, \epsilon), \\ \epsilon y' &= \epsilon A(t,\epsilon)x - (\epsilon A(t,\epsilon) + D^{-1})y + \epsilon r(t, x-y, \epsilon). \end{aligned} \tag{0.9}$$

The problem is to find a family of bounded solutions $(x^*(t,\epsilon), y^*(t,\epsilon))$ of (0.9) such that $(x^*, y^*) \to (0,0)$ as $\epsilon \to 0+$, and to determine the stability properties of the family. This problem is treated in section 4 as an application of our main results, where it is shown that such a family exists and that its stability is determined by the stability properties of the variational equation

$$Z' = A(t,0)Z$$

and by the eigenvalues of D. In particular, if the variational equation is uniformly asymptotically stable and the eigenvalues of D have positive real parts, then the family of solutions $(x^*(t,\epsilon), y^*(t,\epsilon))$ is uniformly asymptotically stable for small $\epsilon > 0$. Moreover, nearby solutions are attracted to $(x^*(t,\epsilon), y^*(t,\epsilon))$ at an ϵ-independent exponential rate α where α is essentially determined by the exponential rate of decay of solutions of the variational equation.

Singularly perturbed second order systems may arise in many ways. Consider, for example, a reaction diffusion system with a traveling wave input

$$u_t = \sigma D u_{xx} + f(u) + h\left(\frac{x}{c} - t\right), \qquad x \in \mathbb{R}, \ t > 0. \tag{0.10}$$

The vector function $u(x,t) \in \mathbb{R}^n$, D is an $n \times n$ positive definite matrix, f is a smooth vector field and h is a bounded function on $\mathbb{R}$ which might be periodic or almost periodic. The parameters $\sigma > 0$ and $c > 0$ are, respectively, a scale factor for diffusion and the speed of the traveling wave. A natural question related to (0.1) is whether there is a traveling wave solution

$$u(x,t) = U(\tau), \qquad \tau = \frac{x}{c} - t, \tag{0.11}$$

where $U(\tau)$ is a bounded (periodic or almost periodic) function on $\mathbb{R}$.

Putting (0.11) into (0.10) gives a second order system for $U(\tau)$,

$$\epsilon^2 D \frac{d^2U}{d\tau^2} + \frac{dU}{d\tau} + f(U) + h(\tau) = 0, \tag{0.12}$$

where $\epsilon = \frac{\sqrt{\sigma}}{c}$. If ϵ is a small parameter, in other words, if the speed c of the traveling wave "forcing" is large compared to a measure of the characteristic speed of diffusion effects, then (0.12) has the required form (0.8) where $f(\tau, U, \epsilon) = -f(U) - h(\tau)$. Our results justify dropping the leading term in (0.12) and looking for bounded (periodic or almost periodic) solutions of the reduced system

$$\frac{dU}{d\tau} + f(U) + h(\tau) = 0. \tag{0.13}$$

If, for example, (0.13) with $h \equiv 0$ has a linearly asymptotically stable critical point x_0 with a substantial domain of attraction, then introducing a "blip" function $h(\tau)$, having compact support K in $\mathbb{R}$, into (0.13) will give rise to a bounded solution U_0 such that $U_0(\tau) = x_0$, $\tau < K$ and $U_0(\tau) \to x_0$ as $\tau \to +\infty$, provided that the forcing h is not so large as to move the solution out of the basin of attraction of x_0 during the "on period" K. Furthermore, if h and f are sufficiently smooth, then $d^2U_0/d\tau^2$ will be bounded and continuous on $\mathbb{R}$. The variational equation about U_0,

$$\frac{dZ}{d\tau} + f'(U_0(\tau))Z = 0,$$

is then uniformly asymptotically stable [5, Ch. III, Thm. 2.2]. Thus, our main results imply the existence of a family of bounded solutions $U(\tau, \epsilon)$ of (0.12) for small ϵ such that $U(\tau, \epsilon) \to U_0(\tau)$ uniformly in $\tau \in \mathbb{R}$. Hence, (0.10) has a traveling wave response to such a "blip" traveling wave input.

Even more interesting than the previously considered case may be the case that h is large enough to move the solution out of the basin of x_0 into the basin of some other attractor before it is shut off. This should give rise to an even more interesting bounded solution U_0. Another interesting case arises when h is T-periodic and (0.13) has a T-periodic (or almost periodic) response.

The present work was in fact motivated by a reaction diffusion system of the form (0.10) which arises in the study of the competition between two microbial populations for a limiting nutrient in a laboratory device called a gradostat which is arranged in a circular configuration. Nutrient supplied to the vessels of the gradostat from a circular configuration of reservoirs is arranged to have the form of a rotating wave. In the continuum limit of a large number of vessels, the ordinary differential equations for the nutrient and microbial population concentrations reduce to a singularly perturbed reaction diffusion system ($\sigma = \epsilon^2$) of the form (0.10), where h represents a rotating wave (x is understood modulo 1) of nutrient concentration, i.e., h is periodic. In [9], we show that there are rotating wave solutions.

The organization of this paper is as follows. In the next section we prove the existence of bounded (periodic or almost periodic) solutions of (0.3). In section 2, the linear homogeneous system (0.3) with $(f, g) = (0, 0)$ is shown to have an exponential dichotomy on $\mathbb{R}$ and this dichotomy is related to the dichotomies (0.6) and (0.7). The stability of the bounded solutions is investigated in section 3 and section 4 contains an application of our results to the second order system (0.8).

Let $\mathcal{B}$ be the Banach space of bounded continuous functions on $\mathbb{R}$ with range in $\mathbb{R}^k$ with norm $\|f\| = \sup\{|f(t)| : t \in \mathbb{R}\}$. The particular value of k $(n, m, m+n)$ should be clear from the context. Let AP denote the Banach subspace of almost periodic functions on $\mathbb{R}$ and let $\mathcal{P}_T$ be the subspace of T-periodic functions on $\mathbb{R}$.

1. Existence of solutions in the spaces $\mathcal{B}, AP$ and $\mathcal{P}_T$. In this section we consider the existence of solutions belonging to one of the spaces $\mathcal{B}, AP$ or $\mathcal{P}_T$ of the nonlinear system (0.3). Our main results will be stated for the space $\mathcal{B}$ with remarks indicating the modifications for the spaces AP and $\mathcal{P}_T$. The following hypotheses are assumed to hold throughout the section.

(H1) $A(t,\epsilon), B(t,\epsilon), C(t,\epsilon), D(t,\epsilon)$ are continuous and bounded matrix functions (of sizes $n \times n$, $n \times m$, $m \times n$, $m \times m$, respectively) defined on $\mathbb{R} \times [0,\epsilon_0]$. Moreover, they are continuous in ϵ, uniformly in $t \in \mathbb{R}$. We let M denote a common bound for the norm of each of these matrices for $(t,\epsilon) \in \mathbb{R} \times [0,\epsilon_0]$.

(H2) $D(t,0) = D_0$ is a constant matrix having no eigenvalues on the imaginary axis; $C(t,0) \equiv 0$.

(H3) The system $z' = A(t,0)z$ has an exponential dichotomy on $\mathbb{R}$ with projection P, exponent $\alpha > 0$ and constant $K > 0$:

$$\begin{aligned} &|Z(t)PZ^{-1}(s)| \le Ke^{-\alpha(t-s)}, \qquad t \ge s, \\ &|Z(t)(I-P)Z^{-1}(s)| \le Ke^{-\alpha(s-t)}, \qquad s \ge t \end{aligned} \tag{1.1}$$

where $Z(t)$ is a fundamental matrix solution.

We find the notation $A_\epsilon(t) \equiv A(t,\epsilon)$ convenient in some calculations of this section.

Lemma 1.1. *There exists $K > 0$ such that for each $g \in \mathcal{B}$, $0 < \epsilon \le \epsilon_0$, there exists a unique solution $u^*(g,\epsilon) \in \mathcal{B}$ of the equation*

$$\epsilon u' = D_0 u + g.$$

Moreover, the map $g \mapsto u^*(g,\epsilon)$ defines a bounded linear operator $K_\epsilon g$ satisfying $\|K_\epsilon\| \le K$, $0 < \epsilon \le \epsilon_0$. The map $\epsilon \mapsto K_\epsilon$ is continuous for $0 < \epsilon \le \epsilon_0$.

Proof. This is just [5, Ch. IV, Lemma 4.2] except for the last statement which trivially follows from the earlier part. Indeed, if we write $u(t,\epsilon) = u^*(g,\epsilon)(t)$ for $0 < \epsilon \le \epsilon_0$, $g \in \mathcal{B}$, then $z(t) = u(t,\epsilon) - u(t,\epsilon') \in \mathcal{B}$ satisfies

$$\epsilon z' = D_0 z + \frac{\epsilon - \epsilon'}{\epsilon'}[D_0 u(t,\epsilon') + g(t)].$$

It follows that

$$\|z\| \le K\frac{|\epsilon-\epsilon'|}{\epsilon'}[\|D_0\|\,\|u(t,\epsilon')\| + \|g\|] \le K\frac{|\epsilon-\epsilon'|}{\epsilon'}[\|D_0\|K+1]\|g\|.$$

This implies the last assertion.

Lemma 1.2. *There exists $N > 0$ such that for each $f \in \mathcal{B}$ there is a unique solution $v(f) \in \mathcal{B}$ of the equation*

$$v' = A_0(t)v + f(t).$$

The map $f \mapsto v(f)$ defines a bounded linear operator satisfying $\|v\| \le N\|f\|$.

Proof. This is a well-known consequence of hypothesis (H3). See [2, Prop. 2, pg. 69]. □

We remark that Lemma 1.1 holds with $\mathcal{B}$ replaced by AP or $\mathcal{P}_T$. Lemma 1.2 also holds with $\mathcal{B}$ replaced by AP or $\mathcal{P}_T$ provided $A_0(t)$ belongs to AP, respectively, $\mathcal{P}_T$. In the AP case, one has the usual module containment relations [5, 2, 7]. In the case that $A_0 \in \mathcal{P}_T$ and $f \in \mathcal{P}_T$ one can replace the assumption (H3) by the assumption that one is not a Floquet multiplier for $z' = A_0(t)z$.

We now consider the inhomogeneous equation

$$\begin{aligned} x' &= A(t,\epsilon)x + B(t,\epsilon)y + f(t), \\ \epsilon y' &= C(t,\epsilon)x + D(t,\epsilon)y + g(t) \end{aligned} \tag{1.2}$$

where $(f,g) \in \mathcal{B}$.

Proposition 1.3. *There exists ϵ_1, $0 < \epsilon_1 \le \epsilon_0$, positive functions $a(\epsilon), b(\epsilon), c(\epsilon)$, $d(\epsilon)$ defined for $0 < \epsilon \le \epsilon_1$, satisfying*

$$\lim_{\epsilon\to 0+} a(\epsilon) = 1 = \lim_{\epsilon\to 0+} d(\epsilon), \ \lim_{\epsilon\to 0+} c(\epsilon) = 0, \quad \lim_{\epsilon\to 0+} b(\epsilon) = M,$$
$$a(\epsilon),\ b(\epsilon),\ c(\epsilon),\ d(\epsilon) \le M + 1$$

such that for each $(f,g) \in \mathcal{B}$, $0 < \epsilon \le \epsilon_1$, there is a unique solution

$$(x(f,g,\epsilon), y(f,g,\epsilon)) \in \mathcal{B}$$

of (1.2). *The solution satisfies*

$$\|x\| \le a(\epsilon)\|f\| + b(\epsilon)\|g\|, \qquad \|y\| \le c(\epsilon)\|f\| + d(\epsilon)\|g\|. \tag{1.3}$$

The map $(f,g) \to (x(f,g,\epsilon), y(f,g,\epsilon))$ defines a bounded linear operator $\mathcal{L}(\epsilon)$ satisfying $\|\mathcal{L}(\epsilon)\| \le 2M + 2$ and $\epsilon \to \mathcal{L}(\epsilon)$ is continuous for $0 < \epsilon \le \epsilon_1$.

Proof. Choose $\epsilon_1 \le \epsilon_0$ so small that

$$N\|A_\epsilon - A_0\| + K\|C_\epsilon\|(1 + MN) \le \frac{1}{2}, \qquad (NM+1)K\|D_\epsilon - D_0\| \le \frac{1}{2},$$
$$(1 - N\|A_\epsilon - A_0\|)^{-1}(1 - K\|D_\epsilon - D_0\|)^{-1}M\|C_\epsilon\| < 1$$

for $0 < \epsilon \le \epsilon_1$. Hypotheses (H1) and (H2) ensure that such an ϵ_1 exists. Given $(f,g) \in \mathcal{B}$, $(x_0, y_0) \in \mathcal{B}$, and $0 < \epsilon \le \epsilon_1$, define $(x,y) \in \mathcal{B}$ as the unique solution in $\mathcal{B}$ of

$$\begin{aligned} x' &= A_0(t)x + [A_\epsilon(t) - A_0(t)]x_0(t) + B_\epsilon(t)y(t) + f(t), \\ \epsilon y' &= D_0 y + [D_\epsilon(t) - D_0]y_0(t) + C_\epsilon(t)x_0(t) + g(t). \end{aligned} \tag{1.4}$$

Note that the second equation in (1.4) is solved first. It has a unique solution $y \in \mathcal{B}$ by Lemma 1.1 and (H1). Then this y is put into the first equation in (1.4) which is then solved for $x \in \mathcal{B}$ using Lemma 1.2 and (H1). Writing $(x,y) = T(x_0, y_0; f, g, \epsilon)$, then solving (1.2) is equivalent to finding a fixed point of $T(\cdot, \cdot; f, g, \epsilon)$.

If $(x,y) = T(x_0, y_0; f, g, \epsilon)$ and $(\bar{x}, \bar{y}) = T(\bar{x}_0, \bar{y}_0; f, g, \epsilon)$ and letting $u = x - \bar{x}$, $v = y - \bar{y}$ we find that u and v satisfy

$$\begin{aligned} u' &= A_0(t)u + [A_\epsilon(t) - A_0(t)](x_0 - \bar{x}_0) + B_\epsilon(t)v, \\ \epsilon v' &= D_0 v + [D_\epsilon(t) - D_0(t)](y_0 - \bar{y}_0) + C_\epsilon(t)(x_0 - \bar{x}_0). \end{aligned}$$

Lemmas 1.1 and 1.2 and (H1) imply that

$$\begin{aligned} \|u\| &\le N[\|A_\epsilon - A_0\| \, \|x_0 - \bar{x}_0\| + M\|v\|], \\ \|v\| &\le K[\|D_\epsilon - D_0\| \, \|y_0 - \bar{y}_0\| + \|C_\epsilon\| \, \|x_0 - \bar{x}_0\|], \end{aligned}$$

and this leads to the estimate

$$\begin{aligned} \|x - \bar{x}\| + \|y - \bar{y}\| &\le [N\|A_\epsilon - A_0\| + (MN+1)K\|C_\epsilon\|]\|x_0 - \bar{x}_0\| \\ &\quad + [(NM+1)K\|D_\epsilon - D_0\|]\|y_0 - \bar{y}_0\| \\ &\le \frac{1}{2}[\|x_0 - \bar{x}_0\| + \|y_0 - \bar{y}_0\|]. \end{aligned}$$

The contraction mapping principle implies that T has a unique fixed point $(x^*, y^*) \in \mathcal{B}$ which is obviously a linear function of $(f,g) \in \mathcal{B}$ and also depends on $\epsilon \in (0, \epsilon_1]$. We may estimate (x^*, y^*) directly from (1.4) using Lemmas 1.1 and 1.2 as follows:

$$\begin{aligned} \|x^*\| &\le N\|A_\epsilon - A_0\| \, \|x^*\| + M\|y^*\| + \|f\|, \\ \|y^*\| &\le K\|D_\epsilon - D_0\| \, \|y^*\| + \|C_\epsilon\| \, \|x^*\| + \|g\| \end{aligned}$$

which imply

$$\begin{aligned} \|x^*\| &\le (1 - N\|A_\epsilon - A_0\|)^{-1}[M\|y^*\| + \|f\|], \\ \|y^*\| &\le (1 - K\|D_\epsilon - D_0\|)^{-1}[\|C_\epsilon\| \, \|x^*\| + \|g\|]. \end{aligned}$$

Putting the second inequality into the first gives

$$\begin{aligned} \|x^*\| \le &[1 - (1 - N\|A_\epsilon - A_0\|)^{-1}M(1 - K\|D_\epsilon - D_0\|)^{-1}\|C_\epsilon\|]^{-1} \\ &\times (1 - N\|A_\epsilon - A_0\|)^{-1}[\|f\| + M(1 - K\|D_\epsilon - D_0\|)^{-1}\|g\|]. \end{aligned}$$

Let $p(\epsilon) = [1 - N\|A_\epsilon - A_0\|)^{-1}$ and $q(\epsilon) = (1 - K\|D_\epsilon - D_0\|)^{-1}$. Then we obtain the estimates (1.3) with

$$\begin{aligned} a(\epsilon) &= (1 - p(\epsilon)q(\epsilon)M\|C_\epsilon\|)^{-1}p(\epsilon), & b(\epsilon) &= a(\epsilon)Mq(\epsilon), \\ c(\epsilon) &= q(\epsilon)\|C_\epsilon\|a(\epsilon), & d(\epsilon) &= q(\epsilon)(\|C_\epsilon\|b(\epsilon) + 1). \end{aligned}$$

The linear operator $(x^*, y^*) = \mathcal{L}(\epsilon)(f, g)$ is bounded with

$$\begin{aligned}\|x^*\| + \|y^*\| &= \|\mathcal{L}(\epsilon)(f, g)\| \le (a(\epsilon) + c(\epsilon))\|f\| + (b(\epsilon) + d(\epsilon))\|g\| \\ &\le (2M + 2)(\|f\| + \|g\|)\end{aligned}$$

provided ϵ_1 is sufficiently small that $a, b, c, d \le M+1$ for $0 < \epsilon \le \epsilon_1$, which we assume is the case. Thus, $\|\mathcal{L}_\epsilon\| \le 2M + 2$. The continuity of the map $\epsilon \mapsto \mathcal{L}(\epsilon)$ can be shown exactly as in Lemma 1.1, although with much more work. □

Proposition 1.3 remains valid if $\mathcal{B}$ is replaced by AP or $\mathcal{P}_T$ provided that A, B, C, D belong to AP, respectively $\mathcal{P}_T$. In the AP case, the usual module containment relations hold. In the $\mathcal{P}_T$ case, (H3) can be relaxed as in the remark following Lemma 1.2.

Later, we will want to consider the case that $C(t, \epsilon) = \epsilon\overline{C}(t, \epsilon)$, $|\overline{C}(t, \epsilon)| \le M$, $\overline{C}(t, \epsilon)$ satisfies the hypothesis (H1), and $g(t) = \epsilon\bar{g}(t)$, $\bar{g} \in \mathcal{B}$, AP or $\mathcal{P}_T$. In this case, the estimates of Proposition 1.3 imply that $y(f, \epsilon g, \epsilon) = \epsilon\bar{y}(f, g, \epsilon)$ where $\bar{y}$ satisfies an estimate like that of the second part of (1.3).

Consider the nonlinear system

$$\begin{aligned}x' &= A(t, \epsilon)x + B(t, \epsilon)y + f(t, x, y, \epsilon), \\ \epsilon y' &= C(t, \epsilon)x + D(t, \epsilon)y + g(t, x, y, \epsilon).\end{aligned} \tag{1.4}$$

The following assumptions are made concerning f and g (see, e.g., [5, Ch. IV]).

(H4) f, g are continuous functions of all four arguments (t, x, y, ϵ) such that $t \in \mathbb{R}$, $|x|, |y| \le \rho_0$, $0 \le \epsilon \le \epsilon_0$ and both functions are continuous in (x, y, ϵ) uniformly in $t \in \mathbb{R}$. Furthermore, there exist nondecreasing functions $M(\epsilon)$ and $\eta(\rho, \epsilon)$, $0 \le \epsilon \le \epsilon_0$, $0 \le \rho \le \rho_0$ satisfying $\lim_{\epsilon\to 0} M(\epsilon) = 0$, $\lim_{(\rho,\epsilon)\to(0,0)} \eta(\rho, \epsilon) = 0$, such that

$$|f(t, 0, 0, \epsilon)| \le M(\epsilon), \quad |g(t, 0, 0, \epsilon)| \le M(\epsilon), \qquad t \in \mathbb{R},\ 0 \le \epsilon \le \epsilon_0,$$

and

$$\begin{aligned}|f(t, x, y, \epsilon) - f(t, \bar{x}, \bar{y}, \epsilon)| &\le \eta(\rho, \epsilon)[|x - \bar{x}| + |y - \bar{y}|], \\ |g(t, x, y, \epsilon) - g(t, \bar{x}, \bar{y}, \epsilon)| &\le \eta(\rho, \epsilon)[|x - \bar{x}| + |y - \bar{y}|]\end{aligned}$$

holds for all $t \in \mathbb{R}$, $|x|, |\bar{x}|, |y|, |\bar{y}| \le \rho$, $0 \le \epsilon \le \epsilon_0$, $\rho \le \rho_0$.

Theorem 1.4. *Assume* (H1)–(H4) *hold. Then there exists* ϵ_2, ρ_1, $0 < \epsilon_2 \le \epsilon_1$ *and* $0 < \rho_1 \le \rho_0$ *such that for each* ϵ *satisfying* $0 < \epsilon \le \epsilon_2$, (1.4) *has a unique solution* $(x^*(t, \epsilon), y^*(t, \epsilon)) \in \mathcal{B}$ *satisfying* $\|x\| \le \rho_1$, $\|y\| \le \rho_1$ *and this solution is continuous in* ϵ *uniformly in* $t \in \mathbb{R}$ *and satisfies* $\|x^*(\epsilon)\| + \|y^*(\epsilon)\| = O(M(\epsilon))$ *as* $\epsilon \to 0$.

Proof. This is a routine application of the uniform contraction principle as in [5, IV.2, Theorem 2.1]. We sketch the proof. Choose ρ_1 and ϵ_2 such that

$$\rho_1 > (2M + 2)[2\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2)], \qquad 2(2M + 2)\eta(\rho_1, \epsilon_2) \le \frac{1}{2}.$$

Given $(x_0, y_0) \in \mathcal{B}$ with $\|x_0\| \le \rho_1$, $\|y_0\| \le \rho_1$ and $0 < \epsilon \le \epsilon_2$, let (x, y) be the unique solution in $\mathcal{B}$ of

$$\begin{aligned} x' &= A(t,\epsilon)x + B(t,\epsilon)y + f(t, x_0(t), y_0(t), \epsilon), \\ \epsilon y' &= C(t,\epsilon)x + D(t,\epsilon)y + g(t, x_0(t), y_0(t), \epsilon). \end{aligned}$$

Such an $(x, y) \in \mathcal{B}$ exists by Proposition (1.3) and the estimate

$$\begin{aligned} |f(t, x_0(t), y_0(t), \epsilon)| &\le \eta(\rho_1, \epsilon_2)[\|x_0\| + \|y_0\|] + M(\epsilon_2) \\ &\le 2\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2),\ t \in \mathbb{R},\ 0 < \epsilon \le \epsilon_2. \end{aligned}$$

In fact, $(x, y) = \mathcal{L}(\epsilon)(f(\cdot, x_0, y_0, \epsilon),\ g(\cdot, x_0, y_0, \epsilon)) \equiv T(x_0, y_0, \epsilon)$. The existence of a solution of (1.4) in $\mathcal{B}$ is equivalent to the existence of a fixed point of the mapping T. We may estimate (x, y) using Proposition 1.3 as

$$\|x\| \le (a(\epsilon) + b(\epsilon))(2\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2)) \le (2M + 2)(2\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2)) < \rho_1$$

and similarly for $\|y\|$. Thus, $T(\cdot, \cdot, \epsilon)$ maps the closed set $F = \{(x_0, y_0) \in \mathcal{B} : \|x_0\| \le \rho_1,\ \|y_0\| \le \rho_1\}$ into itself for each ϵ with $0 < \epsilon \le \epsilon_2$.

Setting $(x, y) = T(x_0, y_0, \epsilon)$ and $(\bar{x}, \bar{y}) = T(\bar{x}_0, \bar{y}_0, \epsilon)$, it is easily shown that

$$\|x - \bar{x}\| \le (2M + 2)\eta(\rho_1, \epsilon_2)[\|x_0 - \bar{x}_0\| + \|y_0 - \bar{y}_0\|]$$

and similarly for $\|y - \bar{y}\|$, yielding

$$\begin{aligned} \|x - \bar{x}\| + \|y - \bar{y}\| &\le 2(2M + 2)\eta(\rho_1, \epsilon_2)[\|x_0 - \bar{x}_0\| + \|y_0 - \bar{y}_0\|] \\ &\le \frac{1}{2}[\|x_0 - \bar{x}_0\| + \|y_0 - \bar{y}_0\|]. \end{aligned}$$

Hence T is a uniform contraction. Since f, g are continuous in (x, y, ϵ) uniformly in $t \in \mathbb{R}$ it follows that $\epsilon \mapsto (f_\epsilon(t), g_\epsilon(t)) \in \mathcal{B}$ is continuous, where $f_\epsilon(t) = f(t, x_0(t), y_0(t), \epsilon)$. Since $\epsilon \to \mathcal{L}(\epsilon)$ is continuous, we conclude that for fixed $(x_0, y_0) \in \mathcal{B}$, the map $\epsilon \mapsto T(x_0, y_0, \epsilon)$ is continuous on $(0, \epsilon_2]$. The uniform contraction principle implies the existence of a unique fixed point $(x^*(\epsilon), y^*(\epsilon)) \in F$ which is a continuous function of ϵ, $0 < \epsilon \le \epsilon_2$.

Finally, (x^*, y^*) can be estimated directly from the defining system as

$$\|x^*(\epsilon)\| \le (2M + 2)[(\|x^*(\epsilon)\| + \|y^*(\epsilon)\|)\eta(\rho_1, \epsilon_2) + M(\epsilon)].$$

This, and a similar estimate for $\|y^*\|$ yields

$$\begin{aligned} \|x^*(\epsilon)\| + \|y^*(\epsilon)\| &\le 2(2M + 2)[(\|x^*(\epsilon)\| + \|y^*(\epsilon)\|)\eta(\rho_1, \epsilon_2) + M(\epsilon)] \\ &\le \frac{1}{2}(\|x^*(\epsilon)\| + \|y^*(\epsilon)\|) + 2(2M + 2)M(\epsilon) \end{aligned}$$

and, hence,

$$\|x^*(\epsilon)\| + \|y^*(\epsilon)\| \le 2M(\epsilon)(2M + 2),$$

completing our proof. □

If $A, B, C, D \in \mathcal{P}_T$ and f, g are T-periodic in t, then $(x^*, y^*) \in \mathcal{P}_T$. In this case (H3) can be relaxed as noted, following Lemma 1.2. If $A, B, C, D \in AP$ and f, g are almost periodic in t for each fixed (x, y, ϵ) with $|x| \leq \rho_0, |y| \leq \rho_0$, $0 < \epsilon \leq \epsilon_0$, then $(x^*, y^*) \in AP$. Indeed, as f, g satisfy the lipschitz condition of (H4), uniformly in $t \in \mathbb{R}$, it follows that f and g are almost periodic in t uniformly in (x, y) satisfying $|x| \leq \rho_0$ and $|y| \leq \rho_0$, for each ϵ, $0 < \epsilon \leq \epsilon_0$.

An important special case of Theorem 1.4 occurs when $C(t, \epsilon) = \epsilon\overline{C}(t, \epsilon)$, $|\overline{C}(t, \epsilon)| \leq M$ and when $g(t, x, y, \epsilon) = \epsilon\bar{g}(t, x, y, \epsilon)$ where both f and $\bar{g}$ satisfy the estimates of (H4). Using the remarks following Proposition 1.3, it is easily shown that in this case the estimates for $(x^*(t, \epsilon), y^*(t, \epsilon))$ can be sharpened as follows:

$$\|x^*(\epsilon)\| = O(M(\epsilon)), \quad \|y^*(\epsilon)\| = \epsilon O(M(\epsilon)) \qquad \text{as } \epsilon \to 0.$$

2. Exponential dichotomies. By virtue of Proposition 1.3, the linear inhomogeneous system (1.2) has a unique bounded solution for each bounded (f, g). Because of (H1), this implies that the homogeneous linear system (1.2) with $(f, g) = 0$ has an exponential dichotomy on $\mathbb{R}$ [7, 2]. In this section we relate the exponential dichotomy to the exponential dichotomies for the systems

$$z' = A(t, 0)z \tag{2.1}$$

and

$$w' = D_0 w. \tag{2.2}$$

Throughout this section, we assume that (H1), (H2) and (H3) hold except that the hypotheses $C(t, 0) \equiv 0$ in (H2) is no longer required since we will consider the equation

$$\begin{aligned} x' &= A(t, \epsilon)x + B(t, \epsilon)y, \\ \epsilon y' &= \epsilon C(t, \epsilon)x + D(t, \epsilon)y. \end{aligned} \tag{2.3}$$

As D_0 has no eigenvalues on the imaginary axis, (2.2) has an exponential dichotomy with projection which we denote by Q and exponent $\beta > 0$ and constant $L > 0$,

$$\begin{aligned} |e^{D_0 t} Q e^{-D_0 s}| &\leq L e^{-\beta(t-s)}, \qquad t \geq s, \\ |e^{D_0 t}(I - Q) e^{-D_0 s}| &\leq L e^{-\beta(s-t)}, \qquad s \geq t. \end{aligned}$$

The next two results are immediate consequences of the stability of exponential dichotomies to perturbation of the linear system [7, 2].

Lemma 2.1. *There exists $\bar{\epsilon}_1 \in (0, \epsilon_0]$, $K' > 0$, $\bar{\alpha} \in (3/4\alpha, \alpha)$ and a continuous family of projections $P(\epsilon)$, $0 \leq \epsilon \leq \bar{\epsilon}_1$, $P(0) = P$, such that*

$$\begin{aligned} |Z(t, \epsilon)P(\epsilon)Z^{-1}(s, \epsilon)| &\leq K' e^{-\bar{\alpha}(t-s)}, \qquad t \geq s, \\ |Z(t, \epsilon)(I - P(\epsilon))Z^{-1}(s)| &\leq K' e^{-\bar{\alpha}(s-t)}, \qquad s \geq t, \end{aligned}$$

where $Z(t, \epsilon)$, $Z(0, \epsilon) = I$, is the fundamental matrix for the linear system

$$z' = A(t, \epsilon)z. \tag{2.4}$$

Lemma 2.2. *By choosing $\bar{\epsilon}_1$ and K' of Lemma 2.1 smaller and larger, respectively, there exists $\bar{\beta} \in (0, \beta)$ and a continuous family of projections $Q(\epsilon)$, $0 < \epsilon \le \bar{\epsilon}_1$, $Q(\epsilon) \to Q$ as $\epsilon \to 0+$, such that*

$$|W(t,\epsilon)Q(\epsilon)W^{-1}(s,\epsilon)| \le K'e^{-\bar{\beta}(t-s)/\epsilon}, \qquad t \ge s,$$
$$|W(t,\epsilon)(I - Q(\epsilon))W^{-1}(s,\epsilon)| \le K'e^{-\bar{\beta}(s-t)/\epsilon}, \qquad s \ge t,$$

where $W(t,\epsilon)$, $W(0,\epsilon) = I$, is the fundamental matrix for the linear system

$$W' = \epsilon^{-1}D(t,\epsilon)W. \tag{2.5}$$

Lemma 2.2 follows easily from the "roughness" of exponential dichotomies by passing to the fast time $\tau = t/\epsilon$. Then (2.5) has the form $dW/d\tau = D(\epsilon\tau, \epsilon)W$ and $D(\epsilon\tau,\epsilon) \to D_0$ uniformly in $\tau \in \mathbb{R}$ by (H1).

For the results of this section, only the conclusion of Lemma 2.2 is used and, hence, we can replace the hypothesis "$D(t,0) \equiv D_0$" in (H2) by any hypotheses which are sufficient for Lemma 2.2 to hold, or just assume that Lemma 2.2 holds.

The change of variables $u = x$, $v = \eta y$ in (2.3), where $\eta > 0$ satisfying $\eta M < \min\{\bar{\alpha}/36(K')^2, \alpha/24K'\}$, is fixed, gives the system

$$\begin{aligned} u' &= A(t,\epsilon)u + \eta^{-1}B(t,\epsilon)v,\\ v' &= \eta C(t,\epsilon)u + \epsilon^{-1}D(t,\epsilon)v. \end{aligned} \tag{2.6}$$

We view (2.6) as a perturbation of the system

$$\begin{aligned} \bar{u}' &= A(t,\epsilon)\bar{u} + \eta^{-1}B(t,\epsilon)\bar{v},\\ \bar{v}' &= \epsilon^{-1}D(t,\epsilon)\bar{v}. \end{aligned} \tag{2.7}$$

The exponential dichotomy for (2.7) will be exhibited explicitly in terms of the dichotomies of Lemmas 2.1 and 2.2 and the dichotomy for (2.6) will be obtained by the perturbation theory for dichotomies. A fundamental matrix for (2.7) has the form

$$\Psi(t,\epsilon) = \begin{bmatrix} Z(t,\epsilon) & X(t,\epsilon)\\ 0 & W(t,\epsilon) \end{bmatrix},$$

where $X(t,\epsilon)$ is an $n \times m$ matrix solution of

$$X' = A(t,\epsilon)X + \eta^{-1}B(t,\epsilon)W(t,\epsilon). \tag{2.8}$$

Lemma 2.3. *There exists $\bar{\epsilon}_2 \in (0, \bar{\epsilon}_1)$ such that for $0 < \epsilon \le \bar{\epsilon}_2$, (2.8) has a matrix solution $X(t,\epsilon) = U(t,\epsilon)W(t,\epsilon)$ where the $n \times m$ matrix function $U(t,\epsilon)$ is continuous in ϵ uniformly in $t \in \mathbb{R}$ and satisfies*

$$|U(t,\epsilon)| \le \frac{4MK'\eta^{-1}\epsilon}{\bar{\beta}}, \qquad t \in \mathbb{R},\ 0 < \epsilon \le \bar{\epsilon}_2.$$

Proof. If $X(t,\epsilon) = U(t,\epsilon)W(t,\epsilon)$ satisfies (2.8) then $U(t,\epsilon)$ satisfies

$$U' = -\epsilon^{-1}UD(t,\epsilon) + A(t,\epsilon)U + \eta^{-1}B(t,\epsilon)$$

and, conversely, if $U(t,\epsilon)$ satisfies the equation above then $X(t,\epsilon) = U(t,\epsilon)W(t,\epsilon)$ satisfies (2.8). The transpose, $U^*(t,\epsilon)$, of $U(t,\epsilon)$ must satisfy

$$U^{*\prime} = -\epsilon^{-1}D^*(t,\epsilon)U^* + U^*A^*(t,\epsilon) + \eta^{-1}B^*(t,\epsilon).$$

We will show that this last equation has a unique bounded solution for all small ϵ. Observe that $\chi(t,\epsilon) = [W^*(t,\epsilon)]^{-1} = [W(t,\epsilon)^{-1}]^*$ is the fundamental matrix, $\chi(0,\epsilon) = I$, of

$$\chi' = -\epsilon^{-1}D^*(t,\epsilon)\chi$$

which is the equation adjoint to (2.5). Obviously, this linear system has an exponential dichotomy with projection $(I - Q^*(\epsilon))$, exponent $\bar\beta/\epsilon$ and constant K', where $Q(\epsilon)$, $\bar\beta$ and K' are as in Lemma 2.2. Choose $\bar\epsilon_2 \le \bar\epsilon_1$ such that $2\bar\epsilon_2 K'M/\bar\beta \le 1/2$. Given an $m \times n$ matrix function $\tilde U_0 \in \mathcal{B}$ and $0 < \epsilon \le \bar\epsilon_2$, let $\tilde U_1(t,\epsilon)$ be the unique solution in $\mathcal{B}$ of

$$\tilde U_1' = -\epsilon^{-1}D^*(t,\epsilon)\tilde U_1 + \tilde U_0 A^*(t,\epsilon) + \eta^{-1}B^*(t,\epsilon).$$

This solution is given by [2, Prop. 2, p. 69]

$$\begin{aligned}\tilde U_1(t,\epsilon) &= \int_{-\infty}^{t} W^*(t,\epsilon)^{-1}(I - Q^*(\epsilon))W^*(s,\epsilon)[\tilde U_0(s)A^*(s,\epsilon) + \eta^{-1}B^*(s,\epsilon)]\,ds \\ &\quad - \int_{t}^{\infty} W^*(t,\epsilon)^{-1}Q^*(\epsilon)W^*(s,\epsilon)[\tilde U_0(s)A^*(s,\epsilon) + \eta^{-1}B^*(s,\epsilon)]\,ds \\ &\equiv T(\tilde U_0,\epsilon).\end{aligned}$$

If $\tilde U_0, \overline{U}_0 \in \mathcal{B}$ and $\tilde U_1 = T(\tilde U_0,\epsilon)$, $\bar U_1 = T(\overline{U}_0,\epsilon)$, then

$$\begin{aligned}&|\tilde U_1(t,\epsilon) - \overline{U}_1(t,\epsilon)| \\ &\le \int_{-\infty}^{t} K'e^{-\bar\beta(t-s)/\epsilon}M\|\tilde U_0 - \overline{U}_0\|\,ds + \int_{t}^{\infty} K'e^{-\bar\beta(s-t)/\epsilon}M\|\tilde U_0 - \overline{U}_0\|\,ds \\ &\le \frac{2K'M\epsilon}{\bar\beta}\|\tilde U_0 - \overline{U}_0\| \le \frac{1}{2}\|\tilde U_0 - \overline{U}_0\|\end{aligned}$$

for $0 < \epsilon \le \bar\epsilon_2$. Thus, $T : \mathcal{B} \times (0,\bar\epsilon_2) \to \mathcal{B}$ is a uniform contraction mapping. It is easy to see that for each fixed $\tilde U_0 \in \mathcal{B}$, the map $\epsilon \to T(U_0,\epsilon)$ is continuous for $0 < \epsilon \le \bar\epsilon_2$. The uniform contraction principle implies that there exists a continuous mapping $\epsilon \to \hat U(\epsilon) \in \mathcal{B}$ such that $T(\hat U(\epsilon),\epsilon) = \hat U(\epsilon)$. We estimate $\hat U(t,\epsilon)$ directly from the integral equation as

$$\begin{aligned}|\hat U(t,\epsilon)| &\le \int_{-\infty}^{t} K'e^{-\bar\beta(t-s)/\epsilon}[M|\hat U(s,\epsilon)| + \eta^{-1}M]\,ds \\ &\quad + \int_{t}^{\infty} K'e^{-\bar\beta(s-t)/\epsilon}[M|\hat U(s,\epsilon)| + \eta^{-1}M]\,ds \\ &\le \frac{2\epsilon K'M}{\bar\beta}[\|\hat U\| + \eta^{-1}],\end{aligned}$$

which implies that

$$\|\hat{U}(\epsilon)\| \leq \big(1 - \frac{2\epsilon K' M}{\bar{\beta}}\big)^{-1} \frac{2\epsilon K' M}{\bar{\beta}} \eta^{-1} \leq \frac{4\epsilon K' M}{\bar{\beta}} \eta^{-1}.$$

If $U(t,\epsilon) = \hat{U}^*(t,\epsilon)$, then $X(t,\epsilon) = U(t,\epsilon)W(t,\epsilon)$ satisfies (2.8) proving the lemma.

Now let

$$\Pi(\epsilon) = \begin{bmatrix} P(\epsilon) & 0 \\ 0 & Q(\epsilon) \end{bmatrix}$$

and observe that

$$\Psi(t,\epsilon)\Pi(\epsilon)\Psi^{-1}(s,\epsilon) =$$
$$\begin{bmatrix} Z(t,\epsilon)P(\epsilon)Z^{-1}(s,\epsilon) & -Z(t,\epsilon)P(\epsilon)Z^{-1}(s,\epsilon)U(s,\epsilon) + U(t,\epsilon)W(t,\epsilon)Q(\epsilon)W^{-1}(s,\epsilon) \\ 0 & W(t,\epsilon)Q(\epsilon)W^{-1}(s,\epsilon) \end{bmatrix}.$$

This implies the estimate

$$\begin{aligned} |\Psi(t,\epsilon)\Pi(\epsilon)\Psi^{-1}(s,\epsilon)| &\leq K'[1 + \|U(\epsilon)\|][e^{-\bar{\alpha}(t-s)} + e^{-\bar{\beta}(t-s)/\epsilon}] \\ &\leq 2K'\Big[\frac{1 + 4MK'\eta^{-1}\epsilon}{\bar{\beta}}\Big] e^{-\bar{\alpha}(t-s)}, \qquad t \geq s \\ &\leq 3K' e^{-\bar{\alpha}(t-s)}, \qquad t \geq s \end{aligned}$$

if $0 < \epsilon \leq \bar{\epsilon}_3$ where $\bar{\epsilon}_3 \leq \bar{\epsilon}_2$ is chosen so that $\bar{\epsilon}_3 \bar{\alpha} \leq \bar{\beta}$ and $4MK'\eta^{-1}\bar{\epsilon}_3/\bar{\beta} \leq 1$. A corresponding estimate is easily obtained for $|\Psi(t,\epsilon)[I - \Pi(\epsilon)]\Psi^{-1}(s,\epsilon)|$ with t and s interchanged. Thus, (2.7) has an exponential dichotomy with projection

$$\tilde{\Pi}(\epsilon) \equiv \Psi(0,\epsilon)\Pi(\epsilon)\Psi^{-1}(0,\epsilon),$$

exponent $\bar{\alpha}$ and constant $3K'$, for $0 < \epsilon \leq \bar{\epsilon}_3$.

The projection $\tilde{\Pi}(\epsilon)$ is given by

$$\tilde{\Pi}(\epsilon) = \begin{bmatrix} P(\epsilon) & -P(\epsilon)U(0,\epsilon) + U(0,\epsilon)Q(\epsilon) \\ 0 & Q(\epsilon) \end{bmatrix}$$

from which it follows that $\epsilon \to \tilde{\Pi}(\epsilon)$ is continuous for $0 < \epsilon \leq \bar{\epsilon}_3$ and $\tilde{\Pi}(\epsilon) = \Pi(\epsilon) + O(\epsilon)$ as $\epsilon \to 0+$. The above considerations, together with the roughness of exponential dichotomies, lead to the main result of this section.

Theorem 2.4. *For $0 < \epsilon \leq \bar{\epsilon}_3$, there exists a continuous family of projections $\sum(\epsilon)$ and a constant $K'' > 0$ such that*

$$\begin{aligned} |\Phi(t,\epsilon)\Sigma(\epsilon)\Phi^{-1}(s,\epsilon)| &\leq K'' e^{-\alpha(t-s)/2}, \qquad t \geq s, \\ |\Phi(t,\epsilon)(I - \Sigma(\epsilon))\Phi^{-1}(s,\epsilon)| &\leq K'' e^{-\alpha(s-t)/2}, \qquad s \geq t \end{aligned}$$

where $\Phi(t,\epsilon)$, $\Phi(0,\epsilon)=I$, *is the fundamental matrix of* (2.3).

Proof It suffices to show that (2.6) has an exponential dichotomy with projections $\Sigma(\epsilon)$ and exponent $\alpha/2$. The coefficient matrix of (2.6) differs from that of (2.7) by the term $\eta C(t,\epsilon)$, the norm of which satisfies

$$|\eta C(t,\epsilon)| \le \eta M < \frac{\bar{\alpha}}{4(3K')^2}$$

by our choice of η. By [2, Prop. 1, p. 34], (2.6) has an exponential dichotomy with projection $\Sigma(\epsilon)$ and exponent $\bar{\alpha} - 2(3K')\eta M > 3\alpha/4 - \alpha/4 = \alpha/2$. This completes the proof.

3. Stable and unstable manifolds for $(x^*(t,\epsilon), y^*(t,\epsilon))$**.** In this section we discuss the stability of the small solutions in $\mathcal{B}, AP$ or $\mathcal{P}_T$ of the system

$$\begin{aligned} x' &= A(t,\epsilon)x + B(t,\epsilon)y + f(t,x,y,\epsilon),\\ y' &= C(t,\epsilon)x + \epsilon^{-1}D(t,\epsilon)y + g(t,x,y,\epsilon), \end{aligned} \tag{3.1}$$

for small positive ϵ. Note that (3.1) has a slightly different form than (1.4); C and g of (1.4) are now ϵC and ϵg in (3.1) and we have divided through by ϵ.

We assume that (H1)–(H3) hold except that we no longer require $C(t,0)=0$ in (H2). In addition, f and g are assumed to satisfy (H4). As remarked following Theorem 1.4, the estimates

$$\|x^*(\epsilon)\| = O(M(\epsilon)), \quad \|y^*(\epsilon)\| = \epsilon O(M(\epsilon)), \qquad \epsilon \to 0,$$

hold for the small solutions $(x^*(t,\epsilon), y^*(t,\epsilon))$ in $\mathcal{B}$ of (3.1). As in previous sections, results will be stated for the space $\mathcal{B}$ and remarks following the results will treat the special cases AP and $\mathcal{P}_T$.

We redefine ϵ_0 so that for $0<\epsilon\le\epsilon_0$, the conclusions of Theorem 1.4 and Theorem 2.4 hold. Furthermore, in order to simplify notation, let

$$\bar{\alpha} = \frac{\alpha}{2}, \qquad \overline{K} = K''$$

in Theorem 2.4. We write $z = \binom{x}{y}$ and rewrite (3.1) as

$$z' = E(t,\epsilon)z + F(t,z,\epsilon). \tag{3.2}$$

Let $z^*(t,\epsilon) = \binom{x^*(t,\epsilon)}{y^*(t,\epsilon)}$ be the solution of (3.2) in $\mathcal{B}$ given by Theorem 1.4 and let $z(t,\sigma,z_0,\epsilon)$ be the maximally extended solution of (3.2) satisfying $z(\sigma,\sigma,z_0,\epsilon)=z_0$.

As in Theorem 2.4, we let $\Phi(t,\epsilon)$, $\Phi(0,\epsilon)=I$, be the fundamental matrix for the linear system

$$z' = E(t,\epsilon)z \tag{3.3}$$

and $\Sigma(\epsilon)$ be the projection corresponding to the exponential dichotomy described in Theorem 2.4. Denote

$$\Sigma(\sigma,\epsilon) = \Phi(\sigma,\epsilon)\Sigma(\epsilon)\Phi^{-1}(\sigma,\epsilon), \qquad \sigma \in \mathbb{R}.$$

Then $\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}$ is the subspace of initial data z_0 such that the solution

$$z(t) = \Phi(t,\epsilon)\Phi^{-1}(\sigma,\epsilon)z_0$$

tends to zero as $t \to +\infty$ at the exponential rate $\bar{\alpha}$ and $(I - \Sigma(\sigma,\epsilon))\mathbb{R}^{n+m}$ is the subspace of initial data for which that solution approaches zero as $t \to -\infty$ at the exponential rate $\bar{\alpha}$.

Given $\delta > 0$, define

$$S(\sigma,\epsilon) = \{\bar{z} \in \mathbb{R}^{n+m} : |\Sigma(\sigma,\epsilon)\bar{z}| < \frac{\delta}{2\overline{K}} \text{ and for } z_0 = \bar{z} + z^*(\sigma,\epsilon), \\ |z(t,\sigma,z_0,\epsilon) - z^*(t,\epsilon)| < \delta,\ t \geq \sigma\},$$

and

$$U(\sigma,\epsilon) = \{\bar{z} \in \mathbb{R}^{n+m} : |(I - \Sigma(\sigma,\epsilon))\bar{z}| < \frac{\delta}{2\overline{K}} \text{ and for } z_0 = \bar{z} + z^*(\sigma;\epsilon), \\ |z(t,\sigma,z_0,\epsilon) - z^*(t,\epsilon)| < \delta,\ t \leq \sigma\}.$$

Let $B(r)$ denote the ball of radius r in $\mathbb{R}^{n+m}$.

The set $S(\sigma,\epsilon)$ describes a set of initial data $z_0 = z^*(\sigma,\epsilon) + \bar{z}$ for (3.2) at time $t = \sigma$ such that the corresponding solution $z(t,\sigma,z_0,\epsilon)$ remains close to $z^*(t,\epsilon)$ for $t \geq \sigma$.

Theorem 3.1. *Assume the hypotheses described in the beginning of this section hold. Then there exists $\delta > 0$ and $\epsilon_1 \in (0,\epsilon_0]$ such that the mapping $\Sigma(\sigma,\epsilon)$ is a homeomorphism of $S(\sigma,\epsilon)$ onto $\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m} \cap B(\delta/2\overline{K})$ and for each $z_0 = \bar{z} + z^*(\sigma,\epsilon)$ with $\bar{z} \in S(\sigma,\epsilon)$, $0 < \epsilon \leq \epsilon_1$,*

$$|z(t,\sigma,z_0,\epsilon) - z^*(t,\epsilon)| \leq 2\overline{K}|\Sigma(\sigma,\epsilon)\bar{z}|e^{-\bar{\alpha}(t-\sigma)/2}, \quad t \geq \sigma.$$

The map

$$g_s(\cdot,\sigma,\epsilon) : \Sigma(\sigma,\epsilon)\mathbb{R}^{n+m} \cap B(\delta/2\overline{K}) \to S(\sigma,\epsilon),$$

inverse to $\Sigma(\sigma,\epsilon)|_{S(\sigma,\epsilon)}$, is Lipschitz with constant $2\overline{K}$. The mapping $I - \Sigma(\sigma,\epsilon)$ is a homeomorphism of $U(\sigma,\epsilon)$ onto $(I - \Sigma(\sigma,\epsilon))\mathbb{R}^{n+m} \cap B(\delta/2\overline{K})$ and for each $z_0 = \bar{z} + z^*(\sigma,\epsilon)$ with $\bar{z} \in U(\sigma,\epsilon)$, $0 < \epsilon \leq \epsilon_1$,

$$|z(t,\sigma,z_0,\epsilon) - z^*(t,\epsilon)| \leq 2\overline{K}|(I - \Sigma(\sigma,\epsilon))\bar{z}|e^{\bar{\alpha}(t-\sigma)/2},\ t \leq \sigma.$$

The map

$$g_U(\cdot,\sigma,\epsilon) : (I - \Sigma(\sigma,\epsilon))\mathbb{R}^{n+m} \cap B(\delta/2\overline{K}) \to U(\sigma,\epsilon),$$

inverse to $(I-\Sigma(\sigma,\epsilon))|_{U(\sigma,\epsilon)}$*, is Lipschitz with constant* $2\overline{K}$.

An immediate consequence of Theorem 3.1 is that $z^*(t,\epsilon)$ is uniformly asymptotically stable if $\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}=\mathbb{R}^{n+m}$ and it is unstable when $\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}\neq\mathbb{R}^{n+m}$. In terms of the linear equations (2.1) and (2.2), $z^*(t,\epsilon)$ is uniformly asymptotically stable provided (2.1) and (2.2) are uniformly asymptotically stable.

In the general case, $S(\sigma,\epsilon)$ and $U(\sigma,\epsilon)$ are Lipschitz manifolds in $\mathbb{R}^{n+m}$, of dimensions $k\equiv\dim\left[\begin{pmatrix} P & O\\ O & Q\end{pmatrix}\mathbb{R}^{n+m}\right]$ and $m+n-k$, respectively, where P and Q are the projections associated with the exponential dichotomies for (2.1) and (2.2), respectively. Solutions of (3.2), $z(t,\sigma,z_0,\epsilon)$ with $z_0=z^*(\sigma,\epsilon)+\bar z$ and $\bar z\in S(\sigma,\epsilon)$ ($\bar z\in U(\sigma,\epsilon)$), approach $z^*(t,\epsilon)$ as $t\to+\infty$ ($t\to-\infty$) at the exponential rate $\bar\alpha/2$.

The proof of Theorem 3.1 follows standard lines (see, e.g., [5, IV.3, Thm. 3.1]) so we will merely sketch the main points. It is convenient to change variables in (3.2) to transform $z^*(t,\epsilon)$ to the trivial solution $z=z^*(t,\epsilon)+\bar z$. Then (3.2) becomes

$$\bar z' = E(t,\epsilon)\bar z+\overline{F}(t,\bar z,\epsilon) \tag{3.4}$$

where $\overline{F}(t,0,\epsilon)\equiv 0$ and the components of $\overline{F}$ satisfy similar bounds and lipschitz conditions to those satisfied by the components of F.

The solutions of (3.4) which are bounded for $t\geq\sigma$, satisfy the integral equation

$$\begin{aligned}\bar z(t) &= \Phi(t,\epsilon)\Phi^{-1}(\sigma,\epsilon)z_s+\int_\sigma^t \Phi(t,\epsilon)\Sigma(\epsilon)\Phi^{-1}(s,\epsilon)\overline{F}(s,\bar z(s),\epsilon)\,ds\\ &\quad-\int_t^\infty \Phi(t,\epsilon)(I-\Sigma(\epsilon))\Phi^{-1}(s,\epsilon)\overline{F}(s,\bar z(s),\epsilon)\,ds,\end{aligned} \tag{3.5}$$

where $z_s\in\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}$ is determined by $\Sigma(\sigma,\epsilon)\bar z(\sigma)=z_s$. Conversely, a bounded solution of the integral equation on $[\sigma,\infty)$ is a bounded solution of (3.4). For each fixed $z_s\in\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}\cap\overline{B(\delta/2\overline{K})}$, one can apply the uniform contraction mapping principle to obtain a unique solution $\hat z(t,\sigma,z_s,\epsilon)$ of (3.5) satisfying $|\hat z(t,\sigma,z_s,\epsilon)|<\delta$ for $t\geq\sigma$ (δ,ϵ_1 sufficiently small). Note that in the notation "$\hat z(t,\sigma,z_s,\epsilon)$," z_s is not the initial value at $t=\sigma$. This solution is continuous in σ,z_s,ϵ. An estimate gives

$$|\hat z(t,\sigma,z_s,\epsilon)-\hat z(t,\sigma,\tilde z_s,\epsilon)|\leq 2\overline{K}e^{-\frac{\bar\alpha}{2}(t-\sigma)}|z_s-\tilde z_s|,\ t\geq\sigma. \tag{3.6}$$

Then

$$S(\sigma,\epsilon)=\{\hat z(\sigma,\sigma,z_s,\epsilon)-z^*(\sigma,\epsilon): z_s\in\Sigma(\sigma,\epsilon)\mathbb{R}^{n+m}\cap B(\delta/2\overline{K})\}$$

and the inverse mapping $g_s(z_s,\sigma,\epsilon)$ is given by

$$g_s(z_s,\sigma,\epsilon)=\hat z(\sigma,\sigma,z_s,\epsilon)-z^*(\sigma,\epsilon)$$

which is Lipschitz with constant $2\overline{K}$ by (3.6). As in [5, IV.3, Thm. 3.1], an estimate yields

$$|g_s(z_s,\sigma,\epsilon)-g_s(\tilde z_s,\sigma,\epsilon)|\geq\frac{|z_s-\tilde z_s|}{2},$$

so g_s is a Lipschitz homeomorphism. Similar arguments apply to obtain the analogous statements for $U(\sigma, \epsilon)$. This completes our sketch of the proof.

4. Application to the Singularly Perturbed Second Order System. In this section our main results are applied to the singularly perturbed second order system (0.8), which we recall for the reader's convenience:

$$\epsilon D u'' + u' - f(t, u, \epsilon) = 0. \tag{4.1}$$

In (4.1), ϵ is a small positive parameter, D is an $n \times n$ matrix described below and f is continuous in all variables, as is $\partial f/\partial u$ and $\partial f/\partial \epsilon$. In addition, $f, \partial f/\partial u, \partial f/\partial \epsilon$ are continuous in (u, ϵ) uniformly in $t \in \mathbb{R}$ and bounded on bounded (u, ϵ) sets uniformly in $t \in \mathbb{R}$.

We assume that D has no purely imaginary eigenvalues except possibly zero and if zero is an eigenvalue then $N(D) = N(D^2)$, where $N(D)$ denotes the nullspace of D. Therefore, D is similar to

$$\left(\begin{array}{c|c} \overline{D} & 0 \\ \hline 0 & O_r \end{array}\right),$$

where $\overline{D}$ and O_r are square matrices, O_r is the $r \times r$ 0 matrix, $r = \dim N(D)$, and $\overline{D}$ is an $(n-r) \times (n-r)$ matrix with no purely imaginary eigenvalues. Of course, we are interested in the case that $r < n$. In turn, $\overline{D}$ is similar to

$$\left(\begin{array}{c|c} D_+ & 0 \\ \hline 0 & D_- \end{array}\right),$$

where D_+ (D_-) is an $m \times m$ ($l \times l$) matrix all of whose eigenvalues have positive (negative) real part and $m \geq 0$, $l \geq 0$, $m + l = n - r$.

In appropriate coordinates $u = (u_1, u_2) \in \mathbb{R}^{n-r} \times \mathbb{R}^r$ (4.1) has the form

$$\begin{aligned} \epsilon \overline{D} u_1'' + u_1' - f_1(t, u_1, u_2, \epsilon) &= 0, \\ u_2' - f_2(t, u_1, u_2, \epsilon) &= 0. \end{aligned} \tag{4.2}$$

We assume that the reduced equation

$$u' = f(t, u, 0) \tag{4.3}$$

has a solution $\hat{u} = (\hat{u}_1, \hat{u}_2)$ defined on $\mathbb{R}$ such that $\hat{u}$ and $\hat{u}_1''$ are bounded on $\mathbb{R}$. The change of variables

$$u = \hat{u} + \bar{u}, \quad x_1 = \epsilon \overline{D} \bar{u}_1' + \bar{u}_1, \quad x_2 = \bar{u}_2, \quad y = \epsilon \overline{D} \bar{u}_1',$$

leads to the system

$$\begin{aligned} x' &= A(t, \epsilon) x + B(t, \epsilon) y + r_1(t, x, y, \epsilon), \\ \epsilon y' &= \epsilon C(t, \epsilon) x - [\overline{D}^{-1} + \epsilon E(t, \epsilon)] y + \epsilon r_2(t, x, y, \epsilon), \end{aligned} \tag{4.4}$$

where

$$A(t,\epsilon) = \frac{\partial f}{\partial u}(t,\hat{u}(t),\epsilon) = \begin{bmatrix} A_{11}(t,\epsilon) & A_{12}(t,\epsilon) \\ A_{21}(t,\epsilon) & A_{22}(t,\epsilon) \end{bmatrix}$$

with

$$\begin{aligned} A_{ij}(t,\epsilon) &= \frac{\partial f_i}{\partial u_j}(t,\hat{u}(t),\epsilon), \quad B(t,\epsilon) = -\begin{bmatrix} A_{11}(t,\epsilon) \\ A_{21}(t,\epsilon) \end{bmatrix}, \\ C(t,\epsilon) &= [A_{11}(t,\epsilon)A_{12}(t,\epsilon)], \quad E(t,\epsilon) = A_{11}(t,\epsilon), \quad r_1(t,x,y,\epsilon) = (s_1,s_2), \\ s_1 &= f_1(t,\hat{u}_1 + x_1 - y, \hat{u}_2 + x_2, \epsilon) - f_1(t,\hat{u}_1,\hat{u}_2,0) - A_{11}(t,\epsilon)(x_1 - y) \\ &\quad - A_{12}(t,\epsilon)x_2 - \epsilon\overline{D}\hat{u}_1'', \\ s_2 &= f_2(t,\hat{u}_1 + x_1 - y, \hat{u}_2 + x_2, \epsilon) - f_2(t,\hat{u}_1,\hat{u}_2,0) - A_{21}(t,\epsilon)(x_1 - y) \\ &\quad - A_{22}(t,\epsilon)x_2, \\ r_2(t,x,y,\epsilon) &= s_1. \end{aligned}$$

We assume that the variational equation for (4.3) about $u_0(t)$,

$$z' = A(t,0)z, \tag{4.5}$$

satisfies an exponential dichotomy as in (H3) of section 1. It is apparent that (4.4) has the form (3.1) considered in Section 3. As an immediate consequence of Theorem 1.4 we have the following.

Corollary 4.1. *There exist positive constants ϵ_2 and ρ_1 such that for each ϵ satisfying $0 < \epsilon \le \epsilon_2$, (4.2) has a unique solution $u^*(t,\epsilon) \in \mathcal{B}$ satisfying $\|u^* - \hat{u}\| \le \rho_1$. This solution is continuous in ϵ uniformly in $t \in \mathbb{R}$ and satisfies the estimates*

$$\|u^*(\epsilon) - \hat{u}\| + \|u_1^{*\prime}(\epsilon) - \hat{u}_1'\| = O(\epsilon) \text{ as } \epsilon \to 0+.$$

Proof. Theorem 1.4 gives a continuous family of solutions $(x^*(t,\epsilon), y^*(t,\epsilon))$ of (4.3) which satisfy $\|x^*(\epsilon)\| = O(M(\epsilon))$ and $\|y^*(\epsilon)\| = \epsilon O(M(\epsilon))$ as $\epsilon \to 0$ (see remarks following Theorem 1.4). It is easily seen that $M(\epsilon) = C\epsilon$ for some $C > 0$. Hence, $u^*(\epsilon) = \hat{u} + (x_1^*(\epsilon), x_2^*(\epsilon)) - (y^*(\epsilon), 0)$ satisfies $\|u^*(\epsilon) - \hat{u}\| = O(\epsilon)$ and $\|u_1^{*\prime}(\epsilon) - \hat{u}_1'\| = \|\bar{u}_1'\| \le \epsilon^{-1}|D^{-1}|\,\|\epsilon D\bar{u}_1'\| = \epsilon^{-1}|D^{-1}|\,\|y^*\| = O(\epsilon)$. $\square$

In case f is T-periodic in t and $\hat{u}$ is T-periodic in t, then the hypothesis (H3) can be relaxed to the assumption that one is not a Floquet multiplier of (4.5). In this case $u^*(t,\epsilon)$ is T-periodic. If f is almost periodic in t and $\hat{u}(t)$ is almost periodic, then $u^*(t,\epsilon)$ is almost periodic. Note that in this case, our hypotheses imply that f is almost periodic in t uniformly in compact (x,ϵ) sets.

Viewed as a first order system of differential equations, (4.2) or (4.4) is $(2n-r)$-dimensional where $r = \dim N(A)$. Let P be the projection onto the stable subspace of (4.5), given in the definition of the exponential dichotomy for (4.5), $k = \dim \mathcal{R}(P) =$ dimension of stable subspace, and $m(l)$ the number of eigenvalues, counting multiplicity, of D having positive (negative) real part. Then the following result describes the stability properties of the solution $u^*(t,\epsilon)$ of (4.2).

Corollary 4.2. *For all sufficiently small* $\epsilon > 0$ *and for all* $\sigma \in \mathbb{R}$, *there is a Lipschitz manifold,* $S(\sigma, \epsilon)(U(\sigma, \epsilon))$, *of initial data at* $t = \sigma$, *having dimension* $k+m$ $(n-k+l)$ *in* $\mathbb{R}^{2n-r}$, *corresponding to solutions of* (4.2) *which are asymptotic to the solution* $(u_1^*(t,\epsilon), u_1^{*\prime}(t,\epsilon), u_2^*(t,\epsilon))$ *with an exponential rate of attraction as* $t \to +\infty$ $(t \to -\infty)$. *In particular, the latter solution is uniformly asymptotically stable if and only if the trivial solution of* (4.5) *is uniformly asymptotically stable and* $m = n - r$ *(so* $l = 0$*).*

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